from "Complex Systems: Fractionality, Time-delay and Synchronization" Springer 2011

Chapter 3
Differential-Delay Equations

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Abstract Periodic motions in DDE (Differential-Delay Equations) are typically created in Hopf bifurcations. In this chapter we examine this process from several points of view. Firstly we use Lindstedt's perturbation method to derive the Hopf Bifurcation Formula, which determines the stability of the periodic motion. Then we use the Two Variable Expansion Method (also known as Multiple Scales) to investigate the transient behavior involved in the approach to the periodic motion. Next we use Center Manifold Analysis to reduce the DDE from an an infinite dimensional evolution equation on a function space to a two dimensional ODE (Ordinary Differential Equation) on the center manifold, the latter being a surface tangent to the eigenspace associated with the Hopf bifurcation. Finally we provide an application to gene copying in which the delay is due to an observed time lag in the transcription process.

3.1 Introduction

Some dynamical processes are modeled as differential-delay equations (abbreviated DDE). An example is

$$\frac{dx(t)}{dt} = -x(t-T) - x(t)^3 \tag{3.1}$$

Here the rate of growth of x at time t is related both to the value of x at time t, and also to the value of x at a previous time, $t-T$.

Applications of DDE include laser dynamics (Wirkus and Rand, 2002), where the source of the delay is the time it takes light to travel from one point to another; machine tool vibrations (Kalmar-Nagy et al., 2001), where the delay is due to the

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dependence of the cutting force on thickness of the rotating workpiece; gene dynamics (Verdugo and Rand, 2008a), where the delay is due to the time required for messenger RNA to copy the genetic code and export it from the nucleus to the cytoplasm; investment analysis (Kot, 1979), where the delay is due to the time required by bookkeepers to determine the current state of the system; and physiological dynamics (Camacho et al., 2004), where the delay comes from the time it takes a substance to travel via the bloodstream from one organ to another.

A generalized version of Eq.(3.1) is

$$\frac{dx(t)}{dt} = \alpha x(t) + \beta x(t-T) + f(x(t), x(t-T)) \tag{3.2}$$

where α and β are coefficients and f is a strictly nonlinear function of $x(t)$ and $x(t-T)$. Here the linear terms $\alpha x(t)$ and $\beta x(t-T)$ have been separated from the strictly nonlinear terms, a step which facilitates stability analysis.

3.2 Stability of equilibrium

Equation (3.2) has the trivial equilibrium solution $x(t) = 0$. Is it stable? In order to find out, we linearize Eq.(3.2) about $x = 0$:

$$\frac{dx(t)}{dt} = \alpha x(t) + \beta x(t-T). \tag{3.3}$$

Since Eq.(3.3) has constant coefficients, we look for a solution in the form $x = e^{\lambda t}$, which gives the characteristic equation:

$$\lambda = \alpha + \beta e^{-\lambda T}. \tag{3.4}$$

Equation (3.4) is a transcendental equation and will in general have an infinite number of roots, which will either be real or will occur in complex conjugate pairs. The equilibrium $x = 0$ will be stable if all the real parts of the roots are negative, and unstable if any root has a positive real part. In the intermediate case in which no roots have positive real part, but some roots have zero real part, the linear stability analysis is inadequate, and nonlinear terms must be considered.

As an example, we consider Eq.(3.1), for which Eq.(3.4) becomes

$$\lambda = -e^{-\lambda T}. \tag{3.5}$$

Since λ will be complex in general, we set $\lambda = \nu + i\omega$, where ν and ω are the real and imaginary parts. Substitution into Eq.(3.5) gives two real equations:

$$\nu = -e^{-\nu T}\cos\omega T, \tag{3.6}$$

$$\omega = e^{-\nu T}\sin\omega T. \tag{3.7}$$

The question of stability will depend upon the value of the delay parameter T. Certainly when $T = 0$ the system is stable. By continuity, as T is increased from zero, there will come a first positive value of T for which $x = 0$ is not (linearly) stable. This can happen in one of two ways. Either a single real root will pass through the origin in the complex-λ plane, or a pair of complex conjugate roots will cross the imaginary axis. Since ν=ω=0 does not satisfy Eqs.(3.6),(3.7), the first case cannot occur.

In order to consider the second case of a purely imaginary pair of roots, we set $\nu = 0$ in Eqs.(3.6),(3.7), giving

$$0 = -\cos\omega T, \tag{3.8}$$

$$\omega = \sin\omega T. \tag{3.9}$$

Equation (3.8) gives ωT=$\pi/2$, whereupon Eq.(3.9) gives $\omega = 1$, from which we conclude that the critical value of delay T= T_{cr}=$\pi/2$. That is, **$\mathbf{x} = \mathbf{0}$ in Eq.(3.1) is stable for $\mathbf{T} < \pi/\mathbf{2}$ and is unstable for $\mathbf{T} > \pi/\mathbf{2}$.** Stability for T= T_{cr}=$\pi/2$ requires consideration of nonlinear terms.

In order to check these results we numerically integrate Eq.(3.1) using the MATLAB package DDE23. Note that this requires that the values of x be given on the entire interval $-T \leq t \leq 0$. Figures 3.1 and 3.2 show the results of numerical integration using the initial condition $x = 0.01$ on $-T \leq t \leq 0$. Figure 3.1 is for $T = \pi/2 - 0.01$ and shows stability, while Fig. 3.2 is for $T = \pi/2 + 0.01$ and shows instability, in agreement with the foregoing analysis.

3.3 Lindstedt's method

The change in stability observed in the preceding example will be accompanied by the birth of a limit cycle in a Hopf bifurcation. In order to obtain an approximation for the amplitude and frequency of the resulting periodic motion, we use Lindstedt's method (Rand and Armbruster, 1987; Rand, 2005).
We begin by stretching time,

$$\tau = \omega t. \tag{3.10}$$

Replacing t by τ as independent variable, Eq.(3.1) may be written in the form:

$$\omega \frac{dx(\tau)}{d\tau} = -x(\tau - \omega T) - x(\tau)^3. \tag{3.11}$$

Next we choose the delay T to be close to the critical value T_{cr}=$\pi/2$:

$$T = \frac{\pi}{2} + \Delta. \tag{3.12}$$

We introduce a perturbation parameter $\varepsilon \ll 1$ by scaling x:

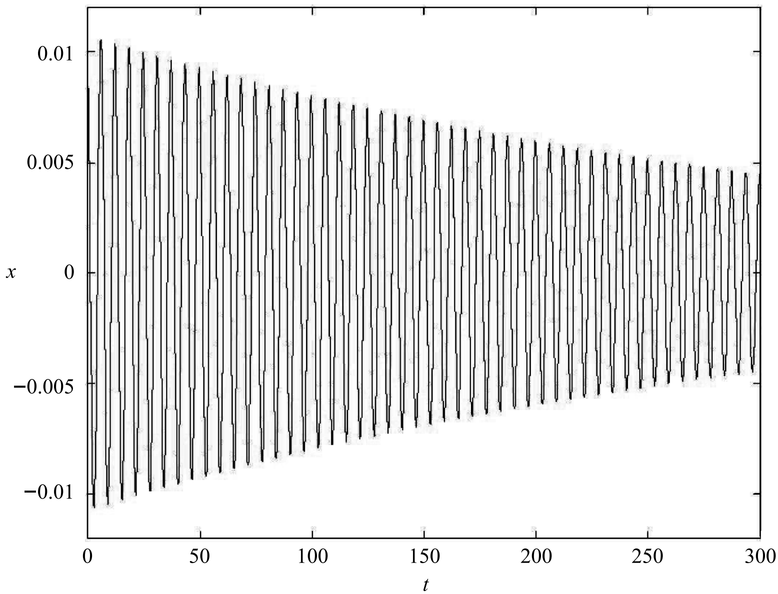

Fig. 3.1 Numerical integration of Eq.(3.1) for the initial condition $x = 0.01$ on $-T \leq t \leq 0$, for $T = \pi/2 - 0.01$.

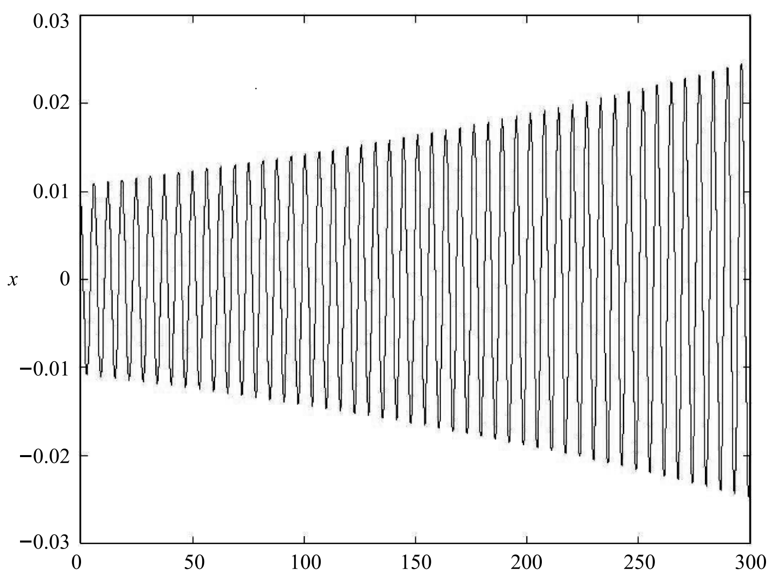

Fig. 3.2 Numerical integration of Eq.(3.1) for the initial condition $x = 0.01$ on $-T \leq t \leq 0$, for $T = \pi/2 + 0.01$.

$$x = \sqrt{\varepsilon}u. \tag{3.13}$$

Using Eq.(3.13), Eq.(3.11) becomes:

$$\omega \frac{du(\tau)}{d\tau} = -u(\tau - \omega T) - \varepsilon u(\tau)^3. \tag{3.14}$$

Next we scale Δ

$$\Delta = \varepsilon\mu, \tag{3.15}$$

and we expand u and ω in power series of ε:

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + O(\varepsilon^2), \tag{3.16}$$
$$\omega = 1 + \varepsilon k_1 + O(\varepsilon^2), \tag{3.17}$$

where we have used the fact that ω=1 when T=T_{cr}.

The delay term $u(\tau - \omega T)$ is handled by expanding it in Taylor series about ε=0:

$$u(\tau - \omega T) = u\left(\tau - (1 + \varepsilon k_1 + O(\varepsilon^2))(\frac{\pi}{2} + \varepsilon\mu)\right) \tag{3.18}$$
$$= u\left(\tau - \frac{\pi}{2} - \varepsilon(k_1\frac{\pi}{2} + \mu) + O(\varepsilon^2)\right) \tag{3.19}$$
$$= u(\tau - \frac{\pi}{2}) - \varepsilon(k_1\frac{\pi}{2} + \mu)\frac{du}{d\tau}(\tau - \frac{\pi}{2}) + O(\varepsilon^2). \tag{3.20}$$

Substituting into Eq.(3.14) and collecting terms gives:

$$\varepsilon^0: \quad \frac{du_0}{d\tau} + u_0(\tau - \frac{\pi}{2}) = 0, \tag{3.21}$$
$$\varepsilon^1: \quad \frac{du_1}{d\tau} + u_1(\tau - \frac{\pi}{2}) = -k_1\frac{du_0}{d\tau} + (k_1\frac{\pi}{2} + \mu)\frac{du_0}{d\tau}(\tau - \frac{\pi}{2}) - u_0^3. \tag{3.22}$$

Since Eq.(3.1) is autonomous, we may choose the phase of the periodic motion arbitrarily. This permits us to take the solution to Eq.(3.21) as:

$$u_0(\tau) = A\cos\tau, \tag{3.23}$$

where A is the approximate amplitude of the periodic motion. Substituting Eq.(3.23) into (3.22), we obtain:

$$\frac{du_1}{d\tau} + u_1(\tau - \frac{\pi}{2}) = k_1 A\sin\tau + (k_1\frac{\pi}{2} + \mu)A\cos\tau - A^3\left(\frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau\right), \tag{3.24}$$

For no secular terms, we equate to zero the coefficients of $\sin\tau$ and $\cos\tau$ on the RHS of Eq.(3.24). This gives:

$$k_1 = 0 \quad \text{and} \quad A = \frac{2}{\sqrt{3}}\sqrt{\mu}. \tag{3.25}$$

Now A is the amplitude in u. In order to obtain the amplitude in x, we multiply the second of Eqs.(3.25) by $\sqrt{\varepsilon}$, which, together with Eqs.(3.13) and (3.15), gives

$$\text{Amplitude of periodic motion in Eq.(3.1)} = \frac{2}{\sqrt{3}}\sqrt{\Delta} = \frac{2}{\sqrt{3}}\sqrt{\left(T - \frac{\pi}{2}\right)}. \quad (3.26)$$

This predicts, for example, that when $T = \pi/2 + 0.01$, the limit cycle born in the Hopf will have approximate amplitude of 0.1155. For comparison, numerical integration gives a value of 0.1145, see Figure 3.3.

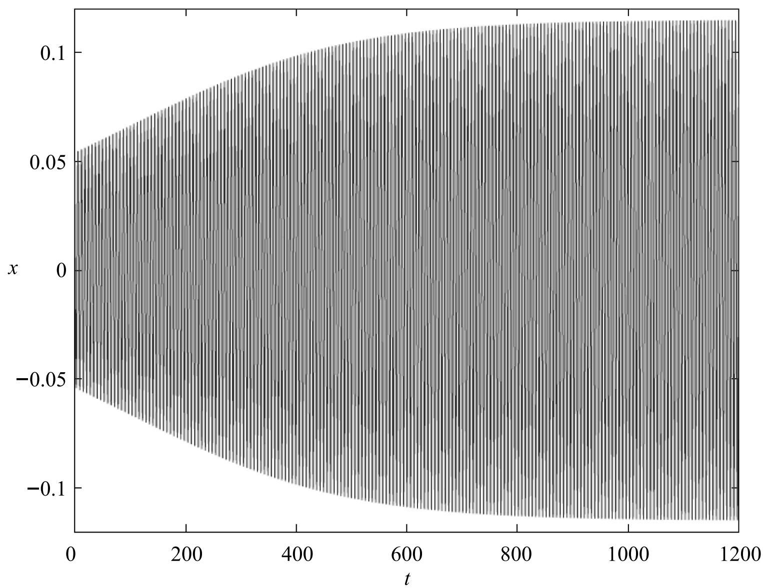

Fig. 3.3 Numerical integration of Eq.(3.1) for the initial condition $x = 0.05$ on $-T \leq t \leq 0$, for $T = \pi/2 + 0.01$.

3.4 Hopf bifurcation formula (Rand and Verdugo, 2007)

The treatment of the Hopf bifurcation in the previous section for Eq.(3.1) can be generalized to apply to a wide class of DDEs. In this section we present a formula for the amplitude of the resulting limit cycle for the DDE:

$$\frac{dx}{dt} = \alpha x + \beta x_d + a_1 x^2 + a_2 x x_d + a_3 x_d^2 + b_1 x^3 + b_2 x^2 x_d + b_3 x x_d^2 + b_4 x_d^3, \quad (3.27)$$

where $x = x(t)$ and $x_d = x(t-T)$. Here T is the delay. Associated with (3.27) is a linear DDE

$$\frac{dx}{dt} = \alpha x + \beta x_d. \tag{3.28}$$

We assume that (3.28) has a critical delay T_{cr} for which it exhibits a pair of pure imaginary eigenvalues $\pm \omega i$ corresponding to the solution

$$x = c_1 \cos \omega t + c_2 \sin \omega t \tag{3.29}$$

Then for values of delay T which lie close to T_{cr},

$$T = T_{cr} + \mu, \tag{3.30}$$

the nonlinear Eq.(3.27) may exhibit a periodic solution which can be written in the approximate form:

$$x = A \cos \omega t, \tag{3.31}$$

where the amplitude A can be obtained from the following expression for A^2:

$$A^2 = \frac{P}{Q} \mu, \tag{3.32}$$

where

$$P = 4\beta^3 (\beta - \alpha)(\beta + \alpha)^2 (-5\beta + 4\alpha), \tag{3.33}$$

$$\begin{aligned}
Q = \; & 15 b_4 \beta^6 T_{cr} + 5 b_2 \beta^6 T_{cr} + 3\alpha b_4 \beta^5 T_{cr} - 15\alpha b_3 \beta^5 T_{cr} + \alpha b_2 \beta^5 T_{cr} \\
& - 15\alpha b_1 \beta^5 T_{cr} - 22 a_3{}^2 \beta^5 T_{cr} - 7 a_2 a_3 \beta^5 T_{cr} - 14 a_1 a_3 \beta^5 T_{cr} - 3 a_2{}^2 \beta^5 T_{cr} \\
& - 7 a_1 a_2 \beta^5 T_{cr} - 4 a_1{}^2 \beta^5 T_{cr} - 12\alpha^2 b_4 \beta^4 T_{cr} - 3\alpha^2 b_3 \beta^4 T_{cr} + 6\alpha^2 b_2 \beta^4 T_{cr} \\
& - 3\alpha^2 b_1 \beta^4 T_{cr} + 12 a_3{}^2 \alpha \beta^4 T_{cr} + 37 a_2 a_3 \alpha \beta^4 T_{cr} + 30 a_1 a_3 \alpha \beta^4 T_{cr} \\
& + 7 a_2{}^2 \alpha \beta^4 T_{cr} + 19 a_1 a_2 \alpha \beta^4 T_{cr} + 18 a_1{}^2 \alpha \beta^4 T_{cr} + 12\alpha^3 b_3 \beta^3 T_{cr} \\
& + 2\alpha^3 b_2 \beta^3 T_{cr} + 12\alpha^3 b_1 \beta^3 T_{cr} + 4 a_3{}^2 \alpha^2 \beta^3 T_{cr} - 20 a_2 a_3 \alpha^2 \beta^3 T_{cr} \\
& - 16 a_1 a_3 \alpha^2 \beta^3 T_{cr} - 12 a_2{}^2 \alpha^2 \beta^3 T_{cr} - 26 a_1 a_2 \alpha^2 \beta^3 T_{cr} - 8 a_1{}^2 \alpha^2 \beta^3 T_{cr} \\
& - 8\alpha^4 b_2 \beta^2 T_{cr} - 4 a_2 a_3 \alpha^3 \beta^2 T_{cr} + 8 a_2{}^2 \alpha^3 \beta^2 T_{cr} + 8 a_1 a_2 \alpha^3 \beta^2 T_{cr} \\
& + 5 b_3 \beta^5 + 15 b_1 \beta^5 - 15\alpha b_4 \beta^4 + \alpha b_3 \beta^4 - 15\alpha b_2 \beta^4 + 3\alpha b_1 \beta^4 - 4 a_3{}^2 \beta^4 \\
& - 9 a_2 a_3 \beta^4 - 18 a_1 a_3 \beta^4 - a_2{}^2 \beta^4 - 9 a_1 a_2 \beta^4 - 18 a_1{}^2 \beta^4 - 3\alpha^2 b_4 \beta^3 \\
& + 6\alpha^2 b_3 \beta^3 - 3\alpha^2 b_2 \beta^3 - 12\alpha^2 b_1 \beta^3 + 26 a_3{}^2 \alpha \beta^3 + 19 a_2 a_3 \alpha \beta^3 \\
& + 30 a_1 a_3 \alpha \beta^3 + 11 a_2{}^2 \alpha \beta^3 + 33 a_1 a_2 \alpha \beta^3 + 12 a_1{}^2 \alpha \beta^3 + 12\alpha^3 b_4 \beta^2 \\
& + 2\alpha^3 b_3 \beta^2 + 12\alpha^3 b_2 \beta^2 - 8 a_3{}^2 \alpha^2 \beta^2 - 32 a_2 a_3 \alpha^2 \beta^2 - 12 a_1 a_3 \alpha^2 \beta^2 \\
& - 14 a_2{}^2 \alpha^2 \beta^2 - 18 a_1 a_2 \alpha^2 \beta^2 - 8\alpha^4 b_3 \beta - 8 a_3{}^2 \alpha^3 \beta + 8 a_2 a_3 \alpha^3 \beta \\
& + 4 a_2{}^2 \alpha^3 \beta + 8 a_2 a_3 \alpha^4
\end{aligned} \tag{3.34}$$

In Eq.(3.32), A is real so that $A^2 > 0$, which means that μ must have the same sign as $\frac{P}{Q}$.

Eq.(3.34) depends on μ, α, β, a_i, b_i and T_{cr}. This equation may be alternately written with T_{cr} expressed as a function of α and β. This relationship may be obtained by considering the linear DDE (3.28). Substituting Eq.(3.31) into Eq.(3.28) and equating to zero coefficients of $\sin(\omega t)$ and $\cos(\omega t)$, we obtain the two equations:

$$\beta \sin(\omega \mathrm{T_{cr}}) = -\omega, \qquad \beta \cos(\omega \mathrm{T_{cr}}) = -\alpha. \tag{3.35}$$

Squaring and adding these we obtain

$$\omega = \sqrt{\beta^2 - \alpha^2}. \tag{3.36}$$

Substituting (3.36) into the second of (3.35), we obtain the desired relationship between T_{cr} and α and β:

$$T_{cr} = \frac{\arccos\left(\frac{-\alpha}{\beta}\right)}{\sqrt{\beta^2 - \alpha^2}}. \tag{3.37}$$

3.4.1 Example 1

As an example, we consider the following DDE:

$$\frac{dx}{dt} = -x - 2x_d - x x_d - x^3. \tag{3.38}$$

This corresponds to the following parameter assignment in Eq.(3.27):

$$\alpha = -1,\ \beta = -2,\ a_1 = a_3 = b_2 = b_3 = b_4 = 0,\ a_2 = b_1 = -1. \tag{3.39}$$

The associated linearized equation (3.28) is stable for zero delay. As the delay T is increased, the origin first becomes unstable when $T = T_{cr}$, where Eq.(3.37) gives that

$$T_{cr} = \frac{\arccos \frac{-1}{2}}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}. \tag{3.40}$$

Substituting (3.39) and (3.40) into (3.32), (3.33), (3.34), we obtain:

$$A^2 = \frac{648\mu}{40\sqrt{3}\pi + 171} = 1.667\mu, \tag{3.41}$$

where we have set

$$T = T_{cr} + \mu = \frac{2\pi}{3\sqrt{3}} + \mu = 1.2092 + \mu. \tag{3.42}$$

Thus the origin is stable for $\mu < 0$ and unstable for $\mu > 0$. In order for A^2 in (3.41) to be positive, we require that $\mu > 0$. Therefore the limit cycle is born out of an unstable equilibrium point. Since the stability of the limit cycle must be the opposite of the stability of the equilibrium point from which it is born, we may conclude that the limit cycle is stable and that we have a *supercritical* Hopf. This result is in agreement with numerical integration of Eq.(3.38).

3.4.2 Derivation

In order to derive the result (3.32), (3.33), (3.34), we use Lindstedt's method. We begin by introducing a small parameter ε via the scaling

$$x = \varepsilon u. \tag{3.43}$$

The detuning μ of Eq.(3.30) is scaled like ε^2:

$$T = T_{cr} + \mu = T_{cr} + \varepsilon^2 \hat{\mu}. \tag{3.44}$$

Next we stretch time by replacing the independent variable t by τ, where

$$\tau = \Omega t. \tag{3.45}$$

This results in the following form of Eq.(3.27):

$$\Omega \frac{du}{d\tau} = \alpha u + \beta u_d + \varepsilon(a_1 u^2 + a_2 u u_d + a_3 u_d^2) + \varepsilon^2(b_1 u^3 + b_2 u^2 u_d + b_3 u u_d^2 + b_4 u_d^3), \tag{3.46}$$

where $u_d = u(\tau - \Omega T)$. We expand Ω in a power series in ε, omitting the $O(\varepsilon)$ term for convenience, since it turns out to be zero:

$$\Omega = \omega + \varepsilon^2 k_2 + \cdots \tag{3.47}$$

Next we expand the delay term u_d:

$$u_d = u(\tau - \Omega T) = u(\tau - (\omega + \varepsilon^2 k_2 + \cdots)(T_{cr} + \varepsilon^2 \hat{\mu})) \tag{3.48}$$

$$= u(\tau - \omega T_{cr} - \varepsilon^2(k_2 T_{cr} + \omega\hat{\mu}) + \cdots) \tag{3.49}$$

$$= u(\tau - \omega T_{cr}) - \varepsilon^2(k_2 T_{cr} + \omega\hat{\mu}) u'(\tau - \omega T_{cr}) + O(\varepsilon^3). \tag{3.50}$$

Finally we expand $u(\tau)$ in a power series in ε:

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \cdots \tag{3.51}$$

Substituting and collecting terms, we find:

$$\omega\frac{du_0}{d\tau}-\alpha u_0(\tau)-\beta u_0(\tau-\omega T_{cr})=0, \tag{3.52}$$

$$\omega\frac{du_1}{d\tau}-\alpha u_1(\tau)-\beta u_1(\tau-\omega T_{cr})$$
$$=a_1u_0(\tau)^2+a_2u_0(\tau)u_0(\tau-\omega T_{cr})+a_3u_0(\tau-\omega T_{cr})^2, \tag{3.53}$$

$$\omega\frac{du_2}{d\tau}-\alpha u_2(\tau)-\beta u_2(\tau-\omega T_{cr})=\cdots \tag{3.54}$$

where $\cdots$ stands for terms in u_0 and u_1, omitted here for brevity. We take the solution of the u_0 equation as (cf. Eq.(3.29) above):

$$u_0(\tau)=\hat{A}\cos(\tau). \tag{3.55}$$

We substitute (3.55) into (3.53) and obtain the following expression for u_1:

$$u_1(\tau)=m_1\sin(2\tau)+m_2\cos(2\tau)+m_3, \tag{3.56}$$

where m_1 is given by the equation:

$$m_1=-\frac{\hat{A}^2\,(2a_3\,\beta+a_2\,\beta-2a_1\,\beta-2a_3\,\alpha)\,\sqrt{\beta^2-\alpha^2}}{2\beta\,(\beta+\alpha)\,(5\beta-4\alpha)}. \tag{3.57}$$

and where m_2 and m_3 are given by similar equations, omitted here for brevity. In deriving (3.57), ω has been replaced by $\sqrt{\beta^2-\alpha^2}$ as in Eq.(3.36).

Next the expressions for u_0 and u_1, Eqs.(3.55),(3.56), are substituted into the u_2 equation, Eq.(3.54), and, after trigonometric simplifications have been performed, the coefficients of the resonant terms, $\sin\tau$ and $\cos\tau$, are equated to zero. This results in Eq.(3.32) for A^2 as well as an expression for k_2 (cf. Eq.(3.47)) which does not concern us here. (Note that $A=\varepsilon\hat{A}$ from Eqs.(3.31),(3.43),(3.55), and $\mu=\varepsilon^2\hat{\mu}$ from (3.44). The perturbation method gives $\hat{A}^2$ as a function of $\hat{\mu}$, but multiplication by ε^2 converts to a relation between A^2 and μ.)

3.4.3 Example 2

As a second example, we consider the case in which the quantity Q in Eqs.(3.32),(3.34) is zero. To generate such an example for the DDE (3.27), we embed the previous example in a one-parameter family of DDE's:

$$\frac{dx}{dt}=-x-2x_d-xx_d-\lambda x^3, \tag{3.58}$$

and we choose λ so that $Q=0$ in Eq.(3.32). This leads to the following critical value of λ:

$$\lambda=\lambda_{cr}=\frac{4\pi+3\sqrt{3}}{18(2\pi+3\sqrt{3})}=0.0859 \tag{3.59}$$

Since this choice for λ leads to $Q = 0$, Eq.(3.32) obviously cannot be used to find the limit cycle amplitude A. Instead we use Lindstedt's method, maintaining terms of $O(\varepsilon^4)$. The correct scalings in this case turn out to be (cf.Eqs.(3.44),(3.47)):

$$T = T_{cr} + \mu = \frac{2\pi}{3\sqrt{3}} + \varepsilon^4 \hat{\mu}, \tag{3.60}$$

$$\Omega = \omega + \varepsilon^2 k_2 + \varepsilon^4 k_4 + \cdots \tag{3.61}$$

We find that the limit cycle amplitude A satisfies the equation:

$$A^4 = -\Gamma \mu, \tag{3.62}$$

where we omit the closed form expression for Γ and give instead its approximate value, Γ=620.477.

The analysis of this example has assumed that the parameter λ exactly takes on the critical value given in Eq.(3.59). Let us consider a more robust model which allows λ to be detuned:

$$\lambda = \lambda_{cr} + \Lambda = \frac{4\pi + 3\sqrt{3}}{18(2\pi + 3\sqrt{3})} + \varepsilon^2 \hat{\Lambda}. \tag{3.63}$$

Using Lindstedt's method we obtain for this case the following equation on A:

$$A^4 + \sigma \Lambda A^2 + \Gamma \mu = 0, \tag{3.64}$$

where we omit the closed form expression for σ and give instead its approximate value, σ=342.689. Eq.(3.64) can have 0,1, or 2 positive real roots for A, each of which is a limit cycle in the original system. Thus the number of limit cycles which are born in the Hopf bifurcation depends on the detuning coefficients Λ and μ. Elementary use of the quadratic formula and the requirement that A^2 be both real and positive gives the following results: If $\mu < 0$ then there is one limit cycle. If $\mu > 0$ and $\sigma\Lambda < -2\sqrt{\Gamma\mu}$ then there are two limit cycles. If $\mu > 0$ and $\sigma\Lambda > -2\sqrt{\Gamma\mu}$ then there are no limit cycles.

3.4.4 Discussion

Although Lindstedt's method is a formal perturbation method, i.e., lacking a proof of convergence, our experience is that it gives the same results as the center manifold approach, which has a rigorous mathematical foundation. The center manifold approach has been described in many places, for example (Hassard et al., 1981; Campbell et al., 1995; Stepan, 1989; Kalmar-Nagy et al., 2001; Rand, 2005), and will be treated later in this Chapter. Since the DDE (3.27) is infinite dimensional (for example the characteristic equation of the linear DDE (3.28) is transcendental rather than polynomial, and hence has an infinite number of complex roots), the center manifold approach involves decomposing the original function space into a two dimensional center manifold (in which the Hopf bifurcation takes place) and an infinite dimensional function space representing the rest of the original phase space. The center manifold procedure is much more complicated than the Hopf calculation. Stepan refers to the center manifold calculation as “long and tedious” ((Stepan, 1989), p.112), and Campbell et al. refer to it as “algebraically daunting” ((Campbell et al., 1995), p.642). Thus the main advantage of the Hopf calculation is that it is simpler to understand and easier to execute than the center manifold approach.

The idea of using Lindstedt’s method on bifurcation problems in DDE goes back to a 1980 paper by Casal and Freedman (Casal and Freedman, 1980). That work provided the algorithm but not the Hopf bifurcation formula. We present the general expression for the Hopf bifurcation, as in Eqs.(3.32)—(3.34), as a convenience for researchers in DDE.

3.5 Transient behavior

We have seen that Lindstedt’s method can be used to obtain an approximation for the periodic motion of a DDE. This section is concerned with the use of perturbation methods to obtain approximate expressions for the *transient* behavior of DDEs, e.g. for the approach to a steady state periodic motion. In the case of ordinary differential equations (ODEs), a very popular method for obtaining transient behavior is the two variable expansion method (also know as multiple scales) (Cole, 1968; Nayfeh, 1973; Rand and Armbruster, 1987; Rand, 2005). In this section we show how this method can be applied to a DDE. See also (Das and Chatterjee, 2002; Wang and Hu, 2003; Das and Chatterjee, 2005; Nayfeh, 2008). Although this approximate method is strictly formal, its use is justified by center manifold considerations. Although the DDE is an infinite dimensional system, a wide class of problems involves the presence of a two dimensional invariant manifold, and it is the approximation of the transient flow on this surface which is the goal of this perturbation method.

3.5.1 Example

In order to illustrate the manner in which this perturbation method may be applied to DDEs, we choose a simple DDE problem, one that has an exact solution, namely:

$$\frac{dx}{dt} = -x(t-T), \quad T = \frac{\pi}{2} + \varepsilon\mu. \tag{3.65}$$

3.5.2 Exact solution

In Eq.(3.65) we set

$$x(t) = \exp(\lambda t), \tag{3.66}$$

giving the characteristic equation

$$\lambda = -\exp(-\lambda T). \tag{3.67}$$

When $\varepsilon = 0$, that is when $T = \pi/2$, Eq.(3.67) has the exact solution $\lambda = i$. Thus for $\varepsilon > 0$ we set

$$\lambda = i + \varepsilon(a + ib). \tag{3.68}$$

Substituting Eq.(3.68) into (3.67) and equating real and imaginary parts to zero, we obtain the following two equations on a and b:

$$a\varepsilon = \exp\left(-a\varepsilon^2\mu - \frac{\pi a\varepsilon}{2}\right)\sin\left(b\varepsilon^2\mu + \varepsilon\mu + \frac{\pi b\varepsilon}{2}\right), \tag{3.69}$$

$$b\varepsilon + 1 = \exp\left(-a\varepsilon^2\mu - \frac{\pi a\varepsilon}{2}\right)\cos\left(b\varepsilon^2\mu + \varepsilon\mu + \frac{\pi b\varepsilon}{2}\right). \tag{3.70}$$

If Eqs.(3.69),(3.70) were to be solved for a and b, we would obtain a solution to (3.65) in the form:

$$x(t) = \exp(\varepsilon at)\begin{cases}\sin & (1+\varepsilon b)t,\\ \cos & (1+\varepsilon b)t.\end{cases} \tag{3.71}$$

In order to obtain a version of the exact solution (3.71) which will be useful for comparing solutions to Eq.(3.65) obtained by the perturbation method, we now derive approximate expressions for a and b. Taylor expanding Eqs.(3.69),(3.70) for small ε, we obtain

$$a\varepsilon + \cdots = \frac{(b\pi + 2\mu)\,\varepsilon}{2} + \cdots \tag{3.72}$$

$$1 + b\varepsilon + \cdots = 1 - \frac{\pi a\varepsilon}{2} + \cdots \tag{3.73}$$

Solving Eqs.(3.72),(3.73) for a and b, we obtain the approximate expressions:

$$a = \frac{4\mu}{\pi^2+4} + O(\varepsilon), \quad b = -\frac{2\pi\mu}{\pi^2+4} + O(\varepsilon) \tag{3.74}$$

3.5.3 Two variable expansion method (also known as multiple scales)

In applying this method to the example of Eq.(3.65), we replace time t by two time variables: regular time $\xi = t$, and slow time $\eta = \varepsilon t$. The dependent variable $x(t)$ is then replaced by $x(\xi, \eta)$, and Eq.(3.65) becomes

$$\frac{\partial x}{\partial \xi} + \varepsilon \frac{\partial x}{\partial \eta} = -x(\xi - T, \eta - \varepsilon T). \tag{3.75}$$

Note that since $T = \pi/2 + \varepsilon\mu$, the delayed term may be expanded for small ε as follows:

$$x(\xi - T, \eta - \varepsilon T) = x(\xi - \frac{\pi}{2}, \eta) - \varepsilon\mu \frac{\partial x_d}{\partial \xi} - \varepsilon \frac{\pi}{2} \frac{\partial x_d}{\partial \eta} + O(\varepsilon^2), \tag{3.76}$$

where x_d is an abbreviation for $x(\xi - \frac{\pi}{2}, \eta)$. Next we expand $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$ and collect terms in Eqs.(3.75),(3.76), giving

$$\frac{\partial x_0}{\partial \xi} + x_0(\xi - \frac{\pi}{2}, \eta) = 0, \tag{3.77}$$

$$\frac{\partial x_1}{\partial \xi} + x_1(\xi - \frac{\pi}{2}, \eta) = \mu \frac{\partial x_{0d}}{\partial \xi} + \frac{\pi}{2} \frac{\partial x_{0d}}{\partial \eta} - \frac{\partial x_0}{\partial \eta}. \tag{3.78}$$

Eq.(3.77) has the periodic solution

$$x_0 = R(\eta)\cos(\xi - \theta(\eta)). \tag{3.79}$$

where as usual in this method, $R(\eta)$ and $\theta(\eta)$ are as yet undetermined functions of slow time η. The next step is to substitute Eq.(3.79) into (3.78). Before doing so, we rewrite (3.78) by noting that (3.77) can be written in the form $x_{0d} = -\partial x_0/\partial \xi$:

$$\frac{\partial x_1}{\partial \xi} + x_1(\xi - \frac{\pi}{2}, \eta) = -\mu \frac{\partial^2 x_0}{\partial \xi^2} - \frac{\pi}{2} \frac{\partial^2}{\partial \eta \partial \xi} x_0 - \frac{\partial x_0}{\partial \eta}. \tag{3.80}$$

Now we substitute (3.79) into (3.80) and require the coefficients of both $\cos(\xi - \theta)$ and $\sin(\xi - \theta)$ to vanish, giving the following slow flow on R and θ:

$$R' + \frac{\pi}{2} R\theta' - \mu R = 0, \tag{3.81}$$
$$\frac{\pi}{2} R' - R\theta' = 0, \tag{3.82}$$

where primes represent differentiation with respect to slow time η. Solving for R' and θ', we get

$$R' = \frac{4\mu}{\pi^2+4}R, \tag{3.83}$$

$$\theta' = \frac{2\pi\mu}{\pi^2+4}, \tag{3.84}$$

from which we obtain

$$R(\eta) = R_0 \exp\left(\frac{4\mu\eta}{\pi^2+4}\right), \tag{3.85}$$

$$\theta(\eta) = \frac{2\pi\mu}{\pi^2+4}\eta + \theta_0. \tag{3.86}$$

Eq.(3.79) thus gives

$$x \approx x_0 = R_0 \exp\left(\frac{4\mu\varepsilon t}{\pi^2+4}\right) \cos\left(t - \frac{2\pi\mu}{\pi^2+4}\varepsilon t - \theta_0\right), \tag{3.87}$$

which agrees with the exact solution given by Eq.(3.71) with a and b given by (3.74).

3.5.4 Approach to limit cycle

Now let us use the two variable method on Eq.(3.1). We have seen by use of Lindstedt's method that this DDE has a limit cycle of amplitude $2\sqrt{\mu}/\sqrt{3}$, see Eq.(3.25). The question arises as to the stability of this limit cycle. This may be determined as follows: After scaling x as in Eq.(3.13), we may obtain Eq.(3.1) by adding the term $-\varepsilon x(t)^3$ to the RHS of Eq.(3.65). This results in the term $-x_0^3$ being added to the RHS of Eqs.(3.78) and (3.80). After trigonometric reduction, this new term causes the term $-3R^3/4$ to be added to the RHS of Eq.(3.81), resulting in the new slow flow

$$R' = \frac{4\mu R - 3R^3}{\pi^2+4}, \tag{3.88}$$

$$\theta' = \frac{2\pi\mu - (3\pi/2)R^2}{\pi^2+4}. \tag{3.89}$$

Here we see that Eq.(3.88) has two equilibria, $R=0$ and $R = 2\sqrt{\mu}/\sqrt{3}$. Treating (3.88) as a flow on the $R-$line immediately shows that for $\mu > 0$ the $R=0$ equilibrium is unstable, a fact which we have already observed via a different approach, since $R=0$ corresponds to the trivial solution of Eq.(3.1), which was investigated in Eqs.(3.5)–(3.9). In addition (3.88) shows that for $\mu > 0$ the equilibrium $R = 2\sqrt{\mu}/\sqrt{3}$ is stable, from which we may conclude that the corresponding limit cycle is stable and that the Hopf bifurcation is supercritical. This conclusion agrees with the numerical integration displayed in Figure 3.3.

3.6 Center manifold analysis

We have seen earlier that the equilibrium solution $x = 0$ in Eq.(3.1) is stable for $T < \pi/2$ and is unstable for $T > \pi/2$. The question remains as to the stability of $x = 0$ when $T = \pi/2$. More generally, in order to determine the stability of the x=0 solution of a DDE in the form of Eq.(3.2),

$$\frac{dx(t)}{dt} = \alpha x(t) + \beta x(t-T) + f(x(t), x(t-T)), \tag{3.90}$$

in the case that the delay T takes on its critical value T_{cr}, it is necessary to take into account the effect of nonlinear terms. This may be accomplished by using a center manifold reduction. In order to accomplish this, the DDE is reformulated as an evolution equation on a function space. Our treatment closely follows that presented in (Kalmar-Nagy et al., 2001).

The idea here is that the initial condition for Eq.(3.90) consists of a function defined on $-T \leq t \leq 0$. As t increases from zero we may consider the piece of the solution lying in the time interval $[-T+t, t]$ as having evolved from the initial condition function. In order to avoid confusion, the variable θ is used to identify a point in the interval $[-T, 0]$, whereupon $x(t+\theta)$ will represent the piece of the solution which has evolved from the initial condition function at time t. From the point of view of the function space, t is a parameter, and it is θ which is the independent variable. To emphasize this, we write:

$$x_t(\theta) = x(t+\theta). \tag{3.91}$$

The evolution equation, which acts on a function space consisting of continuously differentiable functions on $[-T, 0]$, is written:

$$\frac{\partial x_t(\theta)}{\partial t} = \begin{cases} \dfrac{\partial x_t(\theta)}{\partial \theta}, & \text{for } \theta \in [-T, 0), \\ \alpha x_t(0) + \beta x_t(-T) + f(x_t(0), x_t(-T)), & \text{for } \theta = 0. \end{cases} \tag{3.92}$$

Here the DDE (3.90) appears as a boundary condition at $\theta = 0$. The rest of the interval goes along for the ride, which is to say that the equation $\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta}$ is an identity due to Eq.(3.91).

The RHS of Eq.(3.92) may be written as the sum of a linear operator A and a nonlinear operator F:

$$Ax_t(\theta) = \begin{cases} \dfrac{\partial x_t(\theta)}{\partial \theta} & \text{for } \theta \in [-T, 0), \\ \alpha x_t(0) + \beta x_t(-T) & \text{for } \theta = 0. \end{cases} \tag{3.93}$$

$$Fx_t(\theta) = \begin{cases} 0 & \text{for } \theta \in [-T, 0), \\ f(x_t(0), x_t(-T)) & \text{for } \theta = 0. \end{cases} \tag{3.94}$$

We now assume that the delay T is set at its critical value for a Hopf bifurcation, i.e. the characteristic equation has a pair of pure imaginary roots, $\lambda = \pm\omega i$. Corresponding to these eigenvalues are a pair of eigenfunctions $s_1(\theta)$ and $s_2(\theta)$ which satisfy the eigenequation:

$$A(s_1(\theta)+is_2(\theta)) = i\omega(s_1(\theta)+is_2(\theta)). \tag{3.95}$$

That is,

$$As_1(\theta) = -\omega s_2(\theta), \tag{3.96}$$
$$As_2(\theta) = \omega s_1(\theta). \tag{3.97}$$

From the definition (3.93) of the linear operator A we find that $s_1(\theta)$ and $s_2(\theta)$ must satisfy the following differential equations and boundary conditions:

$$\frac{d}{d\theta}s_1(\theta) = -\omega s_2(\theta), \tag{3.98}$$
$$\frac{d}{d\theta}s_2(\theta) = \omega s_1(\theta), \tag{3.99}$$

$$\alpha s_1(0)+\beta s_1(-T) = -\omega s_2(0), \tag{3.100}$$
$$\alpha s_2(0)+\beta s_2(-T) = \omega s_1(0). \tag{3.101}$$

Let's illustrate this process by using Eq.(3.1) as an example. We saw earlier that at T=T_{cr}=$\pi/2$, ω=1, which permits us to solve Eqs.(3.98), (3.99) as:

$$s_1(\theta) = c_1\cos\theta - c_2\sin\theta, \tag{3.102}$$
$$s_2(\theta) = c_1\sin\theta + c_2\cos\theta, \tag{3.103}$$

where c_1 and c_2 are arbitrary constants. In the case of Eq.(3.1), the boundary conditions (3.100), (3.101) become (α=0, β=-1):

$$-s_1(-\pi/2) = -s_2(0), \tag{3.104}$$
$$-s_2(-\pi/2) = s_1(0). \tag{3.105}$$

Equations (3.102), (3.103) identically satisfy Eqs.(3.104), (3.105) so that the constants of integration c_1 and c_2 remain arbitrary at this point.

In preparation for the center manifold analysis, we write the solution $x_t(\theta)$ as a sum of points lying in the center subspace (spanned by $s_1(\theta)$ and $s_2(\theta)$) and of points which do not lie in the center subspace, i.e., the rest of the solution, here called w:

$$x_t(\theta) = y_1(t)s_1(\theta)+y_2(t)s_2(\theta)+w(t)(\theta). \tag{3.106}$$

Here $y_1(t)$ and $y_2(t)$ are the components of $x_t(\theta)$ lying in the directions $s_1(\theta)$ and $s_2(\theta)$ respectively.

The idea of the center manifold reduction is to find w as an approximate function of y_1 and y_2 (the center manifold), and then to substitute $w(y_1,y_2)$ into the equations on $y_1(t)$ and $y_2(t)$. The result is that we will have replaced the original infinite dimensional system with a two dimensional approximation.

In order to find the equations on $y_1(t)$ and $y_2(t)$, we need to project $x_t(\theta)$ onto the center subspace. In this system, projections are accomplished by means of a bilinear form:

$$(v,u) = v(0)u(0) + \int_{-T}^{0} v(\theta+T)\beta u(\theta)d\theta, \tag{3.107}$$

where $u(\theta)$ lies in the original function space, i.e. the space of continuously differentiable functions defined on $[-T,0]$, and where $v(\theta)$ lies in the adjoint function space of continuously differentiable functions defined on $[0,T]$. See the Appendix to this chapter for a discussion of the adjoint operator A^*.

In order to project $x_t(\theta)$ onto the center subspace, we will need the adjoint eigenfunctions $n_1(\theta)$ and $n_2(\theta)$ which satisfy the adjoint eigenequation:

$$A^*(n_1(\theta)+in_2(\theta)) = -i\omega(n_1(\theta)+in_2(\theta)), \tag{3.108}$$

That is,

$$A^*n_1(\theta) = \omega n_2(\theta), \tag{3.109}$$
$$A^*n_2(\theta) = -\omega n_1(\theta), \tag{3.110}$$

where the adjoint operator A^* is defined by

$$A^*v(\theta) = \begin{cases} -\dfrac{dv(\theta)}{d\theta} & \text{for } \theta\in(0,T], \\ \alpha v(0)+\beta v(T) & \text{for } \theta = 0. \end{cases} \tag{3.111}$$

In addition, the adjoint eigenfunctions n_i are defined to be orthonormal to the eigenfunctions s_i:

$$(n_i,s_j) = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j. \end{cases} \tag{3.112}$$

From the definition (3.111) of the linear operator A^* we find that $n_1(\theta)$ and $n_2(\theta)$ must satisfy the following differential equations and boundary conditions:

$$-\frac{d}{d\theta}n_1(\theta) = \omega n_2(\theta), \tag{3.113}$$
$$-\frac{d}{d\theta}n_2(\theta) = -\omega n_1(\theta), \tag{3.114}$$
$$\alpha n_1(0)+\beta n_1(T) = \omega n_2(0), \tag{3.115}$$
$$\alpha n_2(0)+\beta n_2(T) = -\omega n_1(0). \tag{3.116}$$

We continue to illustrate by using Eq.(3.1) as an example. With $\omega = 1$, Eqs.(3.113), (3.114) may be solved as:

$$n_1(\theta) = d_1 \cos\theta - d_2 \sin\theta, \tag{3.117}$$
$$n_2(\theta) = d_1 \sin\theta + d_2 \cos\theta, \tag{3.118}$$

where d_1 and d_2 are arbitrary constants. In the case of Eq.(3.1), the boundary conditions (3.115), (3.116) become (α=0, β=-1):

$$-n_1(\pi/2) = n_2(0), \tag{3.119}$$
$$-n_2(\pi/2) = -n_1(0). \tag{3.120}$$

As in the case of the s_i eigenfunctions, Eqs.(3.117), (3.118) identically satisfy Eqs.(3.119), (3.120) so that the constants of integration d_1 and d_2 remain arbitrary.

The four arbitrary constants c_1, c_2, d_1, d_2 of Eqs.(3.102),(3.103),(3.117),(3.118) will be related by the orthonormality conditions (3.112). Let's illustrate this by computing (n_1, s_1) for the example of Eq.(3.1). Using the definition of the bilinear form (3.107), we obtain:

$$(n_1, s_1) = n_1(0)s_1(0) + \int_{-\frac{\pi}{2}}^{0} n_1\left(\theta + \frac{\pi}{2}\right)(-1)s_1(\theta)d\theta, \tag{3.121}$$

$$(n_1, s_1) = \frac{(2c_2 + \pi c_1)\, d_2 + (2c_1 - \pi c_2)\, d_1}{4} = 1. \tag{3.122}$$

Similarly, we find:

$$(n_1, s_2) = \frac{(\pi c_2 - 2c_1)\, d_2 + (2c_2 + \pi c_1)\, d_1}{4} = 0. \tag{3.123}$$

The other two orthonormality conditions give no new information since it turns out that $(n_2, s_1) = -(n_1, s_2)$ and $(n_2, s_2) = (n_1, s_1)$. Thus Eqs.(3.122) and (3.123) are two equations in four unknowns, c_1, c_2, d_1, d_2. Without loss of generality we take $d_1 = 1$ and $d_2 = 0$, giving $c_1 = \frac{8}{\pi^2+4}, c_2 = -\frac{4\pi}{\pi^2+4}$. Thus the eigenfunctions s_i and n_i for Eq.(3.1) become:

$$s_1(\theta) = \frac{4\pi \sin\theta + 8 \cos\theta}{\pi^2 + 4}, \tag{3.124}$$
$$s_2(\theta) = \frac{8 \sin\theta - 4\pi \cos\theta}{\pi^2 + 4}, \tag{3.125}$$
$$n_1(\theta) = \cos\theta, \tag{3.126}$$
$$n_2(\theta) = \sin\theta. \tag{3.127}$$

Recall that our purpose in solving for n_1 and n_2 was to obtain equations on $y_1(t)$ and $y_2(t)$, the components of $x_t(\theta)$ lying in the directions $s_1(\theta)$ and $s_2(\theta)$ respectively, see Eq.(3.106). We have:

$$y_1(t) = (n_1, x_t), \quad y_2(t) = (n_2, x_t). \tag{3.128}$$

Differentiating (3.128) with respect to t:

$$\dot{y}_1(t) = (n_1, \dot{x}_t), \quad \dot{y}_2(t) = (n_2, \dot{x}_t). \tag{3.129}$$

Let us consider the first of these:

$$\dot{y}_1(t) = (n_1, \dot{x}_t) = (n_1, Ax_t + Fx_t) = (n_1, Ax_t) + (n_1, Fx_t). \tag{3.130}$$

Now by definition of the adjoint operator A^*,

$$(n_1, Ax_t) = (A^* n_1, x_t) = (\omega n_2, x_t) = \omega(n_2, x_t) = \omega y_2. \tag{3.131}$$

So we have

$$\dot{y}_1 = \omega y_2 + (n_1, Fx_t),$$

and similarly (3.132)

$$\dot{y}_2 = -\omega y_1 + (n_2, Fx_t).$$

In Eqs.(3.132), the quantities (n_i, Fx_t) are given by (cf. Eq.(3.107)):

$$(n_i, Fx_t) = n_i(0)Fx_t(0) + \int_{-T}^{0} n_i(\theta + T)\beta Fx_t(\theta)d\theta \tag{3.133}$$

$$= n_i(0)f(x_t(0), x_t(-T)) \text{ since } Fx_t(\theta){=}0 \text{ unless } \theta{=}0, \tag{3.134}$$

in which $x_t = y_1(t)s_1(\theta) + y_2(t)s_2(\theta) + w(t)(\theta)$. Continuing with the example of Eq.(3.1), Eqs.(3.132) become, using $f = -x(t)^3$:

$$\dot{y}_1 = y_2 - \left(\frac{8y_1}{\pi^2+4} - \frac{4\pi y_2}{\pi^2+4} + w(\theta = 0)\right)^3 \quad \text{and} \quad \dot{y}_2 = -y_1, \tag{3.135}$$

where we have used $s_1(0){=}\frac{8}{\pi^2+4}$, $s_2(0){=}\frac{-4\pi}{\pi^2+4}$, $n_1(0){=}1$ and $n_2(0){=}0$ from Eqs.(3.124) –(3.127).

The next step is to look for an approximate expression for the center manifold, which is tangent to the y_1-y_2 plane at the origin, and which may be written in the form:

$$w(y_1, y_2)(\theta) = m_1(\theta)y_1^2 + m_2(\theta)y_1y_2 + m_3(\theta)y_2^2. \tag{3.136}$$

The procedure is to substitute (3.136) into the equations of motion, collect terms, and solve for the unknown functions $m_i(\theta)$. Then the resulting expression is to be substituted into the y_1-y_2 equations (3.132). Note that if this is done for the example of Eq.(3.1), i.e. for Eqs.(3.135), the contribution made by w will be of degree 4 and higher in the y_i. However, stability of the origin will be determined by terms of degree 2 and 3, according to the following formula (obtainable by averaging). Suppose the y_1-y_2 equations are of the form:

$$\dot{y}_1 = \omega y_2 + h(y_1, y_2) \quad \text{and} \quad \dot{y}_2 = -\omega y_1 + g(y_1, y_2). \tag{3.137}$$

Then the stability of the origin is determined by the sign of the quantity Q, where

$$16Q = h_{111} + h_{122} + g_{112} + g_{222} - \frac{1}{\omega}\left[h_{12}(h_{11}+h_{22}) - g_{12}(g_{11}+g_{22}) - h_{11}g_{11} + h_{22}g_{22}\right], \quad (3.138)$$

where subscripts represent partial derivatives, which are to be evaluated at $y_1 = y_2 = 0$.$Q > 0$ means unstable, $Q < 0$ means stable. See (Guckenheimer and Holmes, 1983) pp.154-156, where it is shown that the flow on the y_1-y_2 plane in the neighborhood of the origin can be approximately written in polar coordinates as:

$$\frac{dr}{dt} = Qr^3 + O(r^5). \quad (3.139)$$

Applying this criterion to Eqs.(3.135) (in which w is assumed to be of the form (3.136) and hence contributes terms of higher order in y_i), we find:

$$Q = -\frac{48}{(\pi^2+4)^2} = -0.2495. \quad (3.140)$$

The origin in Eq.(3.1) for $T = T_{cr} = \pi/2$ is therefore predicted to be stable. This result is in agreement with numerical integration, see Fig. 3.4.

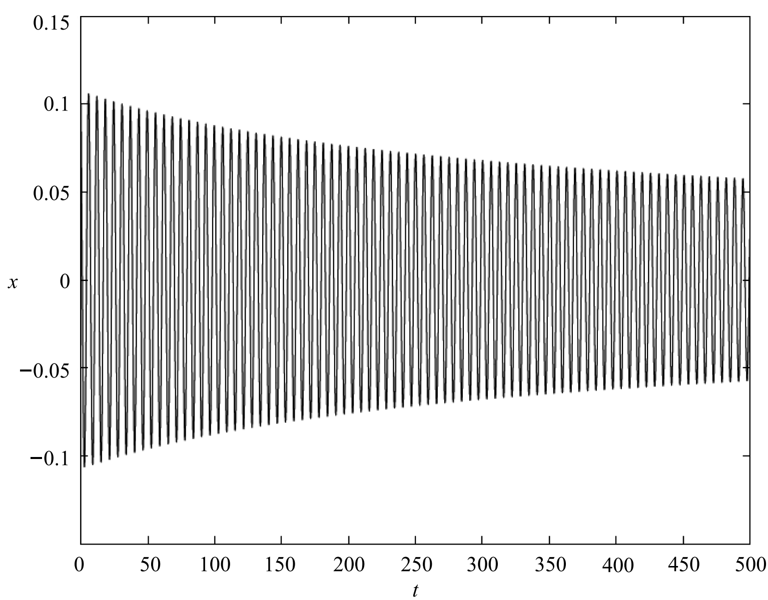

Fig. 3.4 Numerical integration of Eq.(3.1) for the initial condition $x = 0.1$ on $-T \le t \le 0$, for $T = \pi/2$.

Now let's change the example a little so that w plays a significant role in determining the stability of the origin:

$$\frac{dx(t)}{dt} = -x(t-T) - x(t)^2. \tag{3.141}$$

Since the linear parts of this example and of the previous example are the same, our previously derived expressions for s_1, s_2, n_1 and n_2, Eqs.(3.124)–(3.127), still apply. Eqs.(3.135) now become:

$$\dot{y}_1 = y_2 - \left(\frac{8y_1}{\pi^2+4} - \frac{4\pi y_2}{\pi^2+4} + w(\theta=0)\right)^2 \quad \text{and} \quad \dot{y}_2 = -y_1. \tag{3.142}$$

So our goal now is to find the functions $m_i(\theta)$ in the expression for the center manifold (3.136), and then to substitute the result into Eq.(3.142) and use the formula (3.138) to determine stability.

We begin by differentiating the expression for the center manifold (3.136) with respect to t:

$$\frac{\partial w(y_1,y_2)(\theta)}{\partial t} = 2m_1(\theta)y_1\dot{y}_1 + m_2(\theta)(y_1\dot{y}_2 + y_2\dot{y}_1) + 2m_3(\theta)y_2\dot{y}_2. \tag{3.143}$$

We substitute the equations (3.132) on $\dot{y}_1$ and $\dot{y}_2$ in (3.143) and neglect terms of degree higher than 2 in the y_i:

$$\begin{aligned}\frac{\partial w(y_1,y_2)(\theta)}{\partial t} &= 2m_1(\theta)y_1\omega y_2 + m_2(\theta)(-y_1\omega y_1 + y_2\omega y_2) \\ &\quad -2m_3(\theta)y_2\omega y_1 + \cdots \qquad (3.144)\\ &= \omega[-m_2(\theta)y_1^2 + 2(m_1(\theta)-m_3(\theta))y_1y_2 \\ &\quad + m_2(\theta)y_2^2] + \cdots \qquad (3.145)\end{aligned}$$

We obtain conditions on the functions $m_i(\theta)$ by deriving another expression for $\dot{w}$ and equating them. Let us differentiate Eq.(3.106) with respect to t:

$$\frac{\partial x_t(\theta)}{\partial t} = \dot{y}_1(t)s_1(\theta) + \dot{y}_2(t)s_2(\theta) + \frac{\partial w(t)(\theta)}{\partial t}. \tag{3.146}$$

Using the functional DE (3.92)–(3.94), and rearranging terms, we get:

$$\begin{aligned}
\frac{\partial w(t)(\theta)}{\partial t} &= \frac{\partial x_t(\theta)}{\partial t} - \dot{y}_1(t)s_1(\theta) - \dot{y}_2(t)s_2(\theta) && (3.147)\\
&= Ax_t(\theta) + Fx_t(\theta) - \dot{y}_1(t)s_1(\theta) - \dot{y}_2(t)s_2(\theta) && (3.148)\\
&= A(y_1(t)s_1(\theta) + y_2(t)s_2(\theta) + w(t)(\theta))\\
&\quad + Fx_t(\theta) - \dot{y}_1(t)s_1(\theta) - \dot{y}_2(t)s_2(\theta) && (3.149)\\
&= y_1As_1 + y_2As_2 + Aw + Fx_t - \dot{y}_1s_1 - \dot{y}_2s_2 && (3.150)\\
&= y_1(-\omega s_2) + y_2(\omega s_1) + Aw + Fx_t\\
&\quad -(\omega y_2 + (n_1, Fx_t))s_1 - (-\omega y_1 + (n_2, Fx_t))s_2 && (3.151)\\
&= Aw + Fx_t - (n_1, Fx_t)s_1 - (n_2, Fx_t)s_2 && (3.152)
\end{aligned}$$

where we have used Eqs.(3.96), (3.97) and (3.132), and where the quantities (n_i, Fx_t) are given by Eq.(3.134).

Eq.(3.152) is an equation for the time evolution of w. Since the operator A is defined differently for $\theta \in [-T, 0)$ and for $\theta = 0$, we consider each of these cases separately when we substitute Eq.(3.136) for the center manifold. In the $\theta \in [-T, 0)$ case, Eq.(3.152) becomes:

$$\frac{\partial w(t)(\theta)}{\partial t} = m_1' y_1^2 + m_2' y_1 y_2 + m_3' y_2^2 - (n_1, Fx_t)s_1(\theta) - (n_2, Fx_t)s_2(\theta), \quad (3.153)$$

where primes denote differentiation with respect to θ. In the $\theta = 0$ case, Eq.(3.152) becomes:

$$\begin{aligned}
\frac{\partial w(t)(\theta)}{\partial t} &= \alpha(m_1(0)y_1^2 + m_2(0)y_1y_2 + m_3(0)y_2^2)\\
&\quad + \beta(m_1(-T)y_1^2 + m_2(-T)y_1y_2 + m_3(-T)y_2^2)\\
&\quad + f(x_t(0), x_t(-T)) - (n_1, Fx_t)s_1(0) - (n_2, Fx_t)s_2(0). && (3.154)
\end{aligned}$$

Now we equate Eqs.(3.153) and (3.154) to the previously derived expression for $\dot{w}$, Eq.(3.145). Equating (3.153) to (3.145), we get:

$$\begin{aligned}
&m_1' y_1^2 + m_2' y_1 y_2 + m_3' y_2^2 - (n_1, Fx_t)s_1(\theta) - (n_2, Fx_t)s_2(\theta) =\\
&\omega[-m_2 y_1^2 + 2(m_1 - m_3)y_1y_2 + m_2 y_2^2] + \cdots && (3.155)
\end{aligned}$$

Equating (3.154) to (3.145), we get:

$$\begin{aligned}
&\alpha(m_1(0)y_1^2 + m_2(0)y_1y_2 + m_3(0)y_2^2)\\
&+\beta(m_1(-T)y_1^2 + m_2(-T)y_1y_2 + m_3(-T)y_2^2)\\
&+f(x_t(0), x_t(-T)) - (n_1, Fx_t)s_1(0) - (n_2, Fx_t)s_2(0) =\\
&\omega[-m_2(0)y_1^2 + 2(m_1(0) - m_3(0))y_1y_2 + m_2(0)y_2^2] + \cdots && (3.156)
\end{aligned}$$

Now we equate coefficients of y_1^2, y_1y_2 and y_2^2 in Eqs.(3.155) and (3.156) and so obtain 3 first order ODE's on m_1, m_2 and m_3 and 3 boundary conditions. From Eq.(3.134), the nonlinear terms (n_i, Fx_t) become:

$$(n_i, Fx_t) = n_i(0)f(x_t(0), x_t(-T)), \tag{3.157}$$

in which $x_t = y_1(t)s_1(\theta) + y_2(t)s_2(\theta) + w(t)(\theta) \approx y_1(t)s_1(\theta) + y_2(t)s_2(\theta)$. In the case of the example system (3.141) we have $\alpha = 0$, $\beta = -1$, $\omega = 1$, $T = \pi/2$, $f(x(t), x(t-T)) = -x(t)^2$ and

$$f(x_t(0), x_t(-T)) = -x_t(0)^2 = -(y_1 s_1(0) + y_2 s_2(0))^2. \tag{3.158}$$

For this example, Eq.(3.155) becomes

$$\begin{aligned} &m_1' y_1^2 + m_2' y_1 y_2 + m_3' y_2^2 + (y_1 s_1(0) + y_2 s_2(0))^2 s_1(\theta) \\ &= -m_2 y_1^2 + 2(m_1 - m_3) y_1 y_2 + m_2 y_2^2 \end{aligned} \tag{3.159}$$

which gives the following 3 ODE's on $m_i(\theta)$:

$$m_1' + s_1(0)^2 s_1(\theta) = -m_2, \tag{3.160}$$

$$m_2' + 2s_1(0)s_2(0)s_1(\theta) = 2(m_1 - m_3), \tag{3.161}$$

$$m_3' + s_2(0)^2 s_1(\theta) = m_2. \tag{3.162}$$

For this example, Eq.(3.156) becomes

$$\begin{aligned} &-(m_1(-\pi/2)y_1^2 + m_2(-\pi/2)y_1 y_2 + m_3(-\pi/2)y_2^2) \\ &-(y_1 s_1(0) + y_2 s_2(0))^2 + (y_1 s_1(0) + y_2 s_2(0))^2 s_1(0) = \\ &-m_2(0)y_1^2 + 2(m_1(0) - m_3(0))y_1 y_2 + m_2(0)y_2^2, \end{aligned} \tag{3.163}$$

which gives the following 3 boundary conditions on $m_i(\theta)$:

$$-m_1(-\pi/2) - s_1(0)^2 + s_1(0)^3 = -m_2(0), \tag{3.164}$$

$$-m_2(-\pi/2) - 2s_1(0)s_2(0) + 2s_1(0)^2 s_2(0) = 2(m_1(0) - m_3(0)), \tag{3.165}$$

$$-m_3(-\pi/2) - s_2(0)^2 + s_2(0)^2 s_1(0) = m_2(0). \tag{3.166}$$

So we have 3 linear ODE's (3.160)–(3.162) with 3 boundary conditions (3.164)–(3.166) for the $m_i(\theta)$. In these equations, s_1 and s_2 are given by Eqs.(3.124), (3.125). The solution of these equations is algebraically complicated. I used a computer algebra system to obtain a closed form solution for the $m_i(\theta)$. For brevity, a numerical version of the center manifold is given below:

$$\begin{aligned} w &= \\ &(0.20216 \sin 2\theta + 0.16022 \cos 2\theta - 0.6953 \sin\theta + 0.39537 \cos\theta - 0.5768)\, {y_1}^2 \\ &+ (0.32044 \sin 2\theta - 0.40432 \cos 2\theta + 0.09393 \sin\theta + 0.5034 \cos\theta)\, y_1 y_2 \\ &+ (-0.20216 \sin 2\theta - 0.16022 \cos 2\theta + 0.0299 \sin\theta + 0.64984 \cos\theta - 0.5768)\, {y_2}^2. \end{aligned} \tag{3.167}$$

Next we substitute the algebraic version of Eq.(3.167) into the y_1-y_2 Eqs.(3.142) and use Eq.(3.138) to compute the stability parameter Q:

$$Q = \frac{32\,(9-\pi)}{5\,(\pi^2+4)^2} = 0.19491 > 0. \tag{3.168}$$

Thus the center manifold analysis predicts that origin of Eq.(3.141) is unstable. This result is in agreement with numerical integration, see Fig. 3.5.

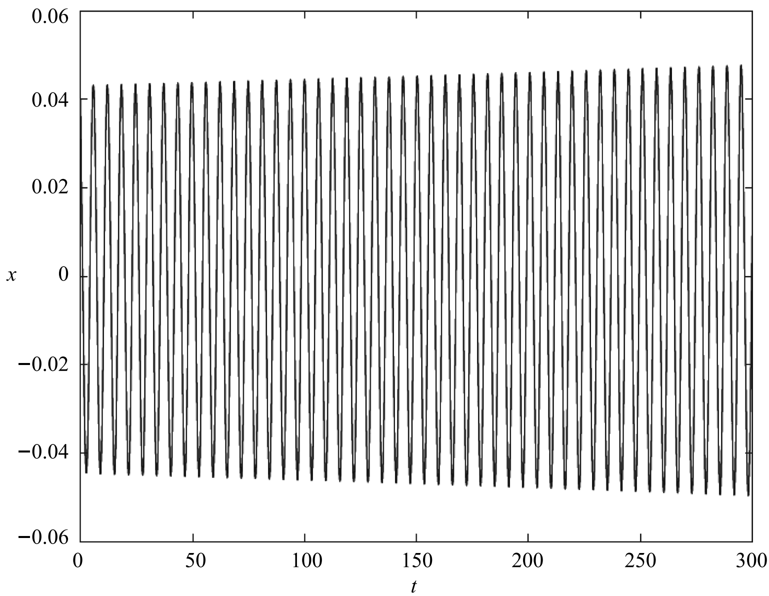

Fig. 3.5 Numerical integration of Eq.(3.141) for the initial condition $x = 0.04$ on $-T \le t \le 0$, for $T = \pi/2$.

3.6.1 Appendix: The adjoint operator A^*

The adjoint operator A^* is defined by the relation:

$$(v, Au) = (A^* v, u), \tag{3.169}$$

where the bilinear form (v,u) is given by Eq.(3.107):

$$(v,u) = v(0)u(0) + \int_{-T}^{0} v(\theta+T)\beta u(\theta)d\theta, \tag{3.170}$$

where $u(\theta)$ lies in the original function space, i.e. the space of continuously differentiable functions defined on $[-T,0]$, and where $v(\theta)$ lies in the adjoint function space of continuously differentiable functions defined on $[0,T]$.

The linear operator A is given by Eq.(3.93):

$$Au(\theta) = \begin{cases} \dfrac{du(\theta)}{d\theta} & \text{for } \theta \in [-T,0), \\ \alpha u(0) + \beta u(-T) & \text{for } \theta = 0, \end{cases} \tag{3.171}$$

from which (v, Au) can be written as follows:

$$(v,Au) = v(0)Au(0) + \int_{-T}^{0} v(\theta+T)\beta Au(\theta)d\theta \tag{3.172}$$

$$= v(0)[\alpha u(0) + \beta u(-T)] + \int_{-T}^{0} v(\theta+T)\beta \frac{du(\theta)}{d\theta}d\theta. \tag{3.173}$$

Using integration by parts, the last term of Eq.(3.173) can be written:

$$\int_{-T}^{0} v(\theta+T)\beta \frac{du(\theta)}{d\theta}d\theta$$

$$= v(\theta+T)\beta u(\theta)|_{-T}^{0} - \int_{-T}^{0} \beta u(\theta)\frac{dv(\theta+T)}{d\theta}d\theta \tag{3.174}$$

$$= v(T)\beta u(0) - v(0)\beta u(-T) - \int_{\phi=0}^{T} \beta u(\phi-T)\frac{dv(\phi)}{d\phi}d\phi \tag{3.175}$$

where $\phi = \theta + T$. Substituting (3.175) into (3.173), we get

$$(v,Au) = [\alpha v(0) + \beta v(T)]u(0) + \int_{\phi=0}^{T} \left(-\frac{dv(\phi)}{d\phi}\right)\beta u(\phi-T)d\phi \tag{3.176}$$

$$= (A^* v, u) \tag{3.177}$$

from which we may conclude that the adjoint operator A^* is given by:

$$A^* v(\phi) = \begin{cases} -\dfrac{dv(\phi)}{d\phi} & \text{for } \phi \in (0,T], \\ \alpha v(0) + \beta v(T) & \text{for } \phi = 0. \end{cases} \tag{3.178}$$

3.7 Application to gene expression (Verdugo and Rand, 2008a)

This section offers a timely example showing how DDEs occur in a mathematical model of gene expression (Monk, 2003). The biology of the problem may be described as follows: A gene, i.e. a section of the DNA molecule, is copied (*tran-*

scribed) onto messenger RNA (mRNA), which diffuses out of the nucleus of the cell into the cytoplasm, where it enters a subcellular structure called a ribosome. In the ribosome the genetic code on the mRNA produces a protein (a process called *translation).* The protein then diffuses back into the nucleus where it represses the transcription of its own gene.

Dynamically speaking, this process may result in a steady state equilibrium, in which the concentrations of mRNA and protein are constant, or it may result in an oscillation. In this section we analyze a simple model previously proposed in the biological literature (Monk, 2003), and we show that the transition between equilibrium and oscillation is a Hopf bifurcation. The model takes the form of two equations, one an ordinary differential equation (ODE) and the other a delayed differential equation (DDE). The delay is due to an observed time lag in the transcription process.

Oscillations in biological systems with delay have been dealt with previously in (Mahaffy, 1988; Mahaffy et al., 1992; Mocek et al., 2005).
The model equations investigated here involve the variables $M(t)$, the concentration of mRNA, and $P(t)$, the concentration of the associated protein (Monk, 2003):

$$\dot{M} = \alpha_m \left(\frac{1}{1+\left(\frac{P_d}{P_0}\right)^n} \right) - \mu_m M \tag{3.179}$$

$$\dot{P} = \alpha_p M - \mu_p P \tag{3.180}$$

where dots represent differentiation with respect to time t, and where we use the subscript d to denote a variable which is delayed by time T, thus $P_d = P(t-T)$. The model constants are as given in (Monk, 2003): α_m is the rate at which mRNA is transcribed in the absence of the associated protein, α_p is the rate at which the protein is produced from mRNA in the ribosome, μ_m and μ_p are the rates of degradation of mRNA and of protein, respectively, P_0 is a reference concentration of protein, and n is a parameter. We assume μ_m=μ_p=μ.

3.7.1 Stability of equilibrium

We begin by rescaling Eqs. (3.179) and (3.180). We set $m = \frac{M}{\alpha_m}$, $p = \frac{P}{\alpha_m \alpha_p}$, and $p_0 = \frac{P_0}{\alpha_m \alpha_p}$, giving:

$$\dot{m} = \frac{1}{1+\left(\frac{p_d}{p_0}\right)^n} - \mu m, \tag{3.181}$$

$$\dot{p} = m - \mu p. \tag{3.182}$$

Equilibrium points, (m^*, p^*), for (3.181) and (3.182) are found by setting $\dot{m} = 0$ and $\dot{p} = 0$

$$\mu m^* = \frac{1}{1+\left(\frac{p^*}{p_0}\right)^n} \quad (3.183)$$

$$m^* = \mu p^* \quad (3.184)$$

Eliminating m^* from Eqs. (3.183) and (3.184), we obtain an equation on p^*:

$$(p^*)^{n+1} + p_0^n p^* - \frac{p_0^n}{\mu^2} = 0. \quad (3.185)$$

Next we define ξ and η to be deviations from equilibrium: $\xi = \xi(t) = m(t) - m^*$, $\eta = \eta(t) = p(t) - p^*$, and $\eta_d = \eta(t-T)$. This results in the nonlinear system:

$$\dot{\xi} = \frac{1}{1+\left(\frac{\eta_d+p^*}{p_0}\right)^n} - \mu(m^*+\xi), \quad (3.186)$$

$$\dot{\eta} = \xi - \mu\eta. \quad (3.187)$$

Expanding for small values of η_d, Eq.(3.186) becomes:

$$\dot{\xi} = -\mu\xi - K\eta_d + H_2\eta_d^2 + H_3\eta_d^3 + \cdots \quad (3.188)$$

where K, H_2 and H_3 depend on p^*, p_0, and n as follows:

$$K = \frac{n\beta}{p^*(1+\beta)^2}, \quad \text{where} \quad \beta = \left(\frac{p^*}{p_0}\right)^n, \quad (3.189)$$

$$H_2 = \frac{\beta n\,(\beta n - n + \beta + 1)}{2\,(\beta+1)^3\, p^{*2}}, \quad (3.190)$$

$$H_3 = -\frac{\beta n\,\left(\beta^2 n^2 - 4\beta n^2 + n^2 + 3\beta^2 n - 3n + 2\beta^2 + 4\beta + 2\right)}{6\,(\beta+1)^4\, p^{*3}}. \quad (3.191)$$

Next we analyze the linearized system coming from Eqs. (3.188) and (3.187):

$$\dot{\xi} = -\mu\,\xi - K\,\eta_d, \quad (3.192)$$

$$\dot{\eta} = \xi - \mu\,\eta. \quad (3.193)$$

Stability analysis of Eqs. (3.192) and (3.193) shows that for $T=0$ (no delay), the equilibrium point (m^*, p^*) is a stable spiral. Increasing the delay, T, in the linear system (3.192)–(3.193), will yield a critical delay, T_{cr}, such that for $T > T_{cr}$, (m^*, p^*) will be unstable, giving rise to a Hopf bifurcation. For $T = T_{cr}$ the system (3.192)–(3.193) will exhibit a pair of pure imaginary eigenvalues $\pm\omega i$ corresponding to the solution

$$\xi(t) = B\cos(\omega t + \phi), \quad (3.194)$$

$$\eta(t) = A\cos\omega t, \quad (3.195)$$

where A and B are the amplitudes of the $\eta(t)$ and $\xi(t)$ oscillations, and where ϕ is a phase angle. Note that we have chosen the phase of $\eta(t)$ to be zero without loss of generality. Then for values of delay T close to T_{cr},

$$T = T_{cr} + \Delta, \tag{3.196}$$

the nonlinear system (3.181)–(3.182) is expected to exhibit a periodic solution (a limit cycle) which can be written in the approximate form of Eqs. (3.194), (3.195). Substituting Eqs. (3.194) and (3.195) into Eqs. (3.192) and (3.193) and solving for ω and T_{cr} we obtain

$$\omega = \sqrt{K-\mu^2}, \tag{3.197}$$

$$T_{cr} = \frac{\arctan\left(\frac{2\mu\sqrt{K-\mu^2}}{K-2\mu^2}\right)}{\sqrt{K-\mu^2}}. \tag{3.198}$$

3.7.2 Lindstedt's method

We use Lindstedt's Method (Rand and Verdugo, 2007) on Eqs. (3.188) and (3.187). We begin by changing the first order system into a second order DDE. This results in the following form:

$$\ddot{\eta} + 2\mu\dot{\eta} + \mu^2\eta = -K\eta_d + H_2\eta_d^2 + H_3\eta_d^3 + \cdots \tag{3.199}$$

where K, H_2 and H_3 are defined by Eqs. (3.189)–(3.191). We introduce a small parameter ε via the scaling

$$\eta = \varepsilon u. \tag{3.200}$$

The detuning Δ of Eq. (3.196) is scaled like ε^2, $\Delta = \varepsilon^2\delta$:

$$T = T_{cr} + \Delta = T_{cr} + \varepsilon^2\delta. \tag{3.201}$$

Next we stretch time by replacing the independent variable t by τ, where

$$\tau = \Omega t. \tag{3.202}$$

This results in the following form of Eq. (3.199):

$$\Omega^2\frac{d^2u}{d\tau^2} + 2\mu\Omega\frac{du}{d\tau} + \mu^2 u = -K u_d + \varepsilon H_2 u_d^2 + \varepsilon^2 H_3 u_d^3, \tag{3.203}$$

where $u_d = u(\tau - \Omega T)$. We expand Ω in a power series in ε, omitting the $O(\varepsilon)$ for convenience, since it turns out to be zero:

$$\Omega = \omega + \varepsilon^2 k_2 + \cdots \tag{3.204}$$

Next we expand the delay term u_d:

$$u_d = u(\tau - \Omega T) = u(\tau - (\omega + \varepsilon^2 k_2 + \cdots)(T_{cr} + \varepsilon^2 \delta)) \tag{3.205}$$
$$= u(\tau - \omega T_{cr} - \varepsilon^2 (k_2 T_{cr} + \omega\delta) + \cdots) \tag{3.206}$$
$$= u(\tau - \omega T_{cr}) - \varepsilon^2 (k_2 T_{cr} + \omega\delta) u'(\tau - \omega T_{cr}) + O(\varepsilon^3) \tag{3.207}$$

Now we expand $u(\tau)$ in a power series in ε:

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \cdots \tag{3.208}$$

Substituting and collecting terms, we find:

$$\omega^2 \frac{d^2 u_0}{d\tau^2} + 2\mu\omega \frac{du_0}{d\tau} + K u_0(\tau - \omega T_{cr}) + \mu^2 u_0 = 0 \tag{3.209}$$
$$\omega^2 \frac{d^2 u_1}{d\tau^2} + 2\mu\omega \frac{du_1}{d\tau} + K u_1(\tau - \omega T_{cr}) + \mu^2 u_1 = H_2 u_0^2(\tau - \omega T_{cr}) \tag{3.210}$$
$$\omega^2 \frac{d^2 u_2}{d\tau^2} + 2\mu\omega \frac{du_2}{d\tau} + K u_2(\tau - \omega T_{cr}) + \mu^2 u_2 = \cdots \tag{3.211}$$

where $\cdots$ stands for terms in u_0 and u_1, omitted here for brevity. We take the solution of the u_0 equation as:

$$u_0(\tau) = \hat{A}\cos\tau, \tag{3.212}$$

where from Eqs. (3.195) and (3.200) we know $A = \hat{A}\varepsilon$. Next we substitute (3.212) into (3.38) and obtain the following expression for u_1:

$$u_1(\tau) = m_1 \sin 2\tau + m_2 \cos 2\tau + m_3, \tag{3.213}$$

where m_1 is given by the equation:

$$m_1 = -\frac{2\hat{A}^2 H_2 \mu \sqrt{K - \mu^2}\,(\mu^2 - K)\,(2\mu^2 - 3K)}{K\,(16\mu^6 - 39K\mu^4 + 18K^2\mu^2 + 9K^3)}, \tag{3.214}$$

and where m_2 and m_3 are given by similar equations, omitted here for brevity. We substitute Eqs. (3.212) and (3.213) into (3.40), and after trigonometric simplifications have been performed, we equate to zero the coefficients of the resonant terms $\sin\tau$ and $\cos\tau$. This yields the amplitude, A, of the limit cycle that was born in the Hopf bifurcation:

$$A^2 = \frac{P}{Q}\Delta, \tag{3.215}$$

where

$$P = -8K^2(\mu^4 - K^2)(16\mu^6 - 39K\mu^4 + 18K^2\mu^2 + 9K^3), \tag{3.216}$$
$$Q = Q_0 T_{cr} + Q_1, \tag{3.217}$$

and

$$\begin{aligned} Q_0 &= 48\,\mathrm{H}_3\,K^2\,\mu^8 + 16\,\mathrm{H}_2{}^2\,K\,\mu^8 - 69\,\mathrm{H}_3\,K^3\,\mu^6 + 32\,\mathrm{H}_2{}^2\,K^2\,\mu^6 \\ &\quad - 63\,\mathrm{H}_3\,K^4\,\mu^4 - 162\,\mathrm{H}_2{}^2\,K^3\,\mu^4 + 81\,\mathrm{H}_3\,K^5\,\mu^2 + 108\,\mathrm{H}_2{}^2\,K^4\,\mu^2 \\ &\quad + 27\,\mathrm{H}_3\,K^6 + 30\,\mathrm{H}_2{}^2\,K^5, \end{aligned} \tag{3.218}$$

$$\begin{aligned} Q_1 &= 96\,\mathrm{H}_3\,K\,\mu^9 + 64\,\mathrm{H}_2{}^2\,\mu^9 - 138\,\mathrm{H}_3\,K^2\,\mu^7 - 16\,\mathrm{H}_2{}^2\,K\,\mu^7 \\ &\quad - 126\,\mathrm{H}_3\,K^3\,\mu^5 - 308\,\mathrm{H}_2{}^2\,K^2\,\mu^5 + 162\,\mathrm{H}_3\,K^4\,\mu^3 + 296\,\mathrm{H}_2{}^2\,K^3\,\mu^3 \\ &\quad + 54\,\mathrm{H}_3\,K^5\,\mu + 12\,\mathrm{H}_2{}^2\,K^4\,\mu. \end{aligned} \tag{3.219}$$

Eq. (3.217) depends on μ, K, H_2, H_3, and T_{cr}. By using Eq. (3.198) we may express Eq. (3.217) as a function of μ, K, H_2, and H_3 only. Removal of secular terms also yields a value for the frequency shift k_2, where, from Eq. (3.204), we have $\Omega = \omega + \varepsilon^2 k_2$:

$$k_2 = -\frac{R}{Q}\,\delta. \tag{3.220}$$

where Q is given by (3.217) and

$$R = \sqrt{K - \mu^2}\,Q_0. \tag{3.221}$$

An expression for the amplitude B of the periodic solution for $\xi(t)$ (see Eq. (3.194)) may be obtained directly from Eq. (3.187) by writing $\xi = \dot{\eta} + \mu\eta$, where $\eta \sim A\cos\omega t$. We find:

$$B = \sqrt{K}A, \tag{3.222}$$

where K and A are given as in (3.189) and (3.215) respectively.

3.7.3 Numerical example

Using the same parameter values as in (Monk, 2003)

$$\mu = 0.03/\mathrm{min}\,,\ p_0 = 100\,,\ n = 5, \tag{3.223}$$

we obtain

$$p^* = 145.9158\,,\ m^* = 4.3774, \tag{3.224}$$

$$K = 3.9089\times 10^{-3}\,,\ H_2 = 6.2778\times 10^{-5}\,,\ H_3 = -6.4101\times 10^{-7}, \tag{3.225}$$

$$T_{cr} = 18.2470\,,\ w = 5.4854\times 10^{-2}\,,\ \frac{2\pi}{w} = 114.5432. \tag{3.226}$$

Here the delay T_{cr} and the response period $2\pi/\omega$ are given in minutes. Substituting (3.224)–(3.226) into (3.215)–(3.222) yields the following equations:

$$A = 27.0215\,\sqrt{\Delta}, \tag{3.227}$$

$$k_2 = -2.4512\times 10^{-3}\,\delta, \tag{3.228}$$

$$B = 1.6894\,\sqrt{\Delta}. \tag{3.229}$$

Note that since Eq. (3.227) requires $\Delta > 0$ for the limit cycle to exist, and since we saw in Eqs. (3.192) and (3.193) that the origin is unstable for $T > T_{cr}$, i.e. for $\Delta > 0$, we may conclude that the Hopf bifurcation is supercritical, i.e., the limit cycle is stable.

Multiplying (3.228) by ε^2 and substituting into (3.204) we obtain:

$$\Omega = 5.4854 \times 10^{-2} - 2.4512 \times 10^{-3}\, \Delta \tag{3.230}$$

where $\Delta = T - T_{cr} = T - 18.2470$. Plotting the period, $\frac{2\pi}{\Omega}$, against the delay, T, yields the graph shown in Figure 3.6. These results are in agreement with those obtained by numerical integration of the original Eqs. (3.179) and (3.180) and with those presented in (Monk, 2003).

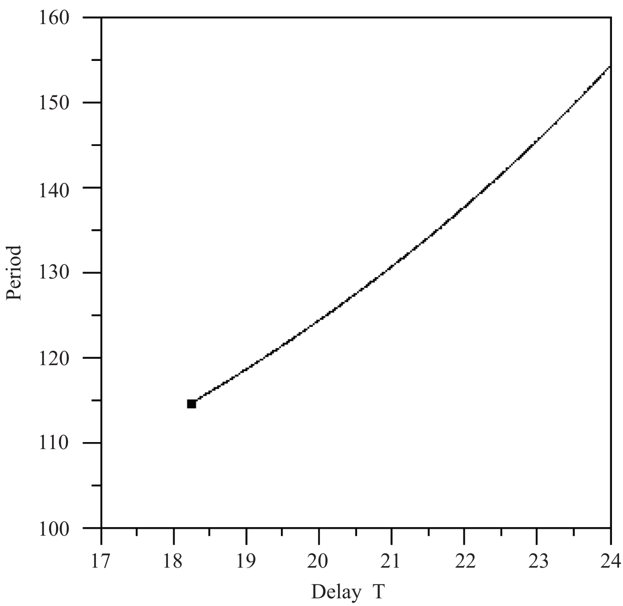

Fig. 3.6 Period of oscillation, $\frac{2\pi}{\Omega}$, plotted as a function of delay T, where Ω is given by Eq.(3.230). The initiation of oscillation at $T = T_{cr} = 18.2470$ is due to a supercritical Hopf bifurcation, and is marked in the Figure with a dot.

3.8 Exercises

Exercise 1
For which values of the delay $T > 0$ is the trivial solution in the following DDE stable?

$$\frac{dx(t)}{dt} = x(t) - 2x(t-T). \tag{3.231}$$

Exercise 2
Use Lindstedt's method to find an approximation for the amplitude of the limit cycle in the following DDE:

$$\frac{dx(t)}{dt} = -x(t-T) + x(t-T)^3. \tag{3.232}$$

Exercise 3
Use the center manifold approach to determine the stability of the x=0 solution in the following DDE:

$$\frac{dx(t)}{dt} = -x(t - \frac{\pi}{2})(1 + x(t)). \tag{3.233}$$

Here is an outline of the steps involved in this complicated calculation:

1. Show that the parameters of the linearized equation

$$\frac{dx(t)}{dt} = -x\left(t - \frac{\pi}{2}\right), \tag{3.234}$$

   have been chosen so that the delay is set at its critical value for a Hopf bifurcation, i.e. the characteristic equation has a pair of pure imaginary roots, $\lambda = \pm\omega i$. Find ω.
2. Find the eigenfunctions $s_1(\theta), s_2(\theta)$ and the adjoint eigenfunctions $n_1(\theta), n_2(\theta)$. These are determined by Eqs.(3.98)–(3.101),(3.113)–(3.116), where the constants c_i, d_i are related by the orthonomality conditions (3.112), in which the bilinear form (v,u) is given by Eq.(3.107).
3. By comparing Eq.(3.233) with the general form (3.90), identify α, β, and f for this system. This will permit you to write down Eqs.(3.155) and (3.156), in which (n_i, Fx_t) is given by Eq.(3.157) and $x_t = y_1(t)s_1(\theta) + y_2(t)s_2(\theta)$.
4. Equate coefficients of y_1^2, y_1y_2 and y_2^2 in Eqs.(3.155) and (3.156) and so obtain 3 first order linear ODE's on $m_1(\theta)$, $m_2(\theta)$ and $m_3(\theta)$, together with 3 boundary conditions.
5. Solve these for $m_i(\theta)$.
6. Substitute the resulting expressions for $m_i(\theta)$ into Eq.(3.136) for the center manifold.
7. Substitute your expression for the center manifold into the y_1-y_2 Eqs.(3.132). Here (n_i, Fx_t) is given by Eq.(3.134) and $x_t = y_1(t)s_1(\theta) + y_2(t)s_2(\theta) + w(t)(\theta)$.
8. Compute Q from Eq.(3.138).
   Answer: $Q = -\frac{4\pi}{5}\frac{(3\pi-2)}{(\pi^2+4)^2}$

3.9 Acknowledgement

The author wishes to thank his associates Anael Verdugo, Si Mohamed Sah, Meghan Suchorsky, and Tamas Kalmar-Nagy for their help in preparing this work.

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