Graham Higman's lectures on januarials

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Abstract

This is an account of a series of lectures of Graham Higman on *januarials*, namely coset graphs for actions of triangle groups which become *2-face maps* when embedded in orientable surfaces.

Spilt Milk

We that have done and thought,
That have thought and done,
Must ramble, and thin out
Like milk spilt on a stone.

The Nineteenth Century and After

Though the great song return no more
There's keen delight in what we have:
The rattle of pebbles on the shore
Under the receding wave.

from *The Winding Stair and Other Poems*, W.B.Yeats, 1933

1 Preamble

Graham Higman gave the lectures on which this article is based, in Oxford in 2001. They are likely to have been the final lectures he gave; he died in April 2008, at the age of 91. He introduced them with the quotes from W.B. Yeats reproduced above, and

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described the work in preparing them as "justifying my old age" and "keeping me relatively sane." The second author attended the lectures, and the first author remembers Higman's work on related topics some years earlier; this account is developed from recollections and from notes taken at the time. As such, any errors are ours, and the presentation and the proofs offered may not be as Higman had in mind. At various points, and as indicated, we have extended Higman's treatment; we also include some of our own observations in an afterword.

Januarials, which we will define in Section 2, are 2-complexes with two distinguished faces, that result from embedding coset graphs for the actions of triangle groups into compact orientable surfaces. They can be viewed as being assembled from two subsurfaces (essentially those two distinguished faces); we give appropriate definitions and tools to explore the complexity of this assembly in Section 3. In Section 4 we give sufficient conditions for actions of PSL(2, p) on projective lines to give rise to januarials. This leads to a number of examples presented in Section 5. Finally Section 6, our afterword, contains some remarks on the coset graph appearing in Norman Blamey's 1984 portrait of Higman, and some further examples of januarials.

It appears that Higman's study of januarials was sparked by his work on *Hurwitz groups*, which are non-trivial finite quotients of the $(2, 3, 7)$-triangle group. Higman used coset diagrams to show that for all sufficiently large n, the alternating group Alt(n) is a Hurwitz group, and his work was taken further by the first author to determine exactly which Alt(n) are Hurwitz, in [1].

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2 Coset graphs, face maps, januarials, and surfaces

Suppose S is a set endowed with an action $s \mapsto s^g$ by a group G, and A is a generating set for G. Define Γ to be the graph with vertex set S, and with an oriented edge labelled by $a \in A$ (called an *a-edge*) from vertex u to vertex v whenever $u^a = v$. We will be concerned with situations where G acts transitively on S, so that Γ is connected. In that event we can identify S with the right cosets $\{ Hg : g \in G \}$ of the stabiliser $H = G_s = \mathrm{Stab}_G(s)$ of any particular $s \in S$, and for this reason, Γ is known as a *coset graph* or *Schreier graph*, or sometimes *coset diagram*, for the action of G on S with respect to A. When the action of G on S is also regular, we can identify S with the underlying set of G, in which case Γ is the *Cayley graph* of G with respect to A.

Paths in the coset graph may be labelled with words on the generating set A (which can be thought of as an alphabet). Suppose that a word w on $A^{\pm 1}$ represents $g \in G$, and that $s \in S$. Let γ be the path in Γ obtained by concatenating the unique edge-paths in Γ from s^{g^i} to $s^{g^{i+1}}$, for each $i \in \mathbb{Z}$, along which one reads w. This tours an orbit of $\langle g \rangle$ and is a (closed) circuit precisely when that orbit is finite. There is one such path for each orbit.

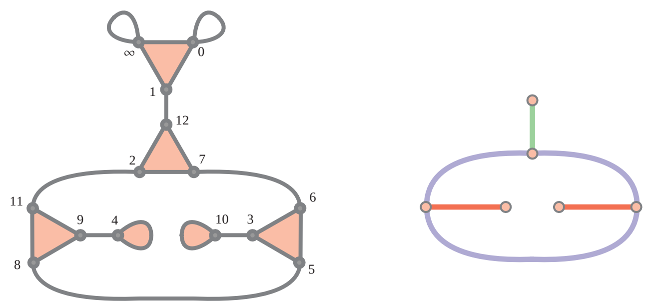

Figure 1: A coset graph arising from an action of the group PSL(2, 13) on $\mathbb{F}_{13} \cup \{\infty\}$, with $x : z \mapsto -z$ and $y : z \mapsto (z-1)/z$, and its companion graph. This results in a 3-janualial of genus 0 and simple type (1, 0, 0).

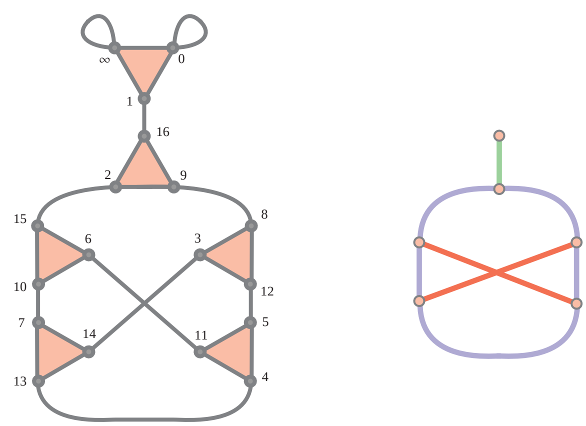

Figure 2: A coset graph arising from an action of the group PSL(2, 17) on $\mathbb{F}_{17} \cup \{\infty\}$, with $x : z \mapsto -z$ and $y : z \mapsto (z-1)/z$, and its companion graph. This results in a 3-janualial of genus 1 and simple type (1, 1, 0).

A *map* is a 2-cell embedding of a connected (multi)graph in some closed surface, with its *faces* (the components of the complement of the graph in the surface) being homeomorphic to open disks in $\mathbb{R}^2$. Examples include triangulations and quadrangulations of the torus, and the Platonic solids (which may be viewed as highly symmetric maps on the sphere), with all vertices having the same valence and all faces having the same size.

A *januarial* is a special instance of a map constructed from embedding a coset graph for an action of the the $(2, k, l)$ *triangle group*

$$\Delta(2, k, l) \;=\; \langle\, x, y \;|\; x^2 = y^k = (xy)^l = 1 \,\rangle,$$

with $A = \{x, y\}$. Because $x^2 = 1$, the x-edges in such a coset graph Γ coming from non-trivial cycles of x occur in pairs: whenever there is an x-edge from u to v, there is an x-edge from v to u. We may identify each such pair, so as to leave an *unoriented* x-edge between u and v. Then for each fixed point s of x, we attach a 2-cell (which we will call an *x-monogon*) along the x-edge which forms a loop at s. Similarly, for each orbit of $\langle y\rangle$, we attach a polygon (which we call a *y-face*) along the path γ given by $\langle y\rangle$ as described above. This gives a 2-complex, many examples of which appear in this article; see Figures 1, 2, 6, 7, 8, 9, and 10. These and similar figures can be displayed without labels on the edges, because we may shade the y-faces so that y-edges are identifiable as those in the boundaries of y-faces, while all the remaining edges are x-edges. We need not indicate orientations on the edges: the x-edges for the reason given above, and the y-edges because we may adopt a convention that all y-edges are oriented anti-clockwise around the corresponding y-faces. Note that the length (the number of sides) of each y-face divides k.

Next, attach a polygon (which we call an xy-face) around each orbit of $\langle xy\rangle$. As shown by the following lemma, the resulting 2-complex J is homeomorphic to a closed orientable surface. We may call the corresponding embedding of Γ an *m-face map*, where m is the number of orbits of $\langle xy\rangle$.

A *januarial* (and more precisely, a *k-januarial*) is the instance when $m = 2$ and the orbits of $\langle xy\rangle$ have the same size $|S|/2$. Two examples of 3-januarials are given in Figure 3.

Lemma 2.1. *The 2-complex J defined above is homeomorphic to a compact orientable surface without boundary.*

Proof. In the construction of J we identified oppositely oriented x-edges in pairs. For this proof, however, it is convenient to revert to the pairs of oriented x-edges, and insert a digon (which we call an *x-digon*) between each pair. We will show that the resulting complex gives an orientable surface without boundary. It will follow that the same is true of a januarial, because we have an embedding in the same surface when the digons (any two of which have no x-edge in common) are successively collapsed to single edges.

Now in this complex, each vertex has valence four: it has both an incoming and an outgoing x-edge (coming from an edge-loop in the event that x fixes the vertex), and both an incoming and an outgoing y-edge (which, similarly, may come from a loop). Each x-edge is incident with exactly one x-face (that is, an x-monogon or an x-digon), and one xy-face. Each y-edge is incident with exactly one y-face and one xy-face. It follows that the complex gives a surface without boundary. Moreover, the surface is

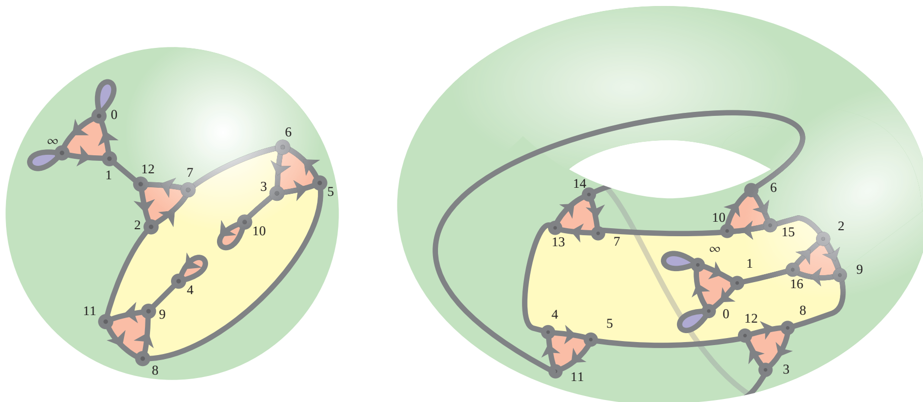

Figure 3: The 3-janurials arising from the coset graphs in Figures 1 and 2. The unoriented edges are x-edges and oriented edges are y-edges.

orientable because the directions of the edges give consistent orientations around the perimeters of all the faces. Finally, since S is finite, the surface is compact. □

3 The topological complexity of janurials

Higman gave a notion of topological complexity which we call *simple type* below. It concerns how a janurial J is assembled from the subspaces S_1 and S_2 that are the closures of its two xy-faces. He recognised that some janurials are beyond the scope of this notion; indeed, he made some ad hoc calculations for the examples in Figures 8 and 10 which show as much. Accordingly, below, we define a more general notion which we call *type*, which applies to all janurials, and we explain how to calculate it in general.

Topological features of J, S_1 and S_2 come into clearer focus when we collapse each x-monogon and each y-face in J to a point. Any two y-faces in a janurial are disjoint. The same is true of any two x-monogons. And an x-monogon can only meet a y-face at a single vertex. So these collapses do not change the homeomorphism types of J, S_1 or S_2.

Let Γ be the 1-skeleton of J — that is, the coset graph. Let $\overline{J}$, $\overline{S_1}$, $\overline{S_2}$ and $\overline{\Gamma}$ be the images of J, S_1, S_2 and Γ under these collapses. We call $\overline{\Gamma}$ a *companion graph*. Then $\overline{J} = \overline{S_1} \cup \overline{S_2}$ is a closed surface obtained by some identification of $\overline{S_1}$ and $\overline{S_2}$ along their boundaries. Taking another perspective, $\overline{J}$ is the result of adding two faces to $\overline{\Gamma}$, via attaching maps ρ_1 and ρ_2 induced by the maps that attach the xy-faces to Γ.

Examples of such Γ and $\overline{\Gamma}$ appear in Figures 1, 2, 7, 8, 10, 12, and 13. Each one is drawn in such a way that the cyclic order in which edges emanate from vertices agrees with that in which x-edges meet y-faces in Γ. So, as the y-edges in Γ are oriented anti-clockwise around the y-faces, one can read off ρ_i by following successive edges in $\overline{\Gamma}$ in such a way that on arriving at a vertex, one exits along the right-most of all the remaining edges (except if the vertex has valence one, in which case one exits along the edge by which one arrived).

As $\langle xy \rangle$ yields exactly two orbits when acting on S, together ρ_1 and ρ_2 traverse each edge in $\overline{\Gamma}$ twice, once in each direction. The edges comprising the subgraph $\mathcal{G} := \overline{S_1} \cap \overline{S_2}$, shown in blue in the figures, are traversed by ρ_1 in one direction and ρ_2 in the other. Those traversed by ρ_1 (resp. ρ_2) in both directions are shown in red (resp. green).

The collapses carrying J to $\overline{J}$ leave only the two xy-faces, those x-edges which are not loops, and one vertex for each y-face in J. These collapses do not alter the Euler characteristic. Since J is a closed orientable surface, we find that the genus of J is readily calculated as follows.

Lemma 3.1. *Twice the genus of a janaurial equals the number of x-edges which are not loops minus the number of y-faces.*

For example, this is $6 - 6 = 0$ in the left-hand example of Figure 3 and is $8 - 6 = 2$ in the right-hand example, giving genera 0 and 1, respectively.

Now we turn to genera associated to S_1 and S_2, or their images $\overline{S_1}$ and $\overline{S_2}$. Defining these requires care, since $\overline{S_1}$ and $\overline{S_2}$ may fail to be sub-surfaces of $\overline{J}$ (and likewise S_1 and S_2 fail to be sub-surfaces of J): they are closed surfaces from which the interiors of some collection of discs have been removed, but the boundaries of those discs need not be disjoint. (Figures 8 and 10 provide such examples.) But if we take a small closed neighbourhood R_i of $\overline{S_i}$ in $\overline{J}$, we get a genuine sub-surface which serves as a suitable proxy:

Lemma 3.2. *R_1 and R_2 are orientable surfaces, and they retract to $\overline{S_1}$ and $\overline{S_2}$, respectively.*

Proof. A small closed neighbourhood of $\mathcal{G}$ (or indeed of any subgraph of the 1-skeleton of a finite cellulation of a closed surface) is a sub-surface with boundary and retracts to $\mathcal{G}$. Similarly R_i, which is the union of $\overline{S_i}$ with a small closed neighbourhood of $\mathcal{G}$, is orientable and retracts to $\overline{S_i}$. It is orientable because $\overline{J}$ is orientable. □

We define the *type* of J to be the pair $((h_1, g_1), (h_2, g_2))$, where g_i and h_i are the genus of R_i and the number of connected components of the boundary of R_i respectively, for $i = 1, 2$. We will not distinguish between types $((h_1, g_1), (h_2, g_2))$ and $((h_2, g_2), (h_1, g_1))$.

The most straightforward way in which R_1 and R_2 can be assembled to make $\overline{J}$ occurs when $R_1 \cap R_2$ is a disjoint union of h annuli, where $h = h_1 = h_2$, or in other words, when $\overline{J}$ is homeomorphic to a join of R_1 and R_2 in which the boundary components are paired off and identified. In this case, we say that the janaurial J is of *simple type* (h, g_1, g_2). We do not distinguish between the simple types (h, g_1, g_2) and (h, g_2, g_1).

Maps in which the graph is embedded in a suitably non-pathological manner (for instance as a subgraph of the 1-skeleton of a finite cellulation of the surface) have the property that a small neighbourhood is a disjoint union of annuli if and only if the graph is a collection of disjoint simple circuits. So, as $R_1 \cap R_2$ is a small neighbourhood of $\mathcal{G}$, one can recognise simple type from the graph $\overline{\Gamma}$:

Lemma 3.3. *J is of simple type if and only if $\mathcal{G}$ is a collection of disjoint simple circuits. In that case, if J has simple type (h, g_1, g_2) then h is the number of circuits.*

The genus of a janiarial J (equivalently, of $\overline{J}$) of simple type is present in the data (h, g_1, g_2). When the handles (that is, the h annuli from $R_1 \cap R_2$) that connect R_1 and R_2 are severed one-by-one, the genus falls by 1 each time, until we only have one handle connecting R_1 and R_2, and hence a surface of genus $g_1 + g_2$. Since J has h handles to begin with, this gives the following:

Lemma 3.4. *The genus of a janiarial J of simple type is $g_1 + g_2 + h - 1$.*

Figures 1, 2, 7 and 12 show examples of coset graphs which give janiarials of simple type, and Figures 8, 10 and 13 show examples which give janiarials that are not of simple type. In each case, the caption indicates the genus of the janiarial and the details of the type. The genus of the janiarial can be established in each case via an Euler characteristic calculation (as per Lemma 3.4 for those of simple type).

For the examples of simple type, h is immediately evident from the companion graph $\overline{\Gamma}$ on account of Lemma 3.3. For those not of simple type, our next lemma gives a means of identifying h_1 and h_2 from $\overline{\Gamma}$. Examples of *partitions of $\mathcal{G}$ into circuits* in the sense of this lemma can be seen in Figures 8, 10 and 13.

Lemma 3.5. *Let $\mathcal{P}$ be the set of all paths that traverse successive edges in $\mathcal{G}$ in the directions they are traversed by ρ_1 (resp. ρ_2), in such a way that whenever such a path reaches a vertex, it continues along the right-most of the other edges in $\mathcal{G}$ incident with that vertex. (The next edge is necessarily traversed by ρ_1 (resp. ρ_2) in that direction.) All such paths close up into circuits, and $\mathcal{P}$ partitions $\mathcal{G}$, in the sense that the union of the circuits is $\mathcal{G}$ and no two share an edge. The cardinality of $\mathcal{P}$ is h_2 (resp. h_1).*

Proof. We will prove the result for ρ_1. The same argument holds for ρ_2 with the subscripts 1 and 2 interchanged.

By construction, the portion of the circuit ρ_1 that falls in $\mathcal{G}$ runs close alongside the boundaries of the h_2 holes in R_2. Consider the situation where ρ_1 is traversing an edge e in $\mathcal{G}$, and let B denote the boundary of the hole that runs alongside — see Figure 4. At the terminal vertex v of e, because of our convention for drawing companion graphs, ρ_1 will continue along the right-most of the other incident edges in $\overline{\Gamma}$. If that edge e' is in $\mathcal{G}$, it also runs alongside B. (This happens at u in the figure.) Suppose, on the other hand, that e' is not in $\mathcal{G}$. Then ρ_1 does not run alongside B, but rather heads into the interior of R_2. (This happens at v in the figure.) At some later time, ρ_1 must return along e' in the opposite direction (perhaps visiting another portion of B in the interim) since the edges ρ_1 traverses exactly once are precisely those in $\mathcal{G}$. Hence ρ_1 arrives back at v and then continues along the (new) right-most edge — which will either be alongside B, or take it back into the interior of R_2, again to return eventually along that same edge. Repeating this, we eventually find the next edge in $\mathcal{G}$ incident with v that continues alongside B. It follows that however many detours into the interior of R_2 are required, it is the *right-most of the edges in* $\mathcal{G}$ incident with v (aside from e) that continues alongside B. So the circuits traversed as explained in the statement of the lemma are those that run alongside the boundaries of the holes in R_2. The remaining claims easily follow from this. □

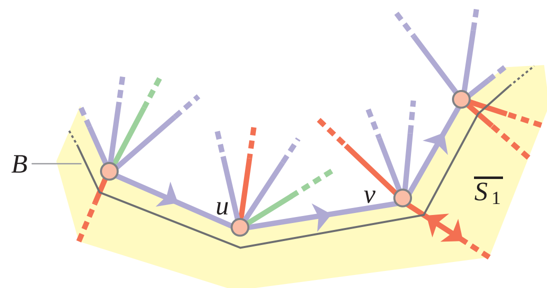

Figure 4: Tracking the boundary of one of the holes in R_2.

Given h_i (for $i = 1$ or 2), one can determine g_i from $\overline{\Gamma}$ via the following observation:

Lemma 3.6. *The genus g_i of R_i satisfies $2 - 2g_i = V_i - E_i + h_i + 1$ where V_i and E_i denote the number of vertices and edges, respectively, in the subgraph of $\overline{\Gamma}$ visited by the attaching map of the face of $\overline{S}_i$.*

Proof. By Lemma 3.2, filling the h_i holes in R_i with discs gives a closed orientable surface of genus g_i which is homotopic to $\overline{S}_i$ with h_i discs attached along circuits in its 1-skeleton. Hence the Euler characteristic $2 - 2g_i$ of R_i is the same as that of $\overline{S}_i$ with the h_i discs attached, namely $V_i - E_i + h_i + 1$. □

Questions 3.7. Higman asked the following questions concerning k-janurials of simple type. For a given k, what are the possible values for and interrelationships between g_1, g_2 and h? Are there arbitrarily large values of k for which there exist examples with $h = 1$? How large can h be, for given k? Similar questions can be asked also about janurials that are not of simple type.

4 Janurials from $\mathrm{PSL}(2, q)$

4.1 $\mathrm{PGL}(2, q)$, $\mathrm{PSL}(2, q)$ and the classical modular group

The *projective linear group* $\mathrm{PGL}(n, \mathbb{F})$ over a field $\mathbb{F}$ is the quotient $\mathrm{GL}(n, \mathbb{F})/Z$ of the group of invertible $n \times n$ matrices by its centre $Z = \{ aI_n \mid a \in \mathbb{F} \smallsetminus \{0\}\}$. Its subgroup, the *projective special linear group* $\mathrm{PSL}(n, \mathbb{F})$, is the quotient of $\mathrm{SL}(n, \mathbb{F})$, the group of all $n \times n$ matrices over $\mathbb{F}$ of determinant one, by its subgroup of all scalar matrices of determinant one.

There is a natural isomorphism between $\mathrm{PGL}(2,\mathbb{F})$ and a group of Möbius transformations, under which the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to the transformation $z \mapsto \dfrac{az+c}{bz+d}$, when multiplication of transformations is read from left to right. This gives actions of $\mathrm{PGL}(2,\mathbb{F})$ and $\mathrm{PSL}(2,\mathbb{F})$ on the projective line $\mathrm{PL}(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$. Also if $\mathbb{F}$ is finite, of order q, then $\mathrm{PGL}(2,\mathbb{F})$ and $\mathrm{PSL}(2,\mathbb{F})$ are denoted by $\mathrm{PGL}(2,q)$ and $\mathrm{PSL}(2,q)$.

A search for 3-januarials may begin with the *classical modular group*

$$\mathrm{PSL}(2,\mathbb{Z}) \;=\; \langle\, x, y \,|\, x^2 = y^3 = 1 \rangle$$

which acts on $\mathbb{Q} \cup \{\infty\}$ by Möbius transformations with $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$. Notice that $xy : z \mapsto z+1$. A portion of the resulting coset diagram is shown in Figure 5.

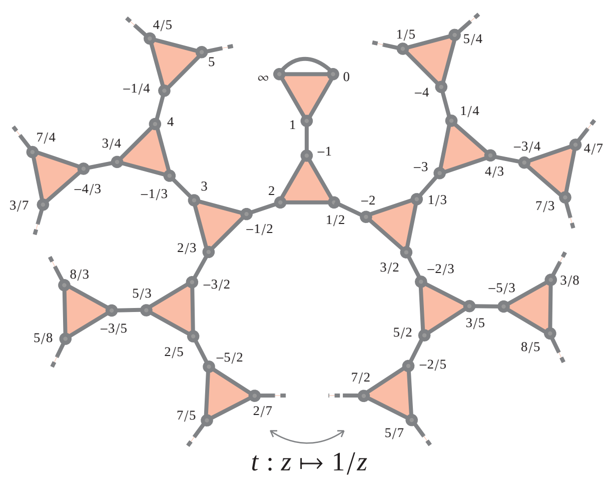

Figure 5: The coset graph arising from the action of the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \langle\, x, y \,|\, x^2 = y^3 = 1 \,\rangle$ on $\mathbb{Q} \cup \{\infty\}$ via $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$.

Suppose p is a prime. Then the group $\mathrm{PSL}(2,p)$ is a homomorphic image of

$$\Delta(2,3,p) \;=\; \langle\, x, y \;|\; x^2 = y^3 = (xy)^p = 1 \,\rangle,$$

whereby $\Delta(2,3,p)$ acts on $\mathbb{F}_p \cup \{\infty\}$ via $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$, with product $xy : z \mapsto z+1$. The resulting coset diagrams Γ are quotients of the diagram in Figure 5; for example, the diagrams for $\mathrm{PSL}(2,13)$ and $\mathrm{PSL}(2,17)$ are shown in Figure 6. But, these coset diagrams do not immediately yield januarials, since the orbits of $\langle xy \rangle$ have lengths 1 and p rather than both $(p+1)/2$.

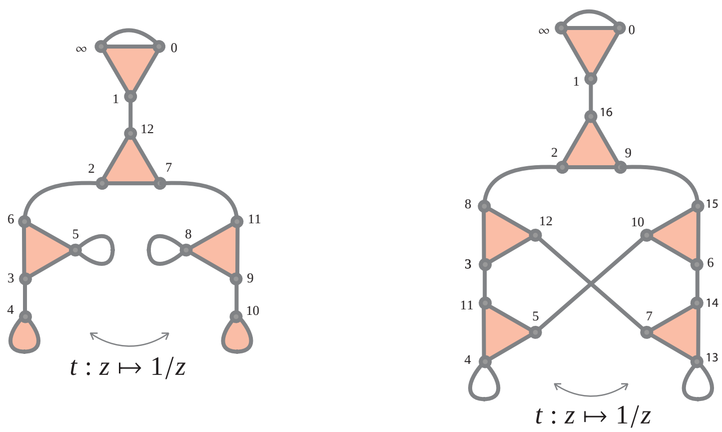

Figure 6: Coset graphs arising from actions of $\Delta(2, 3, 13)$ on $\mathbb{F}_{13} \cup \{\infty\}$ and $\Delta(2, 3, 17)$ on $\mathbb{F}_{17} \cup \{\infty\}$, both via $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$.

4.2 Associates

Here is a potential remedy for the failure of the coset diagrams constructed above from $\mathrm{PSL}(2, p)$ to produce janarials. It applies in the general setting of a finite group G acting on a set S and containing elements x and y satisfying $x^2 = y^k = (xy)^l = 1$ for some $l \in \mathbb{Z}$. Let Γ be the resulting coset graph for the action of $\Delta(2, k, l)$ on S, via G, with respect to the generating set $\{x, y\}$.

Suppose G has an element t of order 2 with the property that $t^{-1}xt = x^{-1}$ $(= x)$ and $t^{-1}yt = y^{-1}$. Conjugation by such an element t reverses every cycle of y and preserves every cycle of the involution x, and hence t induces a reflection of the coset graph Γ.

Note that $(xt)^2 = x(txt) = xx = 1$, which allows us to consider the pair (xt, y) in place of (x, y). If l' is the order of $xt\,y$, then we have an action of $\Delta(2, 3, l')$ on S via $\langle xt, y\rangle$. The resulting coset graph Γ' is called an *associate* of Γ. This graph also admits a reflection via the same involution t, since $t^{-1}(xt)t = x^{-1}t = xt = (xt)^{-1}$ (and $t^{-1}yt = y^{-1}$). For more details about the correspondence $(x, y, t) \mapsto (xt, y, t)$, see [5]. The associate graph Γ' gives a new candidate for a janarial.

Examples 4.1. When G is $\mathrm{PSL}(2, p)$ for some prime $p \equiv 1 \bmod 4$, and $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$, we can take t to be the transformation $z \mapsto 1/z$, which has order 2 and satisfies $t^{-1}xt = x^{-1}$ and $t^{-1}yt = y^{-1}$, as required. [The condition $p \equiv 1 \bmod 4$ ensures that -1 is a square mod p, so that the transformation t (and hence also xt) lies in $\mathrm{PSL}(2, p)$.] In this case, xt is the transformation $z \mapsto -z$. Hence, in particular, the associates of the coset graphs in the cases $p = 13$ and 17 from Figure 6 are precisely those in Figures 1 and 2. The transformation $xty : z \mapsto (z+1)/z$ has two cycles of equal length, and so in both cases the associates are janarials — specifically those depicted in Figure 3.

Like Γ, the associate can fail to yield a januarial if the sizes of the $\langle xy \rangle$-orbits are not the requisite $|S|/2$, but it succeeds in many cases. In Section 4.5 we will explore when it can be successfully applied to the examples from $\mathrm{PSL}(2,p)$. But first we need the following study of conjugacy classes in $\mathrm{PGL}(2,q)$.

4.3 Classifying conjugacy classes in $\mathrm{PGL}(2,q)$

Some of the details of the analysis in this section are similar to that carried out by Macbeath in [6].

Let q be any odd prime-power greater than 3, say $q = p^s$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,q)$, we may define

$$\theta(M) \;=\; \frac{(\mathrm{tr}\, M)^2}{\det M} \;=\; \frac{(a+d)^2}{ad-bc}.$$

The characteristic polynomial of M is $\det(xI_2 - M) = x^2 - \mathrm{tr}(M)x + \det(M)$, and $\mathrm{tr}(M)$ and $\det(M)$, and therefore also $\theta(M)$, are invariant under conjugacy within $\mathrm{GL}(2,q)$. Since $\theta(M)$ is invariant under scalar multiplication of M, it follows that this gives us a well-defined function $\theta : \mathrm{PGL}(2,q) \to \mathbb{F}_q$ that is constant on conjugacy classes of $\mathrm{PGL}(2,q)$.

In fact, the function θ parametrises the conjugacy classes of $\mathrm{PGL}(2,q)$, as follows.

Proposition 4.2. *Let g and h be elements of $\mathrm{PGL}(2,q)$. If $\theta(g) = \theta(h) \notin \{0,4\}$, then g and h are conjugate in $\mathrm{PGL}(2,q)$. In the exceptional cases, there are precisely two conjugacy classes on which $\theta = 0$, namely the class of involutions in $\mathrm{PSL}(2,q)$ and the class of involutions in $\mathrm{PGL}(2,q) \setminus \mathrm{PSL}(2,q)$, and two classes on which $\theta = 4$, namely the class containing the identity element and the class of the transformation $z \mapsto z+1$.*

This proposition can be proved using rational canonical forms, but also we can give a direct proof for the generic case.

Proof for the case where $\theta(g) \notin \{0,4\}$. Suppose the transformation $g \in \mathrm{PGL}(2,q)$ is induced by the matrix $M \in \mathrm{GL}(2,q)$, with $\mathrm{tr}(M) \neq 0$. Then we can choose a vector $u \in \mathbb{F}_q^2$ such that u and uM are linearly independent over $\mathbb{F}_q$. The matrix for g with respect to the basis $\{u, uM\}$ is the of the form $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$, but since the trace is non-zero and a conjugacy invariant, we can change the basis if necessary, so that the matrix for g has entry 1 in the lower-right corner. The matrix for g then becomes

$$M' \;=\; \begin{pmatrix} 0 & 1 \\ -\Delta & 1 \end{pmatrix},$$

where Δ is the determinant. But then $\theta(g) = \theta(M') = 1/\Delta$, so $\Delta = 1/\theta(g)$, and it follows that $\theta(g)$ determines the matrix. Since matrices representing the same linear transformation with respect to different bases are conjugate within $\mathrm{GL}(2,q)$, we find that $\theta(g)$ determines the conjugacy class of g. □

The utility of the parameter θ is enhanced by the following lemma, which gives a number of cases in which the order of an element $y \in \mathrm{PGL}(2,q)$ determines $\theta(y)$.

Elements with trace 0 give involutions in $\mathrm{PGL}(2,q)$, and conversely, while elements with trace -1 and determinant 1 give elements of order 3 in $\mathrm{PGL}(2,q)$, and parabolic elements (which are the conjugates of the matrix $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, or equivalently, the elements with trace 2 and determinant 1), give elements of order p in $\mathrm{PGL}(2,q)$. We also note that every element of order $(q+1)/2$ in $\mathrm{PGL}(2,p)$ is the square of an element of order $q+1$ in $\mathrm{PGL}(2,q)$, and hence lies in $\mathrm{PSL}(2,q)$ and is the product of two cycles of length $(q+1)/2$ in the natural action of $\mathrm{PGL}(2,q)$ on $\mathbb{F}_q \cup \{\infty\}$.

Lemma 4.3. *If y is an element of order* 1, 2, 3, 4 *or* 6 *in* $\mathrm{PGL}(2,q)$, *then* $\theta(y) = 4, 0, 1, 2$ *or* 3, *respectively. Also if y has order p (the prime divisor of q) then* $\theta(y) = 4$.

Proof. Suppose y is induced by the element $M = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}(2,q)$. Then the first three cases are easy consequences of the respective observations that in those cases, M is scalar, or M has trace 0, or M has minimum polynomial $x^2 + x + 1$.

For the next two cases, we note that $M^2 = \begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & d^2+bc\end{pmatrix}$ and therefore

$$\mathrm{tr}(M^2) \;=\; a^2 + d^2 + 2bc \;=\; (a+d)^2 - 2(ad-bc) \;=\; (\mathrm{tr}\,M)^2 - 2\det M.$$

If g has order 4, then M^2 has order 2, and so $0 = \mathrm{tr}(M^2) = (\mathrm{tr}\,M)^2 - 2\det M$, which gives $\theta(y) = \theta(M) = (\mathrm{tr}\,M)^2/\det M = 2$. Similarly, if y has order 6, then since M^2 induces an element of order 3 in $\mathrm{PGL}(2,q)$ we know that

$$((\mathrm{tr}\,M)^2 - 2\det M)^2 \;=\; (\mathrm{tr}(M^2))^2 \;=\; \det(M^2) \;=\; (\det M)^2,$$

and therefore $(\theta(M) - 2)^2 = 1$. But since y does not have order 3, we know that $\theta(M) \neq 1$, and so $\theta(M) - 2 = 1$, which gives $\theta(y) = \theta(M) = 3$.

Finally, if y has order p, then y is parabolic and therefore induced by some conjugate of the matrix $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, which implies that $\theta(y) = (1+1)^2/1 = 4$. □

4.4 How many 3-januarials arise from PSL(2, p)?

We can now proceed further, to consider 3-januarials. Let ϕ be Euler's totient function — that is, let $\phi(n)$ be the number of integers in $\{1,\ldots,n\}$ that are coprime to n.

Lemma 4.4. *The number of conjugacy classes of elements in* $\mathrm{PGL}(2,q)$ *of order* $(q+1)/2$ *is* $\frac{1}{2}\phi((q+1)/2)$*. Moreover, if z is any element of order* $(q+1)/2$ *in* $\mathrm{PGL}(2,q)$*, then every one of these conjugacy classes intersects the subgroup generated by z in* $\left\{z^i, z^{-i}\right\}$ *for exactly one i coprime to* $(q+1)/2$*.*

Proof. This follows easily from the observation that every element of order $(q+1)/2$ in $\mathrm{SL}(2,q)$ is conjugate in $\mathrm{GL}(2,q^2)$ to one of the form $\begin{pmatrix}\lambda^i & 0\\ 0 & \lambda^{-i}\end{pmatrix}$, where λ is an element of order $(q+1)/2$ in the field $\mathbb{F}_{q^2}$, and the traces $\lambda^i+\lambda^{-i}$ are distinct in $\mathbb{F}_q$. □

Lemma 4.5. *For any given conjugacy class C of elements of* $\mathrm{PSL}(2,q)$ *of order* $l \notin \{1,2,p\}$*, there exists a triple* (x,y,xy) *of elements of* $\mathrm{PSL}(2,q)$ *such that x has order* 2*, and y has order* 3*, and xy is in C. Moreover, this triple is unique up to conjugacy in* $\mathrm{PGL}(2,q)$ *whenever* $l \neq 6$*.*

Proof. Every element of order 3 in $\mathrm{PSL}(2,q)$ is conjugate in $\mathrm{PGL}(2,q)$ to the element $y : z \mapsto (z-1)/z$, induced by the matrix $Y = \begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$, while any element of order 2 in $\mathrm{PSL}(2,q)$ is induced by a matrix X of the form $X = \begin{pmatrix}a & b\\ c & -a\end{pmatrix}$, with trace 0 and determinant $-a^2-bc = 1$. Now observe that for any such choice of X and Y, we have $XY = \begin{pmatrix}-a-b & a\\ a-c & c\end{pmatrix}$, which has trace $\mathrm{tr}(XY) = -a-b+c$ and determinant $\det(XY) = \det(X)\det(Y) = 1$.

This can be turned around: we can show that for any given non-zero trace r, there exist a,b and c in $\mathbb{F}_q$ such that $-1 = a^2+bc$ and $r = -a-b+c$, and hence there exists a matrix X of trace zero such that XY has trace r, giving a triple (x,y,xy) of the required type. Note that we need $c-b = r+a$ and $bc = -(1+a^2)$, and therefore $(c+b)^2 = (c-b)^2+4bc = r^2+2ar-3a^2-4$. Now if $p \neq 3$, then we can multiply this by 3 and it becomes $3(c+b)^2 = 4r^2-12-(3a-r)^2$; and then since in $\mathbb{F}_q$ there are $(q+1)/2$ elements of the form $3u^2$ and $(q+1)/2$ elements of the form $s-v^2$ for any given $s \in \mathbb{F}_q$, and any two subsets of size $(q+1)/2$ in $\mathbb{F}_q$ have non-empty intersection, the latter equation can be solved in $\mathbb{F}_q$ for $c+b$ and $3a-r$, and hence for $c+b$ and a (and $c-b = r-a$), and hence for a, b and c (since q is odd). On the other hand, if $p = 3$, then the equation becomes $(c+b)^2 = r^2+2ar-3a^2-4 = r^2+2ar-1$, which is even easier to solve for a,b,c, provided that $r \neq 0$.

The main assertion now follows easily. Uniqueness up to conjugacy in $\mathrm{PGL}(2,q)$ when $l \neq 6$ is left as an exercise. □

Since the automorphism group of $\mathrm{PSL}(2,p)$ is $\mathrm{PGL}(2,p)$ for every odd prime p, we obtain the following:

Corollary 4.6. *For any prime* $p > 3$*, the number of distinct* 3*-janualrials constructible from* $\mathrm{PSL}(2,p)$ *in the way described in sub-sections 4.1 and 4.2 is* $\frac{1}{2}\phi((p+1)/2)$*.*

For example, when $p = 37$ the number of 3-janualrials is $\phi(19)/2 = 9$, and when $p = 53$ the number is $\phi(27)/2 = 9$ as well. A further attractive property is as follows:

Lemma 4.7. *For every triple (x, y, xy) as in Lemma 4.5 with $l \neq 6$, there exists an involution t in* $\mathrm{PGL}(2, q)$ *such that $t^{-1}xt = x^{-1}$ and $t^{-1}yt = y^{-1}$.*

Proof. We may suppose that x and y are induced by the matrices X and Y as in the proof of Lemma *4.5*. In that case, let $T = \begin{pmatrix} -(b+c) & 2a+b \\ 2a-c & b+c \end{pmatrix}$, which has the property that $XT = TX^{-1}$ while $YT = TY^{-1}$. The determinant of T is

$$-(b+c)^2 - (2a+b)(2a-c) \;=\; -4a^2 - b^2 - c^2 - 2ab + 2ac - bc,$$

which equals $3\det(XY) - (\mathrm{tr}(XY))^2$, since $\det(XY) = -a^2 - bc$ while $\mathrm{tr}(XY) = -a - b + c$, and therefore T is invertible if and only if $\theta(XY) = (\mathrm{tr}(XY))^2/\det(XY) \neq 3$, or equivalently, XY does not have order 6. Finally, note that if T is invertible, then since its trace is zero, we have $t^2 = 1$. □

4.5 Necessary conditions for associates to yield janualrials

For any positive integer n, define θ_n to be the set of all values of $\theta(g)$ for elements g of order n in the group $\mathrm{PGL}(2, q)$. Consider the effect of the mapping $r \mapsto (r-1)^2$ on the elements of this set θ_n, for some n, as follows.

Suppose that $r \in \theta_n$, where n is coprime to q, and let g be an element of order n in $\mathrm{PGL}(2, q)$ with $\theta(g) = r$. By taking a conjugate of g if necessary in $\mathrm{PGL}(q^2)$, we may assume that g is the transformation $z \mapsto \rho z$ induced by the matrix

$$M = \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix},$$

where ρ is a primitive nth root of 1 in $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$. In this case, $\mathrm{tr}(M) = \rho + 1$ while $\det(M) = \rho$, and so

$$r \;=\; \theta(g) \;=\; \theta(M) \;=\; \frac{(\rho+1)^2}{\rho} \;=\; \rho + \rho^{-1} + 2.$$

It follows that $(r-2)^2 = (\rho + \rho^{-1})^2 = \rho^2 + \rho^{-2} + 2$. But r is of order n and so ρ will be a primitive nth root of unity. In particular, if n is odd then also ρ^2 is a primitive nth root of unity, in which case $(r-2)^2 = \rho^2 + \rho^{-2} + 2 = \theta(M^2)$, which also belongs to θ_n. Iterating the procedure then yields further elements of θ_n, until we reach a stage where $\rho^{2^i} = \rho^{\pm 1}$, and then $\rho^{2^i} + \rho^{-2^i} + 2 = \rho + \rho^{-1} + 2 = r$.

We now derive two necessary conditions on the values of $\theta(g)$ in θ_n in the special case where $n = (q+1)/2$.

Lemma 4.8. *If g is an element of order $(q+1)/2$ in* $\mathrm{PSL}(2, q)$*, then $\theta(g)$ is a square in F_q while $\theta(g) - 4$ is not a square in F_q.*

Proof. First, g is conjugate in $\mathrm{PGL}(2, q^2)$ to the projective image of $M = \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$ where μ is a primitive $(q+1)$th root of unity in $\mathbb{F}_{q^2}$, and this gives

$$\theta(g) \;=\; \frac{(\operatorname{tr} M)^2}{\det M} \;=\; \frac{(\mu^2+1)^2}{\mu^2} \;=\; \mu^2 + 2 + \mu^{-2}.$$

In particular, $\theta(g) = \mu^2+2+\mu^{-2} = (\mu+\mu^{-1})^2$, which is a square in $\mathbb{F}_q$ (since $\mu+\mu^{-1} \in \mathbb{F}_q$). On the other hand, $\theta(g) - 4 = \mu^2 - 2 + \mu^{-2} = (\mu - \mu^{-1})^2$, which is not a square in $\mathbb{F}_q$, since $\mu - \mu^{-1} = 2\mu - (\mu + \mu^{-1}) \notin \mathbb{F}_q$. ◻

Corollary 4.9. *Suppose $g \in \mathrm{PSL}(2, p)$ has order $(p+1)/2$.*

(a) *If $\theta(g) = -1$, then $p \equiv 13$ or 17 mod 20.*

(b) *If $\theta(g) = -2$, then $p \equiv 17$ or 19 mod 24.*

(c) *If $\theta(g) = -3$, then $p \equiv 13, 19$ or 31 mod 42.*

Proof. In case (a), by Lemma 4.8 we require that $-1 = \theta(g)$ is a square mod p while $-5 = \theta(g) - 4$ is not, and hence also 5 is not. Thus $p \equiv 1$ mod 4, and by quadratic reciprocity, also $p \equiv 2$ or 3 mod 5, giving $p \equiv 13$ or 17 mod 20. Similarly, in case (b) we require that -2 is a square mod p while 3 is not. It follows that $p \equiv 1$ or 3 mod 8, while also $p \not\equiv \pm 1$ mod 12, and therefore $p \equiv 17$ or 19 mod 24 (since we are assuming $p > 3$). Finally, in case (c) we require that -3 is a square mod p while -7 is not, and hence that p is a square mod 3 and a non-square modulo 7, giving $p \equiv 10$, 13 or 19 mod 21, and therefore $p \equiv 13$, 19 or 31 mod 42. ◻

Next, we give what Higman described as the 'Pythagorean Lemma'. One motivation for this choice of name is that for an element X of $\mathrm{SO}(3)$, we could define $\theta(X)$ to be $4\cos^2(\phi/2)$, where ϕ is the angle of rotation of X. Now let a, b and c be half-turns about the three co-ordinate axes, and let d be a half-turn about any unit vector $(\cos\alpha, \cos\beta, \cos\gamma)$. Then the angle of rotation of ad is twice the angle between the axes of a and d, namely 2α, and similarly the angles of rotation of bd and cd are 2β and 2γ. Thus $\theta(ad) + \theta(bd) + \theta(cd) = 4\cos^2\alpha + 4\cos^2\beta + 4\cos^2\gamma = 4$.

Lemma 4.10 (Pythagorean Lemma)**.** *Suppose a, b and c are the non-identity elements of a subgroup of $\mathrm{PSL}(2, q)$ isomorphic to the Klein 4-group. If d is any element of order 2 in $\mathrm{PSL}(2, q)$, then*

$$\theta(ad) + \theta(bd) + \theta(cd) \;=\; 4.$$

Proof. One can take a quadratic extension $\mathbb{F}_{q^2}$ of the ground field $\mathbb{F}_q$, and then in $\mathrm{PSL}(2, \mathbb{F}_{q^2})$, all copies of the Klein 4-group are conjugate, since we are assuming that q is odd. (See the classification of subgroups of $\mathrm{PSL}(2, p^f)$ in [4] for example.) Hence we may assume that our Klein 4-group in $\mathrm{PGL}(2, q)$ is generated by

$$a : z \mapsto -z, \qquad b : z \mapsto -\frac{1}{z}, \quad \text{and} \quad c : z \mapsto \frac{1}{z}.$$

Now any involution d has trace zero and hence is of the form $d : z \mapsto \dfrac{\alpha z + \beta}{\gamma z - \alpha}$.
From this we find that

$$ad : z \mapsto \frac{-\alpha z + \beta}{-\gamma z - \alpha}, \quad bd : z \mapsto \frac{-\alpha + \beta z}{-\gamma - \alpha z}, \quad \text{and} \quad cd : z \mapsto \frac{\alpha + \beta z}{\gamma - \alpha z},$$

and therefore

$$\theta(ad) + \theta(bd) + \theta(cd) = \frac{4\alpha^2}{\alpha^2 + \beta\gamma} + \frac{(\beta - \gamma)^2}{-\beta\gamma - \alpha^2} + \frac{(\beta + \gamma)^2}{\beta\gamma + \alpha^2} = 4,$$

as required. □

Lemma 4.3, the Pythagorean Lemma and Corollary 4.9 combine to give us necessary conditions for associates formed as in Section 4.2 to yield janualrials. The scope of this result is limited to k equal to 3, 4 or 6, since these are the only orders for which Lemma 4.3 applies.

Corollary 4.11. *Consider* $\Delta(2, k, p) = \langle x, y \mid x^2 = y^k = (xy)^p = 1 \rangle$ *acting on* $\mathbb{F}_p \cup \{\infty\}$ *via* $\mathrm{PSL}(2, p)$ *in such a way that* xy *is the transformation* $z \mapsto z + 1$. *Let* t *be an involution in* $\mathrm{PGL}(2, p)$ *such that* $t^{-1}xt = x^{-1}$ *and* $t^{-1}yt = y^{-1}$, *and suppose that the resulting associate coset diagram found by replacing* x *by* xt *yields a* k*-janualrial for* $\mathrm{PSL}(2, p)$ *or* $\mathrm{PGL}(2, p)$, *depending on whether or not* t *lies in* $\mathrm{PSL}(2, p)$. *Then*

(a) *if* $k = 3$, *then* $p \equiv 13$ *or* 17 *mod* 20;

(b) *if* $k = 4$, *then* $p \equiv 17$ *or* 19 *mod* 24;

(c) *if* $k = 6$, *then* $p \equiv 13, 19$ *or* 31 *mod* 42.

Proof. When $k = 3$, 4 or 6, Lemma 4.3 gives us $\theta(y) = 1, 2, 3$, respectively, and in all three cases $\theta(xy) = 4$, since xy is the transformation $z \mapsto z + 1$. Apply the Pythagorean Lemma, by taking a, b, c and d as the four involutions t, x, xt and ty respectively, to give $\theta(y) + \theta(xty) + \theta(xy) = 4$. Note that $\theta(xty) = \theta(xyt)$ because $t^{-1}(xty)t = (t^{-1}xt)yt = xyt$. Thus we find $\theta(xyt) = \theta(xty) = 4 - \theta(y) - \theta(xy) = -\theta(y)$, which equals -1, -2, or -3, respectively, when k is 3, 4, or 6. Finally, for the associate to be a janualrial we need xyt to have order $(p + 1)/2$, and so the constraints on p follow from Corollary 4.9. □

5 Examples

In this section we explore a number of examples of k-janualrials guided by Corollary 4.11.

We begin with $k = 3$. The eight smallest p satisfying the conditions of Corollary 4.11 are 13, 17, 37, 53, 73, 97, 113 and 137. We have already seen how the cases $p = 13$ and $p = 17$ yield the 3-janualrials depicted in Figure 3. The cases where p is 37, 53, 73, 97 or 137 all yield janualrials. On the other hand, the standard construction (with

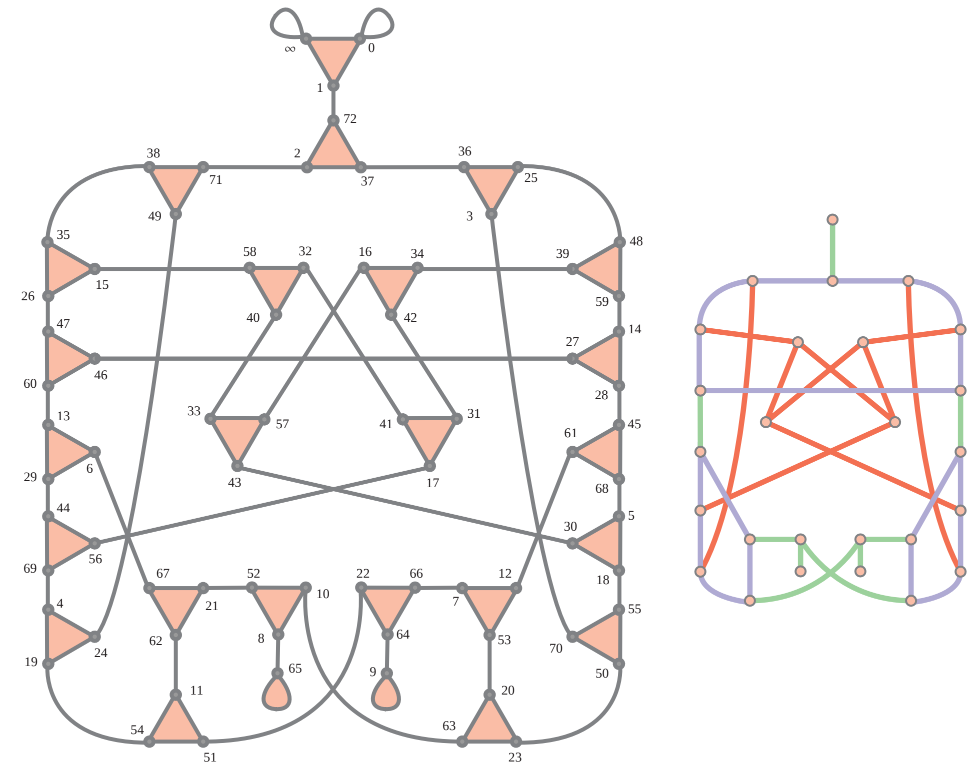

Figure 7: The associate of the standard action of $\mathrm{PSL}(2,73)$ on $\mathbb{F}_{73} \cup \{\infty\}$, giving an action of $\Delta(2,3,37)$ via $z \mapsto -z$ and $z \mapsto (z-1)/z$. This yields a 3-janiarial of genus 5 and simple type $(3,2,1)$.

$x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$) does not yield a janiarial in the case $p = 113$, since in that case $xty : z \mapsto (z+1)/z$, which has order 19, rather than $(113+1)/2 = 57$.

The action of $\mathrm{PSL}(2,73)$ on $\mathbb{F}_{73} \cup \{\infty\}$ via $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$ gives a coset graph, the associate of which is depicted together with its companion graph in Figure 7.

The number of x-edges which are not loops and y-faces in the associate are 36 and 26, respectively, so the genus is $(36-26)/2 = 5$ by Lemma 3.1. The type of the janiarial is apparent from $\overline{\Gamma}$. The blue subgraph (which is the common boundary of $\overline{S_1}$ and $\overline{S_2}$) consists of three disjoint simple closed curves, and so $h = 3$ by Lemma 3.3. There are 22 vertices on $\overline{S_1}$ and 21 on $\overline{S_2}$; there are 26 edges on $\overline{S_1}$ (coloured green and blue), and 27 edges on $\overline{S_2}$ (coloured red and blue); and both $\overline{S_1}$ and $\overline{S_2}$ have 1 face and 3 holes. Hence by Lemma 3.6, the genera of $\overline{S_1}$ and $\overline{S_2}$ are $(2-22+26-3-1)/2 = 1$ and $(2-21+27-3-1)/2 = 2$, respectively. Accordingly, the 3-janiarial is of type $(3,2,1)$, and this gives an alternative means of identifying the genus as $2+1+3-1 = 5$, by Lemma 3.4.

We now turn to 4-januarials.

For the standard construction (with $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$) to give a 4-januarial for $\mathrm{PSL}(2, p)$, Corollary 4.11 requires $p \equiv 17$ or 19 mod 24. The eight smallest values for the prime p satisfying this condition are 17, 19, 41, 43, 67, 89, 113 and 117. But also if we want xt to lie in $\mathrm{PSL}(2, p)$, we need $t : z \mapsto 1/z$ to lie in $\mathrm{PSL}(2, p)$, and then we need -1 to be a square mod p, and so $p \equiv 1$ mod 4. The cases $p = 17$, 89, 113 and 117 all give simple 4-januarials for $\mathrm{PSL}(2, p)$ in this way, while the case $p = 41$ fails, since in that case xy has order 7 rather than the required $(41 + 1)/2 = 21$.

On the other hand, if we are happy to construct 4-januarials for $\mathrm{PGL}(2, p)$ instead of $\mathrm{PSL}(2, p)$, we can relax the requirements and allow x, y or t to lie in $\mathrm{PGL}(2, p) \setminus \mathrm{PSL}(2, p)$. When we do that, we get simple 4-januarials for $\mathrm{PSL}(2, p)$ in the cases $p = 19$ and 43 (but not for $p = 67$). We consider the case $p = 43$ in more detail below.

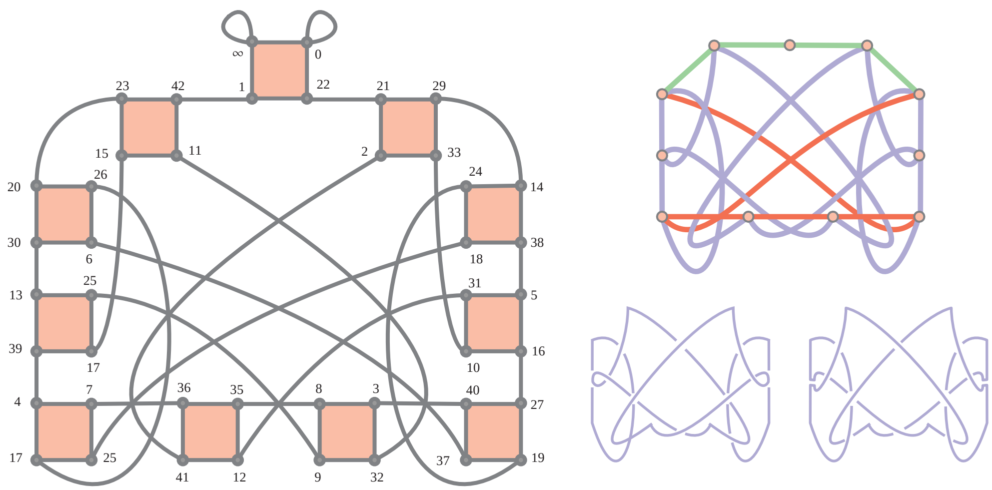

Figure 8: A coset diagram for the action of $\Delta(2, 4, 22)$ on $\mathbb{F}_{43} \cup \{\infty\}$ given by $z \mapsto 21z/22$ and $y : z \mapsto (2z-1)/2z$, its companion graph, and two partitions of the subgraph $\mathcal{G}$ into circuits. This yields a 4-januarial of genus 5 and general type $((4, 3), (2, 2))$.

Letting y be the transformation $z \mapsto (2z-1)/2z$, we can taken x as $z \mapsto 21/z$ and once more get the product xy as the (parabolic) transformation $z \mapsto z + 1$, which fixes ∞ and induces a 43-cycle on the remaining points. The generator y induces a permutation with eleven 4-cycles and no fixed points, while the generator x fixes two points (namely 8 and 35) and induces 21 transpositions on the remaining points.

We have not drawn the resulting coset graph, but note that it is reflexible, via the transformation $t : z \mapsto 22/z$. Its associate, given by the triple (xt, y, xty), shown alongside its companion graph in Figure 8, produces a 4-januarial, since xty is the transformation $z \mapsto (z + 22)/z$, which has two cycles of length 22.

The genus is $(21 - 11)/2 = 5$ by Lemma 3.1. We can use Lemma 3.5 to find h_1 and h_2.

The partition arising from the attaching map ρ_2 (following the green and blue edges) comprises $h_1 = 4$ circuits, while the partition for ρ_1 (following the red and blue edges) comprises $h_2 = 2$ circuits. Both are indicated in the figure. By Lemma 3.6, we have $2 - 2g_1 = 8 - 17 + 4 + 1$ and $2 - 2g_2 = 11 - 16 + 2 + 1$, and so $g_1 = 3$ and $g_2 = 2$.

Finally in this section, we consider 6-janualials.

In this case, Corollary 4.11 requires $p \equiv 13$, 19 or 31 mod 42. Figure 9 shows a coset graph for PGL(2, 31). Its associate, which yields a janualial J, is shown in Figure 10 together with the companion graph. By Lemma 3.1, the genus of J is $(15 - 7)/2 = 4$. Lemma 3.5 gives h_1 and h_2: as shown in the figure, we find that $h_1 = 1$ circuits comprise the partition of $\mathcal{G}$ arising from the attaching map ρ_2 (following the green and blue edges), and $h_2 = 4$ comprise the partition arising from ρ_1 (following the red and blue edges). Hence by Lemma 3.6, we find that $2 - 2g_1 = 5 - 13 + 1 + 1$ and $2 - 2g_2 = 7 - 12 + 4 + 1$, and therefore $g_1 = 4$ and $g_2 = 1$.

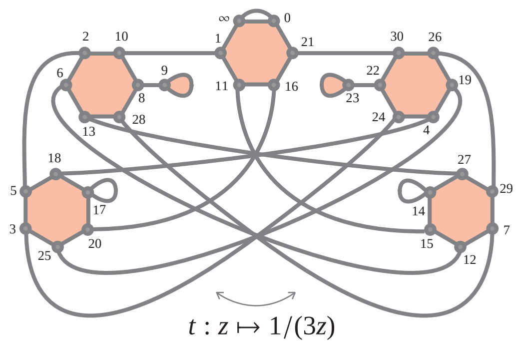

Figure 9: A coset graph for the action of $\Delta(2, 6, 31)$ on $\mathbb{F}_{31} \cup \{\infty\}$ given by $x : z \mapsto 10/z$ and $y : z \mapsto (z + 10)/z$.

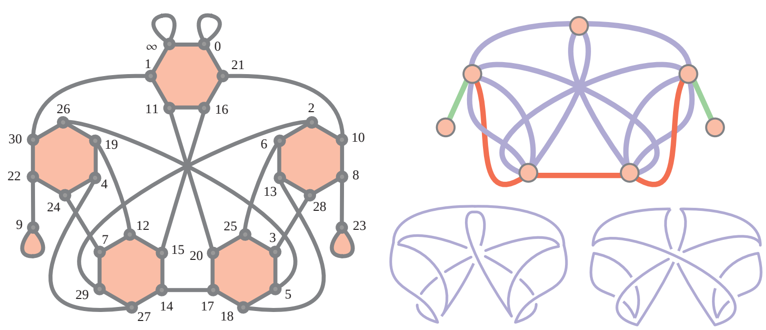

Figure 10: The associate of Figure 9, giving an action of $\Delta(2, 6, 16)$ on $\mathbb{F}_{31} \cup \{\infty\}$ via $x : z \mapsto z/30$ and $y : z \mapsto (z + 10)/z$, together with its companion graph, and two partitions of the subgraph $\mathcal{G}$ into circuits. The resulting 6-janualial is of genus 4 and general type ((1, 4), (4, 1)).

6 Afterword

6.1 Higman's portrait

Higman delivered the lectures on which this account is based in the *Higman Room* of the Mathematical Institute at Oxford University. As he spoke, he could see an image of himself looking on, from his 1984 portrait by Norman Blamey, which is reproduced below.

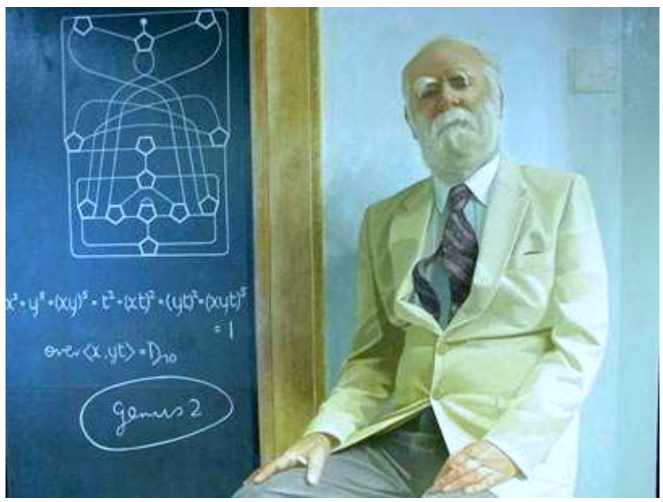

Figure 11: Norman Blamey's 1984 portrait of Graham Higman

The portrait shows Higman beside a coset graph for the action of the group PSL(2, 11) on the cosets of a dihedral subgroup of order 10 and index 66. Equivalently, it gives the natural action of PSL(2, 11) on the 66 unordered pairs of points on the projective line over a field of order 11. The two generators x and y satisfy the relations $x^2 = y^5 = (xy)^5 = 1$, but also the diagram is reflexible about a vertical axis of symmetry, and the reflection is achievable by conjugation by an involution t in the same group.

In fact x and yt may be taken as involutory generators of the stabilizer of the pair $\{0, \infty\}$, such as $z \mapsto -1/z$ and $z \mapsto 2/z$, and t as the transformation $z \mapsto (z+1)/(z-1)$. These choices make y the transformation $z \mapsto (z+2)/(2-z)$. The three generators x, y and t then satisfy the relations written on the blackboard in the portrait, namely

$$x^2 = y^5 = (xy)^5 = t^2 = (xt)^2 = (yt)^2 = (xyt)^5 = 1,$$

which are the defining relations for the group $G^{5,5,5}$ in the notation of Coxeter [2]. Hence in particular, $G^{5,5,5}$ is isomorphic to PSL(2, 11).

The diagram does not give a januarial, but rather a 13-face map. The associated surface has genus 2, since there are 13 pentagons corresponding to the 5-cycles of $\langle y \rangle$, and 28 edges between distinct pairs of such pentagons (from transpositions of x), and 13 faces coming from the 5-cycles of $\langle xy \rangle$, giving Euler characteristic $13 - 28 + 13 = -2$. The isomorphism with $G^{5,5,5}$ also makes $\mathrm{PSL}(2, 11)$ the automorphism group of a *regular map* of type $\{5, 5\}_5$ on a non-orientable surface of Euler characteristic -33 (see [3]), and hence also the automorphism group of a regular 3-polytope of type $[5, 5]$.

6.2 Other sources of januarials

Many januarials can also be constructed from groups other than $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$. For example, the alternating group $\mathrm{Alt}(16)$ is generated by elements $x = (2, 4)(3, 7)(6, 10)(8, 16)(9, 13)(11, 14)$ and $y = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)$, with product $xy = (1, 2, 5, 6, 11, 15, 13, 7)(3, 8, 16, 9, 14, 12, 10, 4)$, which has two cycles of length 8. The resulting coset diagram is shown in Figure 12.

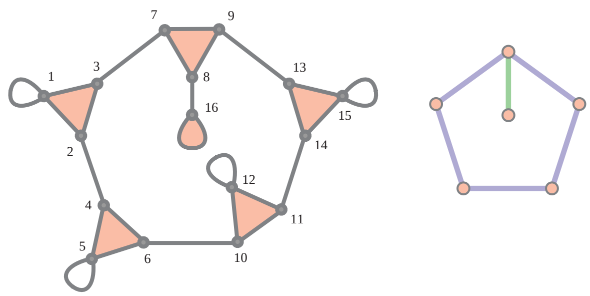

Figure 12: A coset graph for an action of $\mathrm{Alt}(16)$ of degree 16, together with its companion graph. This gives a 3-januarial of genus 0 and simple type $(1, 0, 0)$.

Other examples are obtainable from the groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ without taking the approach that we did in Section 4 which had xy as the transformation $z \mapsto z + 1$. An example is given in Figure 13.

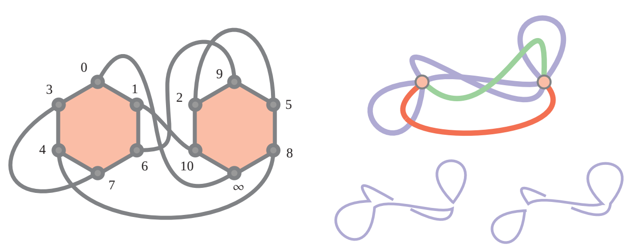

Figure 13: A coset graph for an action of $\mathrm{PSL}(2, 11)$ on $\mathbb{F}_{11} \cup \{\infty\}$, via $x : z \mapsto -1/z$ and $y : z \mapsto (8z - 8)/(z + 1)$, together with its companion graph, and two partitions of the subgraph $\mathcal{G}$. This results in a 6-januarial of genus 1 and general type $((2, 1), (2, 1))$.

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