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Hopf Bifurcations in Delayed Rock–Paper–Scissors Replicator Dynamics

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Abstract We investigate the dynamics of three-strategy (rock–paper–scissors) replicator equations in which the fitness of each strategy is a function of the population frequencies delayed by a time interval T. Taking T as a bifurcation parameter, we demonstrate the existence of (non-degenerate) Hopf bifurcations in these systems and present an analysis of the resulting limit cycles using Lindstedt's method.

Keywords Replicator · Delay · Hopf bifurcation · Limit cycle · Lindstedt

1 Introduction

The field of evolutionary dynamics uses both game theory and differential equations to model population shifts among competing adaptive strategies. There are two main approaches: population models (e.g., Lotka–Volterra) and frequency models such as the replicator equation,

$$\dot{x}_i = x_i(f_i - \phi), \qquad i = 1, \ldots, n \tag{1}$$

where x_i is the frequency or relative abundance and $f_i(x_1, \ldots, x_n)$ is the fitness of strategy i, and $\phi = \sum f_i x_i$ is the average fitness. Note that since the variables x_i represent population frequencies, we have $\sum x_i = 1$.

Hofbauer and Sigmund [3] have shown that the Lotka–Volterra equation with $n-1$ species is equivalent to the replicator equation with n strategies, but the proof requires a rescaling of time, and the correspondence between species and strategies is clearly not one to one.

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In "Appendix 1", we show that the replicator equation can be derived from the (continuous) population growth model

$$\dot{\xi}_i = \xi_i g_i, \qquad i = 1, \ldots, n \tag{2}$$

where ξ_i is the population and $g_i(\xi_1, \ldots, \xi_n)$ the fitness of strategy i. The equivalence simply uses the change of variables $x_i = \xi_i / p$ where p is the total population, with the assumption that the fitness functions depend only on the frequencies and not on the populations directly.

The game-theoretic component of the replicator model lies in the choice of fitness functions. Take the payoff matrix $A = (a_{ij})$, where a_{ij} is the expected reward for strategy i when it competes with strategy j. Then, the fitness f_i is the total expected payoff of strategy i versus all strategies, weighted by their frequency:

$$f_i = (A \cdot \mathbf{x})_i. \tag{3}$$

where

$$\mathbf{x} = (x_1, \ldots, x_n). \tag{4}$$

In this work, we generalize the replicator model to systems in which the fitness of each strategy depends only on the expected payoffs at time $t - T$, as in [4,8]. If we write $\bar{x}_i \equiv x_i(t - T)$ and define

$$\bar{\mathbf{x}} \equiv (\bar{x}_1, \ldots, \bar{x}_n) \tag{5}$$

then the total expected payoff—i.e., the fitness—for strategy i is given by

$$f_i = (A \cdot \bar{\mathbf{x}})_i. \tag{6}$$

The use of delayed fitness functions makes the replicator equation into the delay differential equation (DDE)

$$\dot{x}_i = x_i(f_i - \phi) \tag{7}$$

where

$$\phi = \sum_i x_i f_i = \sum_i x_i (A \cdot \bar{\mathbf{x}})_i. \tag{8}$$

As a system of ODEs, the standard replicator equation is an $(n-1)$-dimensional problem, since $n - 1$ of the x_i are required to specify a point in phase space, in view of the fact that $\sum x_i = 1$. The delayed replicator equation, by contrast, is an infinite-dimensional problem [1] whose solution is a flow on the space of functions on the interval $[-T, 0)$.

A concrete interpretation of this model is that it represents a *social-type* time delay [4]. There is a large, finite pool of players, each of whom uses a particular strategy at any given time. The population is well mixed, and one-on-one contests between players happen continuously. Each player continually decides whether to switch teams, based on the latest information they have about the expected payoff of each strategy. This information is delayed by an interval T.

Previous works on replicator systems with delay [4,8] have examined two-strategy systems which have a stable interior equilibrium point (i.e., both strategies coexist) when there is no delay. It has been shown that for such systems, there is a critical delay T_c at which the interior equilibrium x^* changes stability; for delay greater than T_c solutions oscillate about x^*.

In this work, we prove a similar result for RPS systems. Moreover, we use nonlinear methods to analyze the resulting limit cycles' amplitude and frequency.

2 Three-Strategy Games: Rock–Paper–Scissors

2.1 Derivation

Recall the form of the replicator equation, Eq. (7) with delayed fitness functions (8),

$$\dot{x}_i = x_i(f_i - \phi) \tag{9}$$

where $f_i = (A \cdot \bar{\mathbf{x}})_i$ and

$$\phi = \sum_i x_i f_i = \sum_i x_i (A \cdot \bar{\mathbf{x}})_i . \tag{10}$$

where the bar indicates delay.

We analyze a subset of the space of three-strategy delayed evolutionary games: those known as rock–paper–scissors (RPS) games. RPS games have three strategies, each of which is neutral versus itself and has a positive expected payoff versus one of the other strategies and a negative expected payoff versus the remaining strategy. The payoff matrix A thus has the form

$$A = \begin{pmatrix} 0 & -b_2 & a_1 \\ a_2 & 0 & -b_3 \\ -b_1 & a_3 & 0 \end{pmatrix} \tag{11}$$

where the a_i and b_i are all positive. For ease of notation, write $(x_1, x_2, x_3) = (x, y, z)$. Then

$$\dot{x} = x(a_1\bar{z} - b_2\bar{y} - \phi) \tag{12}$$
$$\dot{y} = y(a_2\bar{x} - b_3\bar{z} - \phi) \tag{13}$$
$$\dot{z} = z(a_3\bar{y} - b_1\bar{x} - \phi) \tag{14}$$

where

$$\phi = x(a_1\bar{z} - b_2\bar{y}) + y(a_2\bar{x} - b_3\bar{z}) + z(a_3\bar{y} - b_1\bar{x}). \tag{15}$$

Now, since x, y, z are the relative abundances of the three strategies, the region of interest is the three-dimensional simplex in $\mathbb{R}^3$

$$\Sigma \equiv \left\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1 \text{ and } x, y, z \geq 0\right\}. \tag{16}$$

Therefore, we can eliminate z using $z = 1 - x - y$. The region of interest is then S, the projection of Σ into the $x - y$ plane:

$$S \equiv \{(x, y) \in \mathbb{R}^2 : (x, y, 1 - x - y) \in \Sigma\} \tag{17}$$

See Fig. 1. Equations (12) and (13) become

$$\dot{x} = x(a_1(1 - \bar{x} - \bar{y}) - b_2\bar{y} - \phi) \tag{18}$$
$$\dot{y} = y(a_2\bar{x} - b_3(1 - \bar{x} - \bar{y}) - \phi) \tag{19}$$

where

$$\phi = x(a_1(1 - \bar{x} - \bar{y}) - b_2\bar{y}) + y(a_2\bar{x} - b_3(1 - \bar{x} - \bar{y})) + (1 - x - y)(a_3\bar{y} - b_1\bar{x}). \tag{20}$$

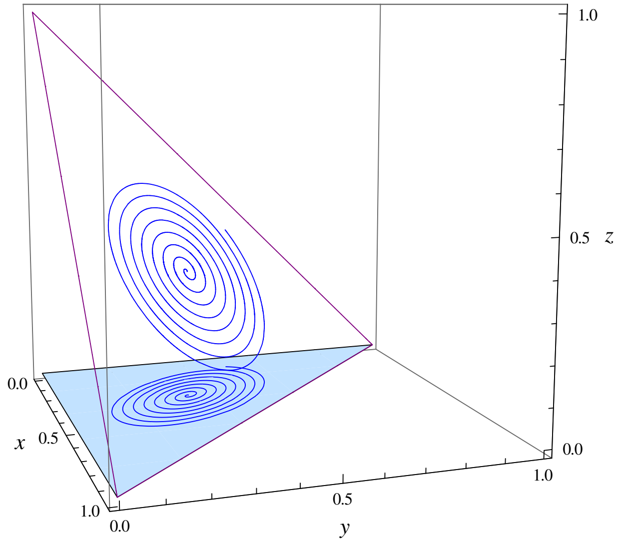

Fig. 1 A curve in Σ and its projection in S

2.2 Stability of Equilibria

The system (18)–(19) has seven equilibria: the corners of the triangle S,

$$(x, y) = (0, 0), \quad (x, y) = (0, 1), \quad (x, y) = (1, 0) \tag{21}$$

one point in the interior of S,

$$(x, y) = \left(\frac{b_3 (a_3 + b_2) + a_1 a_3}{a_1 (a_2 + a_3 + b_1) + a_2 (a_3 + b_2) + b_3 (a_3 + b_1 + b_2) + b_1 b_2}, \right. \\ \left. \frac{a_1 (a_2 + b_1) + b_1 b_3}{a_1 (a_2 + a_3 + b_1) + a_2 (a_3 + b_2) + b_3 (a_3 + b_1 + b_2) + b_1 b_2} \right) \tag{22}$$

and three other points:

$$(x, y) = \left(0, \frac{b_3}{b_3 - a_3}\right), \tag{23}$$

$$(x, y) = \left(\frac{a_1}{a_1 - b_1}, 0\right), \tag{24}$$

$$(x, y) = \left(\frac{b_2}{b_2 - a_2}, \frac{a_2}{a_2 - b_2}\right). \tag{25}$$

Note that since the payoff coefficients $a_1, \ldots, b_3$ are positive, the nonzero coordinate(s) of the last three equilibria are either negative or greater than 1. In either case, these points lie outside of S and we will not consider them further.

We linearize about the three corner equilibrium points to determine their stability. In all three cases, the linearization is independent of the delayed variables $\bar{x}$ and $\bar{y}$; that is, the linearized system about each corner point is an ordinary differential equation. Therefore, the stability of each corner point is determined by the eigenvalues of the Jacobian.

At the point $(x, y) = (0, 0)$, the eigenvalues and eigenvectors of the Jacobian are

$$\lambda_1 = a_1, \qquad \mathbf{v}_1 = [1, 0] \tag{26}$$
$$\lambda_2 = -b_3, \qquad \mathbf{v}_2 = [0, 1]. \tag{27}$$

Similarly, at the point $(x, y) = (1, 0)$, the eigenvalues and eigenvectors of the Jacobian are

$$\lambda_1 = a_2, \qquad \mathbf{v}_1 = [-1, 1] \tag{28}$$
$$\lambda_2 = -b_1, \qquad \mathbf{v}_2 = [1, 0]. \tag{29}$$

Finally, at the point $(x, y) = (0, 1)$, the eigenvalues and eigenvectors of the Jacobian are

$$\lambda_1 = a_3, \qquad \mathbf{v}_1 = [0, 1] \tag{30}$$
$$\lambda_2 = -b_2, \qquad \mathbf{v}_2 = [-1, 1]. \tag{31}$$

Therefore, as in the non-delayed RPS system [1], each corner of S is a saddle point, and its eigenvectors lie along the two edges of S adjacent to it. (Since the lines containing the edges of S are invariant, these lines are in fact the stable and unstable manifolds of the three corner equilibria.)

Next, consider the interior equilibrium (22). Let (x^*, y^*) be the coordinates of the equilibrium point. It is known [5] that in the case of no delay ($T = 0$), this point is globally stable if

$$\det A = a_1a_2a_3 - b_1b_2b_3 > 0. \tag{32}$$

All trajectories starting from interior points of S converge to (x^*, y^*). Similarly, if $T = 0$ and $\det A < 0$, the equilibrium point is unstable and all trajectories starting from other points converge to the boundary of S. If $T = 0$ and $\det A = 0$, then S is filled with periodic orbits.

If $T > 0$, however, then in contrast to the corner equilibria, the linearization about (x^*, y^*) depends only on the *delayed* variables, and it is reasonable to expect that its stability will depend on the delay T. So, we analyze the system for a Hopf bifurcation, taking T as the bifurcation parameter.

Define the translated variables u and v via

$$u = x - x^*, \quad v = y - y^*. \tag{33}$$

Then, the linearization about $(u, v) = (0, 0)$ is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \equiv J \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \tag{34}$$

where the entries $(\alpha, \beta, \gamma, \delta)$ of the matrix J are rational functions of the payoff coefficients $a_1, \ldots, b_3$. See Eqs. (124)–(127) in "Appendix 2".

Set $u = re^{\lambda t}$ and $v = se^{\lambda t}$ to obtain the characteristic equations

$$\lambda r = e^{-\lambda T}(\alpha r + \beta s) \tag{35}$$
$$\lambda s = e^{-\lambda T}(\gamma r + \delta s). \tag{36}$$

Rearranging, we obtain

$$\begin{pmatrix} \lambda - \alpha e^{-\lambda T} & -\beta e^{-\lambda T} \\ -\gamma e^{-\lambda T} & \lambda - \delta e^{\lambda T} \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{37}$$

For brevity, write

$$M \equiv \begin{pmatrix} \lambda - \alpha e^{-\lambda T} & -\beta e^{-\lambda T} \\ -\gamma e^{-\lambda T} & \lambda - \delta e^{\lambda T} \end{pmatrix}. \tag{38}$$

Then, for a non-trivial solution to Eq. (37), we require

$$\det M = 0. \tag{39}$$

This occurs when

$$\beta\gamma = \left(\alpha - \lambda e^{\lambda T}\right)\left(\delta - \lambda e^{\lambda T}\right). \tag{40}$$

At the critical value of delay for a Hopf bifurcation, the eigenvalues are pure imaginary. So, we set $T = T_0$ and $\lambda = i\omega_0$. Substituting this into Eq. (40) and separating the real and imaginary parts, we obtain

$$\beta\gamma = -\alpha\delta - \omega_0^2 \cos(2\omega_0 T_0) + (\alpha + \delta)\sin(\omega_0 T_0) \tag{41}$$

$$0 = \omega_0 \cos(\omega_0 T_0)(\alpha + \delta + 2\omega_0 \sin(\omega_0 T_0)). \tag{42}$$

In terms of the matrix J, these equations are

$$\det J = \omega_0^2 \cos(2\omega_0 T_0) - (\operatorname{tr} J)\sin(\omega_0 T_0) \tag{43}$$

$$0 = \omega_0 \cos(\omega_0 T_0)(\operatorname{tr} J + 2\omega_0 \sin(\omega_0 T_0)). \tag{44}$$

Solving these equations for $\det J$ and $\operatorname{tr} J$, we get

$$\det J = \omega_0^2, \quad \operatorname{tr} J = -2\omega_0 \sin(\omega_0 T_0). \tag{45}$$

Thus, ω_0 and T_0 are given by

$$\omega_0 = \sqrt{\det J}, \quad T_0 = \frac{-1}{\sqrt{\det J}} \sin^{-1}\left(\frac{\operatorname{tr} J}{2\sqrt{\det J}}\right). \tag{46}$$

We have found the critical delay and frequency associated with a Hopf bifurcation. In the next subsection, we use Lindstedt's method to approximate the form of the limit cycle that is born in this bifurcation.

2.3 Approximation of Limit Cycle

Recall that we have the system

$$\dot{x} = x(a_1(1 - \bar{x} - \bar{y}) - b_2\bar{y} - \phi) \tag{47}$$

$$\dot{y} = y(a_2\bar{x} - b_3(1 - \bar{x} - \bar{y}) - \phi) \tag{48}$$

where

$$\phi = x(a_1(1 - \bar{x} - \bar{y}) - b_2\bar{y}) + y(a_2\bar{x} - b_3(1 - \bar{x} - \bar{y})) + (1 - x - y)(a_3\bar{y} - b_1\bar{x}) \tag{49}$$

with the interior equilibrium point

$$(x^*, y^*) = \left(\frac{b_3(a_3 + b_2) + a_1 a_3}{a_1(a_2 + a_3 + b_1) + a_2(a_3 + b_2) + b_3(a_3 + b_1 + b_2) + b_1 b_2}, \frac{a_1(a_2 + b_1) + b_1 b_3}{a_1(a_2 + a_3 + b_1) + a_2(a_3 + b_2) + b_3(a_3 + b_1 + b_2) + b_1 b_2}\right). \tag{50}$$

We have introduced the translated coordinates u and v, defined by

$$u = x - x^*, \quad v = y - y^* \tag{51}$$

and we have determined in Eq. (46) the critical delay T_0 and frequency ω_0 associated with a Hopf bifurcation of the point $(u, v) = (0, 0)$.

Substituting in u and v, the system (47)–(48) can be written as

$$\begin{aligned}\dot{u} &= \alpha\bar{u} + \beta\bar{v} + c_1 u\bar{u} + c_2 u\bar{v} + c_3 v\bar{u} + c_4 v\bar{v} \\ &\quad + d_1 u^2\bar{u} + d_2 u^2\bar{v} + d_3 uv\bar{u} + d_4 uv\bar{v}\end{aligned} \tag{52}$$

$$\begin{aligned}\dot{v} &= \gamma\bar{u} + \delta\bar{v} + h_1 u\bar{u} + h_2 u\bar{v} + h_3 v\bar{u} + h_4 v\bar{v} \\ &\quad + j_1 v^2\bar{u} + j_2 v^2\bar{v} + j_3 uv\bar{u} + j_4 uv\bar{v}\end{aligned} \tag{53}$$

where $\alpha, \beta, \gamma, \delta$ are as in the linearization Eq. (34). The other coefficients $c_1, \ldots, j_4$ are also rational functions of the payoff coefficients $a_1, \ldots, b_3$; see Eqs. (129)–(144) in "Appendix 2".

Now we use Lindstedt's method to approximate the form of the limit cycle generated by this bifurcation.

We are looking for periodic solutions with delay close to T_0 and frequency close to ω_0. First, we rescale time via $\tau = \omega t$, so

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \omega\frac{\mathrm{d}u}{\mathrm{d}\tau} \equiv \omega u' \tag{54}$$

$$\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \omega\frac{\mathrm{d}v}{\mathrm{d}\tau} \equiv \omega v' \tag{55}$$

and, considering u and v to be functions of τ,

$$\bar{u} = u(\tau - \omega T), \quad \bar{v} = v(\tau - \omega T). \tag{56}$$

Next, expand the delay and frequency in ϵ:

$$T = T_0 + \epsilon^2\mu_1 + \epsilon^3\mu_2 \tag{57}$$

$$\omega = \omega_0 + \epsilon^2 k_1 + \epsilon^3 k_2 \tag{58}$$

Note that there is no $O(\epsilon^1)$ term in T or ω because of the presence of quadratic terms in Eqs. (52) and (53). Removal of secular terms at the appropriate order of ϵ will require any $O(\epsilon^1)$ terms in Eqs. (57) and (58) to vanish.

We expand the functions u and v similarly:

$$u = \epsilon u_0 + \epsilon^2 u_1 + \epsilon^3 u_2 \tag{59}$$

$$v = \epsilon v_0 + \epsilon^2 v_1 + \epsilon^3 v_2. \tag{60}$$

Then, we substitute the expanded functions and parameters into Eqs. (52) and (53) and collect like orders of ϵ. This includes expanding $\bar{u}$ and $\bar{v}$ in Taylor series:

$$\begin{aligned}\bar{u} &= u(\tau - \omega T) \\ &= \epsilon u_0(\tau - \omega_0 T_0) + \epsilon^2 u_1(\tau - \omega_0 T_0) \\ &\quad + \epsilon^3\left(u_2(\tau - \omega_0 T_0) - (T_0 k_1 + \omega_0\mu_1)u_0'(\tau - \omega_0 T_0)\right) + \ldots\end{aligned} \tag{61}$$

$$\begin{aligned}\bar{v} &= v(\tau - \omega T)\\ &= \epsilon v_0(\tau - \omega_0 T_0) + \epsilon^2 v_1(\tau - \omega_0 T_0)\\ &\quad + \epsilon^3 \left(v_2(\tau - \omega_0 T_0) - (T_0 k_1 + \omega_0 \mu_1) v_0'(\tau - \omega_0 T_0)\right) + \ldots \end{aligned} \tag{62}$$

Since the only remaining delayed terms are of the form $u(\tau - \omega_0 T_0)$ or $v(\tau - \omega_0 T_0)$, we introduce the notation

$$\tilde{u} \equiv u(\tau - \omega_0 T_0), \qquad \tilde{v} \equiv v(\tau - \omega_0 T_0). \tag{63}$$

The resulting equations are

$$O(\epsilon^1): \quad \omega_0 u_0' = \alpha \tilde{u}_0 + \beta \tilde{v}_0 \tag{64}$$

$$\omega_0 v_0' = \gamma \tilde{u}_0 + \delta \tilde{v}_0 \tag{65}$$

$$O(\epsilon^2): \quad \omega_0 u_1' = \alpha \tilde{u}_1 + \beta \tilde{v}_1 + \tilde{u}_0(c_1 u_0 + c_3 v_0) + \tilde{v}_0(c_2 u_0 + c_4 v_0) \tag{66}$$

$$\omega_0 v_1' = \gamma \tilde{u}_1 + \delta \tilde{v}_1 + \tilde{u}_0(h_1 u_0 + h_3 v_0) + \tilde{v}_0(h_2 u_0 + h_4 v_0) \tag{67}$$

$$\begin{aligned} O(\epsilon^3): \quad \omega_0 u_2' &= \alpha \tilde{u}_2 + \beta \tilde{v}_2 + \tilde{u}_1(c_1 u_0 + c_3 v_0) + \tilde{v}_1(c_2 u_0 + c_4 v_0)\\ &\quad + \tilde{u}_0(c_1 u_1 + c_3 v_1 + d_1 u_0^2 + d_3 u_0 v_0)\\ &\quad + \tilde{v}_0(c_2 u_1 + c_4 v_1 + d_2 u_0^2 + d_4 u_0 v_0)\\ &\quad - k_1 u_0' - \alpha(T_0 k_1 + \omega_0 \mu_1)\tilde{u}_0' - \beta(T_0 k_1 + \omega_0 \mu_1)\tilde{v}_0' \end{aligned} \tag{68}$$

$$\begin{aligned} \omega_0 v_2' &= \gamma \tilde{u}_2 + \delta \tilde{v}_2 + \tilde{u}_1(h_1 u_0 + h_3 v_0) + \tilde{v}_1(h_2 u_0 + h_4 v_0)\\ &\quad + \tilde{u}_0(h_1 u_1 + h_3 v_1 + j_1 v_0^2 + j_3 u_0 v_0)\\ &\quad + \tilde{v}_0(h_2 u_1 + h_4 v_1 + j_2 v_0^2 + j_4 u_0 v_0)\\ &\quad - k_1 v_0' - \gamma(T_0 k_1 + \omega_0 \mu_1)\tilde{u}_0' - \delta(T_0 k_1 + \omega_0 \mu_1)\tilde{v}_0'. \end{aligned} \tag{69}$$

We must solve the equations for each order of ϵ successively, substituting in the results from the lower-order equations as we proceed.

2.3.1 *Solve for u_0 and v_0*

As seen above, the ϵ^1 equations are linear:

$$\omega_0 u_0' = \alpha \tilde{u}_0 + \beta \tilde{v}_0 \tag{64}$$

$$\omega_0 v_0' = \gamma \tilde{u}_0 + \delta \tilde{v}_0. \tag{65}$$

Up to a phase shift, the solution has the form

$$u_0 = A_0 \sin \tau \tag{70}$$

$$v_0 = A_0(r \sin \tau + s \cos \tau) \tag{71}$$

for some constants r and s. We substitute these solutions into Eqs. (64) and (65) and use the angle-sum identities to obtain

$$\begin{aligned} \omega_0 \cos \tau &= (s\beta \cos(\omega_0 T_0) - (\alpha + r\beta)\sin(\omega_0 T_0)) \cos \tau\\ &\quad + (s\beta \sin(\omega_0 T_0) + (\alpha + r\beta)\cos(\omega_0 T_0)) \sin \tau \end{aligned} \tag{72}$$

$$\begin{aligned} \omega_0(r \cos \tau - s \sin \tau) &= (s\delta \cos(\omega_0 T_0) - (\gamma + r\delta)\sin(\omega_0 T_0)) \cos \tau\\ &\quad + (s\delta \sin(\omega_0 T_0) + (\gamma + r\delta)\cos(\omega_0 T_0)) \sin \tau. \end{aligned} \tag{73}$$

Setting the coefficients of $\cos\tau$ and $\sin\tau$ equal to 0 in both equations gives us

$$r = \frac{\delta - \alpha}{2\beta}, \quad s = \frac{\sqrt{-4\beta\gamma - (\alpha - \delta)^2}}{2\beta}. \tag{74}$$

Thus,

$$u_0 = A_0 \sin\tau \tag{75}$$

$$v_0 = A_0 \frac{1}{2\beta}\left((\delta - \alpha)\sin\tau + \sqrt{-4\beta\gamma - (\alpha - \delta)^2}\cos\tau\right). \tag{76}$$

(Note that the coefficient of $\cos\tau$ above is real for the values of $\alpha, \beta, \gamma, \delta$ given in "Appendix 2".)

2.3.2 Solve for u_1 and v_1

Next we solve for u_1 and v_1 using the solutions for u_0 and v_0 above. Recall that they satisfy the equations

$$\omega_0 u_1' = \alpha\tilde{u}_1 + \beta\tilde{v}_1 + \tilde{u}_0(c_1 u_0 + c_3 v_0) + \tilde{v}_0(c_2 u_0 + c_4 v_0) \tag{66}$$

$$\omega_0 v_1' = \gamma\tilde{u}_1 + \delta\tilde{v}_1 + \tilde{u}_0(h_1 u_0 + h_3 v_0) + \tilde{v}_0(h_2 u_0 + h_4 v_0). \tag{67}$$

Using Eqs. (75) and (76), and the values of the various coefficients given in "Appendix 2", these become

$$\omega_0 u_1' = \alpha\tilde{u}_1 + \beta\tilde{v}_1 + A_0^2\left(B_1 \sin 2\tau + B_2 \cos 2\tau\right) \tag{77}$$

$$\omega_0 v_1' = \gamma\tilde{u}_1 + \delta\tilde{v}_1 + A_0^2\left(B_3 \sin 2\tau + B_4 \cos 2\tau\right). \tag{78}$$

The constant coefficients $B_1, \ldots, B_4$ are given in Eqs. (145)–(148) in "Appendix 2". Note that there are no resonant terms to eliminate, and the homogeneous solutions are unnecessary because they will have the same form as u_0 and v_0. Thus, we expect solutions of the form

$$u_1 = A_0^2(r_1 \sin 2\tau + s_1 \cos 2\tau) \tag{79}$$

$$v_1 = A_0^2(r_2 \sin 2\tau + s_2 \cos 2\tau). \tag{80}$$

Substituting into Eqs. (77)–(78) gives

$$\begin{aligned}&[B_2 - \sin(2T_0\omega_0)(\alpha r_1 + \beta r_2) - 2r_1\omega_0 + \cos(2T_0\omega_0)(\alpha s_1 + \beta s_2)]\cos 2\tau \\ &\quad + [B_1 + \cos(2T_0\omega_0)(\alpha r_1 + \beta r_2) + \sin(2T_0\omega_0)(\alpha s_1 + \beta s_2) + 2s_1\omega_0]\sin 2\tau = 0\end{aligned} \tag{81}$$

$$\begin{aligned}&\left[B_4 - \sin(2T_0\omega_0)(\gamma r_1 + \delta r_2) - 2r_2\omega_0 + \cos(2T_0\omega_0)(\gamma s_1 + \delta s_2)\right]\cos 2\tau \\ &\quad + \left[B_3 + \cos(2T_0\omega_0)(\gamma r_1 + \delta r_2) + \sin(2T_0\omega_0)(\gamma s_1 + \delta s_2) + 2s_2\omega_0\right]\sin 2\tau = 0.\end{aligned} \tag{82}$$

We set the coefficients of $\sin 2\tau$ and $\cos 2\tau$ equal to 0. This gives four linear equations in r_1, r_2, s_1, and s_2, which can be solved easily:

$$\begin{pmatrix} r_1 \\ r_2 \\ s_1 \\ s_2 \end{pmatrix} = C^{-1}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \tag{83}$$

where

$$C = \begin{pmatrix} \alpha\cos & \beta\cos & 2\omega_0 + \alpha\sin & \beta\sin \\ -2\omega_0 - \alpha\sin & -\beta\sin & \alpha\cos & \beta\cos \\ \gamma\cos & \delta\cos & \gamma\sin & 2\omega_0 + \delta\sin \\ -\gamma\sin & -2\omega_0 - \delta\sin & \gamma\cos & \delta\cos \end{pmatrix} \tag{84}$$

where the argument of each sin and cos is $2\omega_0 T_0$. However, the expressions for $r_1, \ldots, s_2$ are cumbersome and are omitted here for brevity.

2.3.3 Use the u_2 and v_2 Equations to Find A_0 and k_1 in Terms of μ_1

As in the previous steps, we substitute the solutions found above for u_0, v_0, u_1 and v_1 into the equations satisfied by u_2 and v_2. Recall that

$$\begin{aligned} \omega_0 u_2' = {} & \alpha\tilde{u}_2 + \beta\tilde{v}_2 + \tilde{u}_1(c_1u_0 + c_3v_0) + \tilde{v}_1(c_2u_0 + c_4v_0) \\ & + \tilde{u}_0(c_1u_1 + c_3v_1 + d_1u_0^2 + d_3u_0v_0) \\ & + \tilde{v}_0(c_2u_1 + c_4v_1 + d_2u_0^2 + d_4u_0v_0) \\ & - k_1u_0' - \alpha(T_0k_1 + \omega_0\mu_1)\tilde{u}_0' - \beta(T_0k_1 + \omega_0\mu_1)\tilde{v}_0' \end{aligned} \tag{68}$$

$$\begin{aligned} \omega_0 v_2' = {} & \gamma\tilde{u}_2 + \delta\tilde{v}_2 + \tilde{u}_1(h_1u_0 + h_3v_0) + \tilde{v}_1(h_2u_0 + h_4v_0) \\ & + \tilde{u}_0(h_1u_1 + h_3v_1 + j_1v_0^2 + j_3u_0v_0) \\ & + \tilde{v}_0(h_2u_1 + h_4v_1 + j_2v_0^2 + j_4u_0v_0) \\ & - k_1v_0' - \gamma(T_0k_1 + \omega_0\mu_1)\tilde{u}_0' - \delta(T_0k_1 + \omega_0\mu_1)\tilde{v}_0'. \end{aligned} \tag{69}$$

Using Eqs. (75), (76), (79) and (80), these become

$$\omega_0 u_2' = \alpha\tilde{u}_2 + \beta\tilde{v}_2 + K_1\cos\tau + K_2\sin\tau + L_1\cos 3\tau + L_2\sin 3\tau \tag{85}$$

$$\omega_0 v_2' = \gamma\tilde{u}_2 + \delta\tilde{v}_2 + K_3\cos\tau + K_4\sin\tau + L_3\cos 3\tau + L_4\sin 3\tau. \tag{86}$$

The coefficients $K_1, \ldots, L_4$ are omitted for brevity.

The $\sin 3\tau$ and $\cos 3\tau$ terms are non-resonant, so the L_i will not give any information about A_0 or k_1. The $\sin\tau$ and $\cos\tau$ terms are resonant, so we use the method detailed in "Appendix 3" to eliminate secular terms. The existence of a periodic solution to Eqs. (85) and (86) requires

$$K_3 = \frac{K_1(\delta - \alpha) - K_2\sqrt{-(\alpha-\delta)^2 - 4\beta\gamma}}{2\beta} \tag{87}$$

$$K_4 = \frac{K_1\sqrt{-(\alpha-\delta)^2 - 4\beta\gamma} + K_2(\delta - \alpha)}{2\beta}. \tag{88}$$

We find that the K_i have the form

$$K_i = A_0(q_{i1}A_0^2 + q_{i2}k_1 + q_{i3}\mu_1). \tag{89}$$

Substituting (89) into Eqs. (87) and (88) gives two simultaneous equations on A_0, k_1 and μ_1. We solve these for A_0 and k_1 in terms of μ_1.

As expected, A_0 is proportional to $\sqrt{\mu_1}$. If the proportionality constant is real, the limit cycle exists for $\mu_1 > 0$, and its stability is the same as that of the interior equilibrium (x^*, y^*) when $T = 0$.

2.4 Example

Consider the RPS system

$$\dot{x}_i = x_i(f_i - \phi) \tag{90}$$

where $f_i = (A \cdot \bar{\mathbf{x}})_i$ and

$$\phi = \sum_i x_i f_i = \sum_i x_i (A \cdot \bar{\mathbf{x}})_i \tag{91}$$

with

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \tag{92}$$

Following Sect. 2.2, we see that in this case, det $A = 1$, so the interior equilibrium point $(x^*, y^*) = (\frac{1}{3}, \frac{5}{12})$ is stable when $T = 0$. The critical delay and frequency are

$$\omega_0 = \frac{1}{2}\sqrt{\frac{5}{3}} \approx 0.64550, \qquad T_0 = 2\sqrt{\frac{3}{5}} \sin^{-1}\left(\frac{1}{4\sqrt{15}}\right) \approx 0.10007. \tag{93}$$

Using the method of Sects. 2.3.1 and 2.3.2, we find that

$$u_0 = A_0 \sin\tau \tag{94}$$

$$v_0 = A_0(-0.671875 \sin\tau - 0.72467 \cos\tau) \tag{95}$$

and

$$u_1 = A_0^2(0.235279 \sin 2\tau - 0.430682 \cos 2\tau) \tag{96}$$

$$v_1 = A_0^2(0.203199 \sin 2\tau - 0.0397297 \cos 2\tau). \tag{97}$$

Then, as in Sect. 2.3.3,

$$\omega_0 u_2' = \alpha\tilde{u}_2 + \beta\tilde{v}_2 + K_1 \cos\tau + K_2 \sin\tau + L_1 \cos 3\tau + L_2 \sin 3\tau \tag{98}$$

$$\omega_0 v_2' = \gamma\tilde{u}_2 + \delta\tilde{v}_2 + K_3 \cos\tau + K_4 \sin\tau + L_3 \cos 3\tau + L_4 \sin 3\tau. \tag{99}$$

where

$$\alpha = -\frac{23}{36}, \quad \beta = -\frac{8}{9}, \quad \gamma = \frac{125}{144}, \quad \delta = \frac{5}{9} \tag{100}$$

and

$$K_1 = A_0^2(-0.957018 A_0^2 - k_1) \tag{101}$$

$$K_2 = A_0^2(-0.146492 A_0^2 + 0.0645946 k_1 + 0.416667 \mu_1) \tag{102}$$

$$K_3 = A_0^2(0.573076 A_0^2 + 0.625065 k_1 - 0.301946 \mu_1) \tag{103}$$

$$K_4 = A_0^2(-0.472711 A_0^2 - 0.768069 k_1 - 0.279948 \mu_1). \tag{104}$$

Therefore, using Eqs. (87) and (88), the condition to eliminate secular terms is

$$A_0 = 2.26293\sqrt{\mu_1}, \qquad k_1 = -4.46834\mu_1. \tag{105}$$

This means that the limit cycle exists when $\mu_1 > 0$, so the bifurcation is supercritical and the limit cycle is stable (Fig. 2).

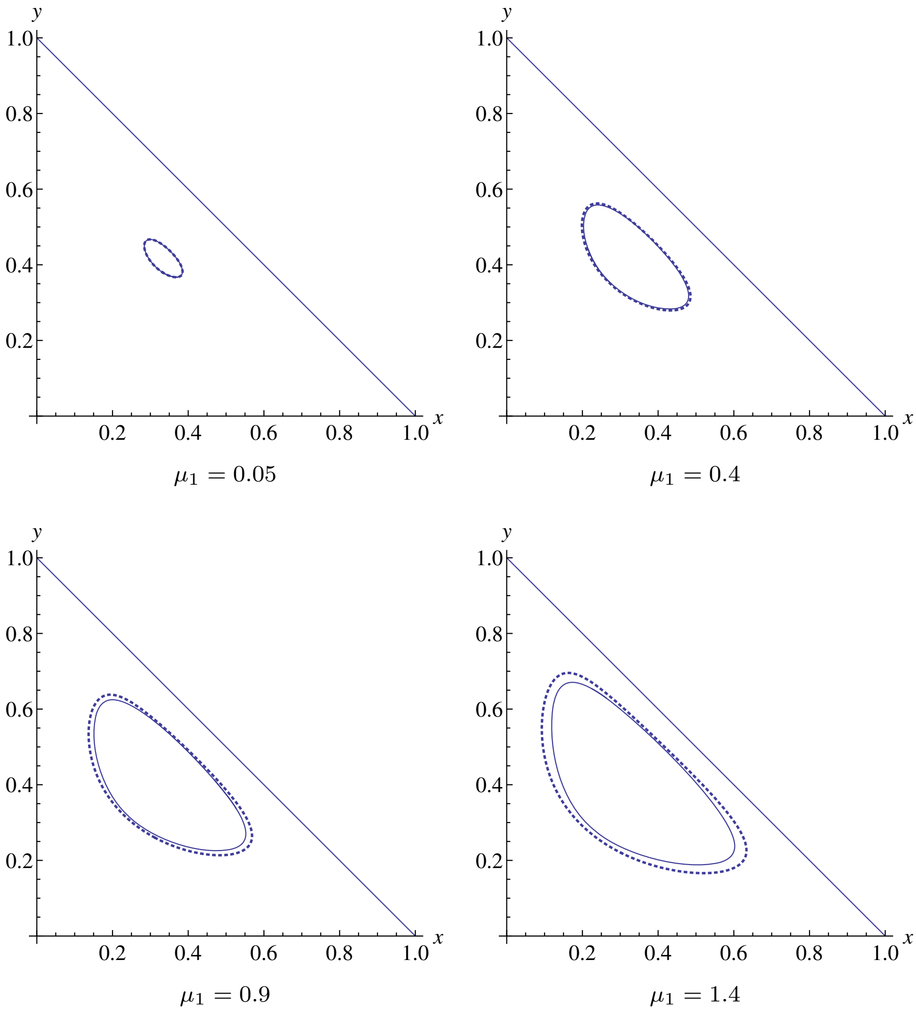

Fig. 2 Limit cycle given by Lindstedt (*dotted*) and numerical integration (*solid*) for $\epsilon = 0.1$ and varying values of μ_1. Recall that $T = T_0 + \epsilon^2 \mu_1$

To evaluate the results of Lindstedt's method qualitatively, we compute the average radius of the limit cycle (i.e., the radius of the circle with the same enclosed area). For the limit cycle predicted by Lindstedt's method, this is simply

$$r_{\text{Lind}} = \left[\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left(u(t)^2 + v(t)^2 \right) \mathrm{d}t \right]^{1/2} \tag{106}$$

where u and v are as in Eqs. (94)–(97). Recall that $\tau = \omega t$ where $\omega = \omega_0 + \epsilon^2 k_1$, where ω_0 is given by Eq. (93) and k_1 by Eq. (105).

We compare this to the average radius of the approximate limit cycle given by numerical integration. To find this, we integrate the original system given in Eqs. (90)–(92), using NDSolve in Mathematica. This is a versatile method that can handle ordinary, partial or delay

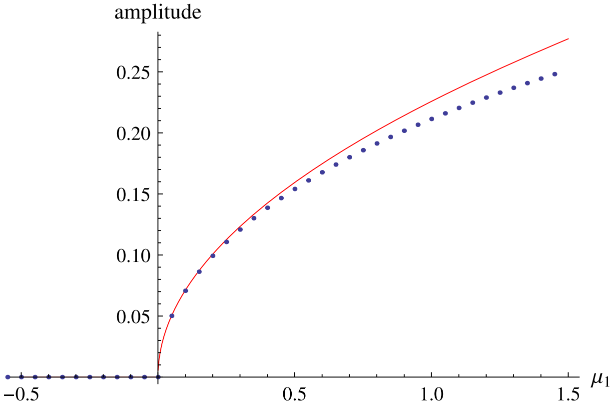

Fig. 3 Average radius of the limit cycle given by Lindstedt (*solid*) and numerical integration (*dotted*) for $\epsilon = 0.1$ as a function of μ_1

differential equations, and which adaptively chooses from among several solving routines. For 40 values of μ_1 between -0.5 and 1.5, we integrated the system up to $t = 3{,}000$, with the assumption that the solutions were constant for $t < 0$. We found that for $t > 2{,}900$, the numerical solutions were nearly periodic: in all cases tested, the peak-to-peak times of the first cycle after $t = 2{,}900$ and the last cycle before $t = 3{,}000$ differed by less than one part in 10^{-7}. This gave the desired approximation to the limit cycle.

Thus, the average radius for the numerical limit cycle is

$$r_{\text{numer}} = \left[\frac{1}{p(\mu_1)} \int_{t_0}^{t_0+p(\mu_1)} \left((x(t) - x^*)^2 + (y(t) - y^*)^2 \right) \mathrm{d}t \right]^{1/2} \tag{107}$$

where $p(\mu_1)$ is the period of the limit cycle, obtained using FindRoot in Mathematica, and t_0 is chosen large enough that the numerical solutions are close to the limit cycle.

We compare the average radius given by Lindstedt's method to that found by numerical integration and observe from Fig. 3 that the two methods are in relatively good agreement for small μ_1.

3 Conclusion

We have investigated the dynamics of rock–paper–scissors systems of the form

$$\dot{x}_i = x_i(f_i - \phi), \tag{108}$$

where $f_i = (A \cdot \bar{\mathbf{x}})_i$ is the (delayed) fitness of strategy i.

It is known that limit cycles cannot occur in non-delayed rock–paper–scissors systems; the phase space is filled with either decreasing, increasing or neutral oscillations, depending on the determinant of the payoff matrix A.

In this work, we have shown using nonlinear methods that, by introducing a social-type delay in the fitnesses of the strategies, it is possible to find rock–paper-scissors systems which exhibit non-degenerate Hopf bifurcations and limit cycles. We have analyzed the resulting limit cycles using Lindstedt's method, finding an approximation of their frequency and amplitude. We have demonstrated a choice of parameters for which a rock–paper–scissors system undergoes a supercritical Hopf bifurcation and exhibits a stable limit cycle. For this choice of parameters, the prediction of Lindstedt's method is found to agree with numerical integration for T close to T_0.

This generalization of the replicator model may be useful in modeling natural or social systems in which each player has a delayed estimate of the expected payoff of each strategy.

Appendix 1: Derivation of replicator equation

Consider an exponential model of population growth,

$$\dot{\xi}_i = \xi_i g_i \quad (i = 1, \ldots, n) \tag{109}$$

where ξ_i is a real-valued function that approximates the population of strategy i and $g_i(\xi_1, \ldots, \xi_n)$ is the fitness of that strategy. The replicator Eq. [7] results from Eq. (109) by changing variables from the populations ξ_i to the relative abundances, defined as $x_i \equiv \xi_i / p$ where p is the total population:

$$p(t) = \sum_i \xi_i(t). \tag{110}$$

We see that

$$\dot{p} = \sum_i \dot{\xi}_i = \sum_i \xi_i g_i \tag{111}$$

$$= p \sum_i \frac{\xi_i}{p} g_i = p \sum_i x_i g_i \tag{112}$$

$$= p\phi \tag{113}$$

where $\phi \equiv \sum_i x_i g_i$ is the average fitness of the whole population.

By the product rule,

$$\dot{x}_i = \frac{\dot{\xi}_i}{p} - \frac{\xi_i \dot{p}}{p^2} \tag{114}$$

$$= \frac{\xi_i}{p} g_i - \frac{\xi_i}{p} \frac{\dot{p}}{p} \tag{115}$$

$$= x_i (g_i - \phi). \tag{116}$$

Therefore,

$$\sum_i \dot{x}_i = \sum_i x_i g_i - \phi \sum_i x_i \tag{117}$$

$$= \sum_i x_i g_i - \sum_j x_j g_j \sum_i x_i. \tag{118}$$

So, using the fact that

$$\sum_i x_i = \frac{\sum_i \xi_i}{p} = \frac{p}{p} \equiv 1 \tag{119}$$

Equation (118) reduces to the identity

$$\sum_i \dot{x}_i = 0. \tag{120}$$

The fitness of a strategy is assumed to depend only on the relative abundance of each strategy in the overall population, since the model only seeks to capture the effect of competition between strategies, not any environmental or other factors. Therefore, we assume that g_i has the form

$$g_i(\xi_1, \ldots, \xi_n) = f_i\left(\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\right) = f_i(x_1, \ldots, x_n). \tag{121}$$

Under this assumption, Eq. (116) is the replicator equation,

$$\dot{x}_i = x_i(f_i - \phi), \tag{122}$$

where ϕ is now expressed entirely in terms of the x_i, as

$$\phi = \sum_i x_i f_i. \tag{123}$$

Mathematically, ϕ is a coupling term that introduces dependence on the abundance and fitness of other strategies.

Appendix 2: Coefficients generated in the RPS problem

The entries of the matrix J from Eq. (34) are

$$\alpha = x^* \left((a_1 - b_1)(x^* - 1) - (a_2 + b_1 + b_3)y^*\right) \tag{124}$$

$$\beta = x^* \left((a_1 + a_3 + b_2)(x^* - 1) + (a_3 - b_3)y^*\right) \tag{125}$$

$$\gamma = y^* \left((a_1 - b_1)x^* - (a_2 + b_1 + b_3)(y^* - 1)\right) \tag{126}$$

$$\delta = y^* \left((a_1 + a_3 + b_2)x^* + (a_3 - b_3)(y^* - 1)\right) \tag{127}$$

where x^* and y^* are the coordinates of the interior equilibrium point,

$$(x^*, y^*) = \left(\frac{b_3(a_3 + b_2) + a_1 a_3}{a_1(a_2 + a_3 + b_1) + a_2(a_3 + b_2) + b_3(a_3 + b_1 + b_2) + b_1 b_2}, \frac{a_1(a_2 + b_1) + b_1 b_3}{a_1(a_2 + a_3 + b_1) + a_2(a_3 + b_2) + b_3(a_3 + b_1 + b_2) + b_1 b_2}\right). \tag{128}$$

The coefficients in Eqs. (52) and (53) are

$$c_1 = (a_1 - b_1)(2x^* - 1) - (a_2 + b_1 + b_3)y^* \tag{129}$$

$$c_2 = (a_1 + a_3 + b_2)(2x^* - 1) + (a_3 - b_3)y^* \tag{130}$$

$$c_3 = -(a_2 + b_1 + b_3)x^* \tag{131}$$

$$c_4 = (a_3 - b_3)x^* \tag{132}$$

$$d_1 = a_1 - b_1 \tag{133}$$

$$d_2 = a_1 + a_3 + b_2 \tag{134}$$

$$d_3 = -(a_2 + b_1 + b_3) \tag{135}$$

$$d_4 = a_3 - b_3 \tag{136}$$

$$h_1 = (a_1 - b_1)y^* \tag{137}$$

$$h_2 = (a_1 + a_3 + b_2)y^* \tag{138}$$

$$h_3 = (a_1 - b_1)x^* - (a_2 + b_1 + b_3)(2y^* - 1) \tag{139}$$

$$h_4 = (a_1 + a_3 + b_2)x^* - (a_3 - b_3)(2y^* - 1) \tag{140}$$
$$j_1 = -(a_2 + b_1 + b_3) \tag{141}$$
$$j_2 = a_3 - b_3 \tag{142}$$
$$j_3 = a_1 - b_1 \tag{143}$$
$$j_4 = a_1 + a_3 + b_2. \tag{144}$$

The coefficients $B_1, \ldots, B_4$ in Eqs. (77) and (78) are

$$B_1 = \frac{1}{2} [s\,(2c_4 r + c_2 + c_3) \cos(\omega_0 T_0) - (c_4(r - s)(r + s) + (c_2 + c_3)\,r + c_1) \sin(\omega_0 T_0)] \tag{145}$$
$$B_2 = \frac{1}{2} [-s\,(2c_4 r + c_2 + c_3) \sin(\omega_0 T_0) - (c_4(r - s)(r + s) + (c_2 + c_3)\,r + c_1) \cos(\omega_0 T_0)] \tag{146}$$
$$B_3 = \frac{1}{2} [s\,(2h_4 r + h_2 + h_3) \cos(\omega_0 T_0) - (h_4(r - s)(r + s) + (h_2 + h_3)\,r + h_1) \sin(\omega_0 T_0)] \tag{147}$$
$$B_4 = \frac{1}{2} [-s\,(2h_4 r + h_2 + h_3) \sin(\omega_0 T_0) - (h_4(r - s)(r + s) + (h_2 + h_3)\,r + h_1) \cos(\omega_0 T_0)] \tag{148}$$

where r and s are as in Eq. (74).

Appendix 3: Removal of secular terms in Lindstedt's method with delay

Consider a system of differential delay equations of the form

$$\omega \frac{\mathrm{d}u}{\mathrm{d}t} = \alpha \bar{u} + \beta \bar{v} + K_1 \sin t + K_2 \cos t \tag{149}$$
$$\omega \frac{\mathrm{d}v}{\mathrm{d}t} = \gamma \bar{u} + \delta \bar{v} + K_3 \sin t + K_4 \cos t. \tag{150}$$

where $\bar{u} = u(t - \omega T)$ and $\bar{v} = v(t - \omega T)$, and where ω and T are such that the associated homogeneous problem,

$$\omega \frac{\mathrm{d}u}{\mathrm{d}t} = \alpha \bar{u} + \beta \bar{v} \tag{151}$$
$$\omega \frac{\mathrm{d}v}{\mathrm{d}t} = \gamma \bar{u} + \delta \bar{v} \tag{152}$$

admits solutions of the form $\sin t$ and $\cos t$, or equivalently e^{it}.

Substituting $u = r\mathrm{e}^{it}$ and $v = s\mathrm{e}^{it}$ into Eqs. (151) and (152), we obtain the characteristic equations

$$i r \omega = \mathrm{e}^{-i\omega T} (\alpha r + \beta s) \tag{153}$$
$$i s \omega = \mathrm{e}^{-i\omega T} (\gamma r + \delta s). \tag{154}$$

Rearranging, these become

$$\begin{pmatrix} \alpha e^{-i\omega T} - i\omega & \beta e^{-i\omega T} \\ \gamma e^{-i\omega T} & \delta e^{-i\omega T} - i\omega \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{155}$$

Define

$$R \equiv \begin{pmatrix} \alpha e^{-i\omega T} - i\omega & \beta e^{-i\omega T} \\ \gamma e^{-i\omega T} & \delta e^{-i\omega T} - i\omega \end{pmatrix}. \tag{156}$$

A non-trivial solution for r and s requires that $\det R = 0$. Separating the real and imaginary parts, this means that

$$Re(\det R) = \cos(2\omega T)(\alpha\delta - \beta\gamma) - \omega((\alpha + \delta)\sin(\omega T) + \omega) = 0 \tag{157}$$

$$Im(\det R) = -\cos(\omega T)(\sin(\omega T)(2\alpha\delta - 2\beta\gamma) + \omega(\alpha + \delta)) = 0. \tag{158}$$

Equation (158) tells us that

$$\sin(\omega T) = \frac{\omega(\alpha + \delta)}{2(\beta\gamma - \alpha\delta)}. \tag{159}$$

(We neglect the alternate possibility that $\cos(\omega T) = 0$.) Then, we substitute this back into Eq. (157) to obtain

$$\omega^2 = \alpha\delta - \beta\gamma. \tag{160}$$

Under the conditions (159) and (160), the solutions to Eqs. (149) and (150) will in general have secular terms:

$$u = m_1 \cos t + m_2 \sin t + n_1 t \cos t + n_2 t \sin t \tag{161}$$

$$v = m_3 \cos t + m_4 \sin t + n_3 t \cos t + n_4 t \sin t. \tag{162}$$

We wish to derive conditions on the K_i in Eqs. (149) and (150) such that the n_i are all equal to 0.

We substitute the solutions (161) and (162) into Eqs. (149) and (150), and set the coefficients of $\sin t$, $\cos t$, $t \sin t$ and $t \cos t$ separately equal to 0 in both equations.

The coefficients of $\sin t$ and $\cos t$ give us a system of linear equations on the m_i and n_i, of the form

$$M \cdot \mathbf{m} + N \cdot \mathbf{n} = -\mathbf{k} \tag{163}$$

where $\mathbf{m} = (m_1, \ldots, m_4)^{\mathrm{T}}$, $\mathbf{n} = (n_1, \ldots, n_4)^{\mathrm{T}}$ and $\mathbf{k} = (K_1, \ldots, K_4)^{\mathrm{T}}$.

Similarly, the coefficients of $t \sin t$ and $t \cos t$ give us a system of linear equations on the n_i, of the form

$$S \cdot \mathbf{n} = \mathbf{0}. \tag{164}$$

By row reducing in Mathematica, we find that both M and S have rank 2. To eliminate the n_i, we proceed as follows:

- Without loss of generality, set $m_3 = m_4 = 0$.

– Solve any two independent rows of Eq. (164) for n_3 and n_4 in terms of n_1 and n_2. The result is

$$n_3 = \frac{n_2\omega\cos(\omega T) - n_1(\alpha + \omega\sin(\omega T))}{\beta} \tag{165}$$

$$n_4 = -\frac{n_1\omega\cos(\omega T) + n_2(\alpha + \omega\sin(\omega T))}{\beta} \tag{166}$$

– Substitute these expressions for n_3 and n_4 into Eq. (163). This is now a *full-rank* linear system of equations on m_1, m_2, n_1 and n_2. Solve this system to obtain expressions for m_1, m_2, n_1 and n_2 in terms of the K_i.
– Substitute the expressions for n_1 and n_2 from the previous step into Eqs. (165) and (166). Now we have all the n_i in terms of the K_i.
– Set the n_i expressions equal to 0. This gives a rank-2 system of equations on the K_i, so it is possible to solve for K_3 and K_4 in terms of K_1 and K_2. The result is

$$K_3 = \frac{\gamma(K_1(\alpha + \omega\sin(\omega T)) + K_2\omega\cos(\omega T))}{\alpha^2 + 2\alpha\omega\sin(\omega T) + \omega^2} \tag{167}$$

$$K_4 = \frac{\gamma(K_2(\alpha + \omega\sin(\omega T)) - K_1\omega\cos(\omega T))}{\alpha^2 + 2\alpha\omega\sin(\omega T) + \omega^2}. \tag{168}$$

Using Eqs. (159) and (160), these reduce to

$$K_3 = \frac{K_1(\delta - \alpha) - K_2\sqrt{-(\alpha - \delta)^2 - 4\beta\gamma}}{2\beta} \tag{169}$$

$$K_4 = \frac{K_1\sqrt{-(\alpha - \delta)^2 - 4\beta\gamma} + K_2(\delta - \alpha)}{2\beta}. \tag{170}$$

If Eqs. (169) and (170) hold, then there are solutions of Eqs. (149) and (150) with no secular terms.

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