1. Tim Riley
The Dehn function of $\mathrm{SL}_n(\mathbb{Z})$

For a word w on ${a_1}^{\pm 1}, \ldots, {a_m}^{\pm 1}$ representing 1 in a finite presentation $\mathcal{P} = \langle a_1, \ldots, a_m \mid \mathcal{R} \rangle$ of a group Γ, define $\mathrm{Area}(w)$ to be the minimal $A \in \mathbb{N}$ such that there is an equality $w = \prod_{i=1}^{A} {u_i}^{-1} {r_i}^{\varepsilon_i} u_i$ in the free group $F(a_1, \ldots, a_m)$ for some $\varepsilon_i = \pm 1$, some words u_i, and some $r_i \in \mathcal{R}$. Equivalently, $\mathrm{Area}(w)$ is the minimal A such that there is a van Kampen diagram for w over $\mathcal{P}$ with at most A 2-cells. Defining $\mathrm{Area}(n)$ to be the maximum of $\mathrm{Area}(w)$ over all w that have length at most n and represent 1 in Γ, gives the Dehn function $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{P}$. Whilst $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$ is defined for $\mathcal{P}$, a different finite presentation $\mathcal{P}'$ for Γ will yield a Dehn function $\mathrm{Area}' : \mathbb{N} \to \mathbb{N}$ that is qualitatively the same — for example, $(\exists C > 1,\ \forall n,\ (1/C)n^2 \leq \mathrm{Area}'(n) \leq Cn^2)$ if and only if the same is true for $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$. (The C may differ.)

Question 1.1. *Is the Dehn function of $\mathrm{SL}_n(\mathbb{Z})$ quadratic when $n \geq 4$?*

Presenting this as a question, rather than a claim, conjecture, or the like, may be unduly conservative. In his 1993 survey article[1], Gersten describes the quadratic Dehn function as an assertion of W.P.Thurston.

I am not even aware of a proof that the Dehn function of $\mathrm{SL}_n(\mathbb{Z})$ is bounded above by a polynomial when $n \geq 4$. By contrast, the Dehn function of $\mathrm{SL}_2(\mathbb{Z})$ is known to grow linearly – $\mathrm{SL}_2(\mathbb{Z})$ is hyperbolic – and that of $\mathrm{SL}_3(\mathbb{Z})$ grows like $n \mapsto \exp(n)$: Epstein & Thurston[2] proved the lower bound and a result sketched by Gromov[3] gives the upper bound. (An elementary proof might be a step towards **??**.)

Of course, **??** presupposes $\mathrm{SL}_n(\mathbb{Z})$ is finite presentable, but that has been long known. The $n^2 - n$ matrices e_{ij} with 1's on the diagonal, the off-diagonal ij-entry 1, and all others 0, generate $\mathrm{SL}_n(\mathbb{Z})$. Milnor[4], following J.R.Silvester and in turn Nielsen and Magnus, explains that the Steinberg relations $\{[e_{ij}, e_{kl}] = 1\}_{i \neq l, j \neq k}$ and $\{[e_{jk}, e_{kl}] = e_{jl}\}_{j \neq l}$ together with $\{(e_{ij} {e_{ji}}^{-1} e_{ij})^4 = 1\}_{i \neq j}$ are defining relations. A proof of **??** would be an exacting quantitative proof of finite presentability.

One can regard **??** as a higher dimensional version of the Lubotzky-Mozes-Raghunathan Theorem, establishing the existence of efficient words representing elements g of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, that is, words of length like the log of the maximum of the absolute values of the

[1]Isoperimetric and isodiametric functions. In G.Niblo and M.Roller, eds., *Geometric group theory I*, no. 181 in LMS lecture notes, C.U.P., 1993.

[2]D.B.A.Epstein et al., *Word Processing in Groups*, Jones and Bartlett, 1992.

[3]Asymptotic invariants of infinite groups. In G. Niblo and M. Roller, eds., *Geometric group theory II*, no. 182 in LMS lecture notes, C.U.P., 1993. See §2B$_1$, §5A_7, §5A_9, §5D(5)(c).

[4]*Introduction to algebraic K-theory*, vol. 72 of *Annals of Mathematical Studies*, Princeton University Press, 1971.

matrix entries.[5] As a word representing g amounts to a path in the Cayley graph from 1 to g, the L.-M.-R. Theorem can be thought of as saying that 0-spheres admit efficient fillings by 1-discs. A word w representing 1 in a finite presentation $\mathcal{P}$ corresponds to a loop ρ_w in the Cayley graph; a van Kampen diagram for w can be regarded as a combinatorial homotopy disc for ρ_w in the Cayley 2-complex of $\mathcal{P}$. So **??** is, roughly speaking, the claim that 1-spheres admit efficient fillings by 2-discs in $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 4$. Gromov[3] takes this further and suggests that in $\mathrm{SL}_n(\mathbb{Z})$, Euclidean isoperimetric inequalities concerning filling k-spheres by $(k+1)$-discs persist up to $k = n - 3$. (For $k = n - 2$, the exponential lower bound of Epstein & Thurston[2] applies.)

One attack on **??** is that whilst $\mathrm{SL}_n(\mathbb{Z})$ is not a *cocompact* lattice in the symmetric space $X := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, and so the quadratic isoperimetric inequality enjoyed by X does not immediately pass to $\mathrm{SL}_n(\mathbb{Z})$, open horoballs can be removed from X to give a space X_0 on which $\mathrm{SL}_n(\mathbb{Z})$ acts cocompactly. Druţu[6] and Leuzinger & Pittet[7] have made progress in this direction, including a quadratic isoperimetric inequality for the boundary horosphere of each removed horoball. They work in the more general setting of lattices in semisimple Lie groups, and establish results towards Gromov's assertion[3] that "solvable groups of high real rank are expected to satisfy a polynomial isoperimetric inequality."

Chatterji has asked whether for $n \geq 4$, $\mathrm{SL}_n(\mathbb{Z})$ enjoys her property L_δ for some $\delta \geq 0$, which would imply a sub-cubic Dehn function[8].

The author's efforts towards **??** have, to date, yielded[9] a version of L.-M.-R. giving explicit efficient words. This may aid the construction of van Kampen diagrams, but that remains to be seen. However it has led to progress elsewhere.[10]

Finally, we mention that for $n > 3$, the Dehn functions of the cousins $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ of $\mathrm{SL}_n(\mathbb{Z})$ are also unknown.[11]

---

[5]Cyclic subgroups of exponential growth and metrics on discrete groups, C.R. Acad. Sci. Paris, Série 1, 317:723–740, 1993. The word and Riemannian metrics on lattices of semisimple groups, *I.H.É.S. Publ. Math.*, 91:5–53, 2000.

[6]Filling in solvable groups and in lattices in semisimple groups, *Topology*, 43:983–1033, 2004.

[7]On quadratic Dehn functions, *Math. Z.*, 248(4):725—755, 2004.

[8]M.Elder, L_δ groups are almost convex and have a sub-cubic Dehn function, *Algebr. Geom. Topol.*, 4:23–29 (electronic), 2004.

[9]Navigating the Cayley graphs of $\mathrm{SL}_N(\mathbb{Z})$ and $\mathrm{SL}_N(\mathbb{F}_p)$, *Geometriae Dedicata*, 113(1):215–229, 2005.

[10]M.Kassabov and T.R.Riley, Diameters of Cayley graphs of Chevalley groups, to appear in Eur. J. Comb.

[11]M.R.Bridson and K.Vogtmann, Automorphism groups of free, surface, and free abelian groups, `arXiv:math.GR/0507612`.