The Spider Algorithm

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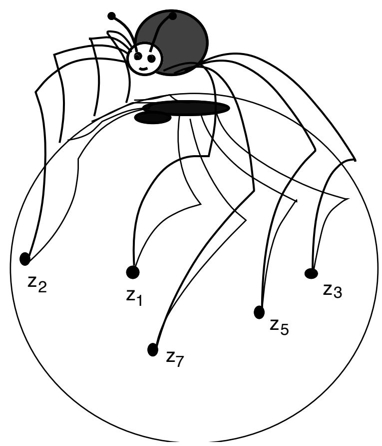

Charlotte [W] casts a 43/255-shadow.

One of the reasons complex analytic dynamics has been such a successful subject is the deep relation that has surfaced between conformal mapping, dynamics and combinatorics. The object of the spider algorithm is to construct polynomials with assigned combinatorics.

This shows up when you try to understand the Mandelbrot set. For this discussion we will write our quadratic polynomials $Q_c(z) = z^2 + c$. Every such polynomial has a *filled in Julia set* K_c, formed of the points with bounded orbits under iteration of Q_c.

A result of Fatou asserts that if the critical point $0 \in K_c$, then K_c is connected, and if $0 \notin K_c$, then K_c is a Cantor set. By definition, the Mandelbrot set M is the set of c for which K_c is connected.

Let $\mathbb{D}$ denote the open unit disc, and let $\Phi_M : \overline{\mathbb{C}} - M \to \overline{\mathbb{C}} - \mathbb{D}$ be the conformal mapping which maps ∞ to ∞ and is tangent to the identity at infinity. The existence of this mapping is not obvious; it is proved to exist at the same time as the

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Mandelbrot set M is shown to be connected. Now call

$$R_M(\theta) = \Phi_M^{-1}\left(\{re^{2\pi i\theta} \mid r > 1\}\right)$$

the *external ray of M at angle θ*. When the limit

$$\lim_{r\searrow 1} \Phi_M^{-1}(re^{2\pi i\theta})$$

exists, we say that the ray at angle θ *lands* at the limit point.

Something which sounds like magic occurs: If θ is rational, then the ray at angle θ does land at some point c_θ, and the dynamics of Q_{c_θ} reflects the digits of θ written in base 2. This is proved in [DH1]; a simpler proof can be found in [S]. A couple of examples should bring out what this means.

Example 1. Consider the polynomial $z^2 + i$, which is at the end of the ray at angle 1/6 turn. The number 1/6 is written in base 2

$$1/6 = .0 \quad 0\,1 \quad 0\,1 \quad 0\,1 \quad 0\,1 \quad \ldots$$

and under iteration of $z \mapsto z^2 + i$, the orbit of 0 is

$$i \quad -1+i \quad -i \quad -1+i \quad -i \quad -1+i \quad -i \quad \ldots$$

Both after one term repeat with period 2.

Example 2. The external rays of M at angles 1/3 and 2/3 land at -0.75, the root of the component of $\overset{\circ}{M}$ in which all polynomials have an attractive cycle of period 2;

$$1/3 = .\overline{01} \quad \text{and} \quad 2/3 = .\overline{10}$$

are the numbers which in base 2 have digits which repeat with period 2.

Similarly, the rays at angle 3/7 and 4/7 land at $-7/4$, which is the root of a component of $\overset{\circ}{M}$ in which the polynomials have an attractive cycle of period 3. Note that

$$3/7 = .\overline{011} \quad \text{and} \quad 4/7 = .\overline{100} \quad ;$$

both have digits which repeat with period 3. This pattern extends to all hyperbolic components of the interior of M (presumably, all components are hyperbolic).

These examples illustrate the sort of combinatorics polynomials may have under iteration. We will restrict our attention to polynomials for which the critical point has a finite orbit under iteration, which means it becomes eventually periodic. We will be interested in the lengths of periodic cycles, and also in the order in which points of a cycle appear. This has an obvious meaning when the polynomials and cycles are real (see Section 1) and the appropriate generalization to the complex case can be found in Section 2.

The spider algorithm is an answer both to the question of which combinatorial patterns are realized by polynomials and how to find them. The finiteness of the critical orbit gives an algebraic equation, but this equation does not distinguish the combinatorics of its many solutions. For instance, if you look for quadratic polynomials $P_c(z) = z^2 + c$ for which the critical point is periodic of period p, where p is prime, you land on an equation of degree $2^{p-1} - 1$ after removing the trivial factor, but the algebra does not distinguish between the solutions.

The underlying techniques of the spider algorithm are due to Bill Thurston and have been elaborated on by many others: Ben Bielefeld, Adrien Douady, Yuval Fisher, Lisa Goldberg, Janet Head, Silvio Levi, Jiaqi Luo, Jack Milnor, Curt McMullen, Alfredo Poirier, Mary Rees, Mitsuhiro Shishikura, Dennis Sullivan, Tan Lei, Ben Wittner, and no doubt others to whom we owe apologies. Yuval Fisher and Ben Bielefeld have written programs implementing the spider algorithm.

Thurston's theorem reduces the problem of finding a rational function with assigned combinatorics to finding a fixed point for a mapping in an appropriate Teichmüller space. He shows that either such a fixed point exists and is unique, or there is a *Thurston Obstruction*, which consists of a set of simple closed curves having special properties. The proof, written in [DH2], is deep and difficult.

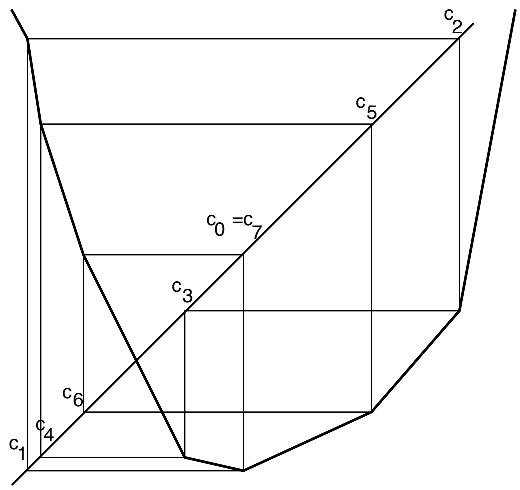
In Section 7 we will give a proof of a special case in which there is no obstruction. This proof is much easier and can serve as an introduction to the general case. We also show with an example how an obstruction can prevent such a fixed point from existing.

Let the reader be reassured: this paper does not require any knowledge of Teichmüller theory, or quasiconformal mappings, etc. The Teichmüller space is replaced by the (more intuitive) space of spiders, and the Thurston mapping by the spider map.

1. Real Kneading Sequences and Quadratic Polynomials

Just what it means for a polynomial to have assigned combinatorics is delicate to define, but for real quadratic polynomials such that the orbit of the critical point is finite this is quite easy to understand, and even to program.

Suppose that we want to find a quadratic polynomial, normalized to the form $x^2 + c$, which realizes the combinatorics sketched in the following picture.



More generally, you could start with any continuous real function which has a unique minimum, which is strictly monotone on both sides of the minimum, and for which the forward orbit of the minimum is finite. The task is to find a quadratic polynomial for which the critical point will have a finite orbit with the same number of points, and such that these points will appear in the same order.

In our case, we want a real quadratic polynomial P whose critical point c_0 is periodic of period 7, and such that the points $c_j = P^j(c_0)$ appear in the order

$$c_1 < c_4 < c_6 < c_3 < c_0 = c_7 = 0 < c_5 < c_2$$

(here, P^j stands for the j-th iterate of P).

The name of the game will be to choose correct square roots; the necessary combinatorial information will be coded in what is called a *kneading sequence*, which says for every point on the critical orbit $c_1, c_2, \dots$ if it is on the left, on the right, or equal to the critical point. Labeling these cases by L, R, and C respectively, we obtain in our case the periodic kneading sequence

$$\overline{L\ R\ L\ L\ R\ L\ C}.$$

In this setting, called "real unimodal", one can verify that there is at most one order on the points of the orbit which is compatible with a kneading sequence. This means that most orders of points on the orbit are not permissible; if one order is permissible, it carries no more information than the kneading sequence. The analogous statement will not be true for the complex case.

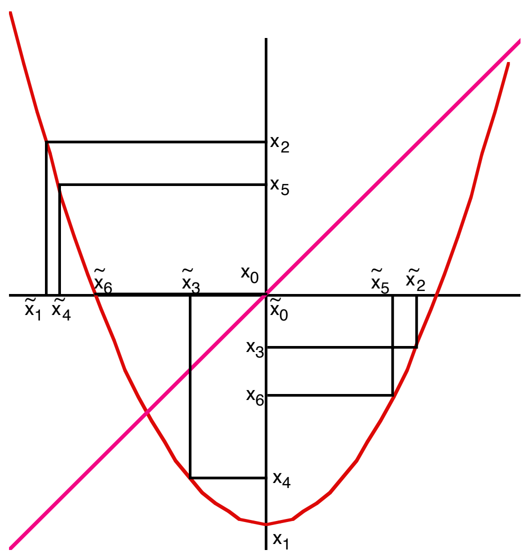
Consider the set of all sequences of real numbers

$$x_1 < x_4 < x_6 < x_3 < x_0 = x_7 = 0 < x_5 < x_2.$$

We will associate to such a sequence a new sequence

$$\tilde{x}_1 < \tilde{x}_4 < \tilde{x}_6 < \tilde{x}_3 < \tilde{x}_0 = \tilde{x}_7 = 0 < \tilde{x}_5 < \tilde{x}_2$$

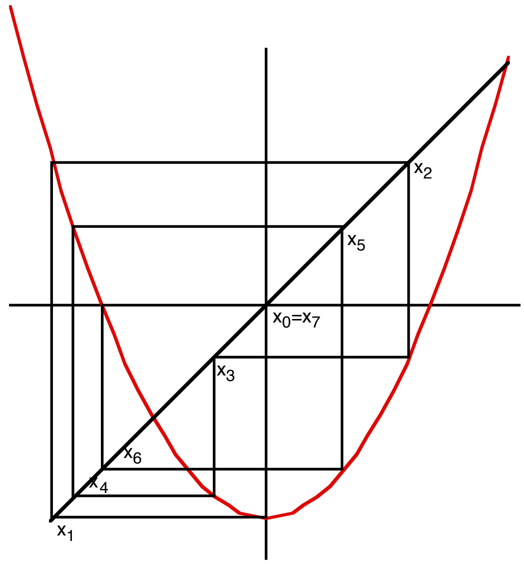
defined as follows: let P be the polynomial $x^2 + x_1$, and let $\tilde{x}_j$ be the point of $P^{-1}(x_{j+1})$ which is to the left or the right of 0, according to whether the jth term of the kneading sequence is L or R (for $j = 1, \ldots, 6$). As the only inverse image of x_1 is 0, we obtain $\tilde{x}_0 = \tilde{x}_7 = 0$ in accordance with the entry C in the kneading sequence. This procedure is illustrated in the following picture.



We will call the mapping

$$(x_1, \ldots, x_7) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_7)$$

the real spider map associated to the combinatorial data. It should be clear that if it has a fixed point $(x_1, \ldots, x_7)$, then the polynomial $P(x) = x^2 + x_1$ is the answer to our question. In the particular case above, it does converge, and the limiting x_1 is approximately $-1.674066\ldots$. The graph of that polynomial, with the critical orbit drawn in, is represented in the following figure.



It is not too difficult to show, using the Brouwer fixed point theorem, that the real spider map has a fixed point, at least if you allow some of the points to coalesce. As far as we know, no one has been able to prove the uniqueness of the fixed point by purely real methods; complex analysis appears to be required. In the next section we will set up an analog of the spider mapping above in a complex context; a proof of its convergence in the case where the critical point is periodic, which applies in particular to the case above, appears in Section 7.

2. Complex Kneading Sequences and Standard Spiders

In this section, we will describe generalizations of the spaces of ordered sequences $(x_1, \ldots, x_n)$ and the kneading sequences above. Since sequences of complex numbers do not have a natural order, "left" and "right" do not make sense, so the notions will be somewhat more elaborate.

A new convention. We will break the usual convention of writing quadratic polynomials as $z^2 + c$, and will instead write them as

$$P_\lambda(z) = \lambda \left(1 + \frac{z}{2}\right)^2 .$$

In this normalization, the critical point of P_λ is -2, and the critical value is 0; for our purposes, this works better than the usual normalization, which is focused on the behavior at infinity. It also generalizes nicely to

$$\lambda \left(1 + \frac{z}{d}\right)^d$$

(critical point $-d$, critical value 0), and ultimately to λe^z. The spider construction generalizes to all of these, and introduces unexpected correspondences between polynomials of different degree. Investigating these correspondences is a lot of fun.

We will restrict ourselves to degree 2, although everything goes through essentially without change for the above mentioned polynomials of any degree with a single critical point, and can be adapted to go through in much greater generality; see Section 8.

Recall that we only deal with polynomials for which the critical point has a finite orbit under iteration; the corresponding condition on angles θ is that they have a finite forward orbit under angle doubling, which is the same thing as saying that the angles are rational. As we will see, there are two quite different cases which occur: the case where θ is periodic under angle doubling, and the case where it is preperiodic. For instance, $4/15$ is periodic of period 4:

$$\frac{4}{15} \mapsto \frac{8}{15} \mapsto \frac{1}{15} \mapsto \frac{2}{15} \mapsto \frac{4}{15} \quad \text{(we are working in } \mathbb{R}/\mathbb{Z}\text{, i.e., mod 1);}$$

but $1/6$ is not periodic: under angle doubling, we have

$$\frac{1}{6} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3} \dots \quad \text{which never returns to } \frac{1}{6}.$$

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the set of angles, counted in full turns, and let $\mathbb{T}^{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$ be the rational ones among them.

Exercise. Show that for every $\theta \in \mathbb{T}^{\mathbb{Q}}$ there are unique smallest integers $l \geq 0$ and $k \geq 1$ and an integer a such that

$$\theta = \frac{a}{2^l(2^k - 1)}$$

(this fraction is not necessarily in lowest terms). Show that the binary expansion of θ has exactly l preperiodic digits, after which it becomes periodic of period k; similarly, the sequence θ, 2θ, $2^2\theta$, $2^3\theta$, ... becomes, after exactly l steps, periodic with period k. Periodic angles are exactly those which, when written as a fraction in lowest terms, have odd denominator.

All our angles θ will be rational. l and k will always denote preperiod and period, respectively; we will also set $n = k + l$ throughout. For periodic angles, n will be equal to the period length.

Given a rational angle $\theta \in \mathbb{T}^{\mathbb{Q}}$, we will build two combinatorial objects: The first is the standard θ-spider $\mathbb{S}_\theta \subset \overline{\mathbb{C}}$, which is the set

$$\left\{ re^{2\pi i 2^{j-1}\theta} \,|\, r \geq 1, j = 1, 2, \dots \right\} \cup \{\infty\}.$$

We might think of this as a daddy-long-legs (as in the first picture of the paper) with its body out at infinity, and with legs streching out to the unit circle. Note that the number of legs of the spider $\mathbb{S}_\theta$ is $n = k + l$ and in particular finite. For the endpoints of the legs of the standard spider $\mathbb{S}_\theta$ we will write $x_j = e^{2\pi i 2^{j-1}\theta}$.

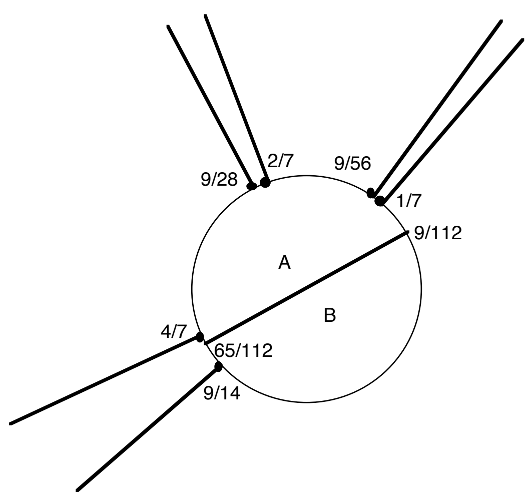
The other piece of combinatorial information is the *kneading sequence* of θ. Cut $\mathbb{T}$ at the halves of θ: $\theta/2$ and $(\theta+1)/2$. Label A the open component of $\mathbb{T}$ which contains θ, and the other B. Now the θ-*itinerary* $k_\theta(\alpha)$ of an angle α is the sequence of symbols $a_1, a_2, \ldots$, where

$$a_j = \begin{cases} A & \text{if } 2^{j-1}\alpha \in A \\ B & \text{if } 2^{j-1}\alpha \in B. \end{cases}$$

It may happen that one of the angles $2^{j-1}\alpha$ equals one of the boundary points $(\theta+1)/2$ and $\theta/2$. In these cases, we use the labels $*_1$ for the point which is at the counterclockwise end of A, and $*_2$ for the other.

The kneading sequence of θ is $K(\theta) = k_\theta(\theta)$. It contains one of the symbols $*_i$ if and only if θ is periodic under angle doubling.

Example 3. Consider $\theta = 9/56$. Then the standard θ-spider $\mathbb{S}_{9/56}$ is the following graph.



The diameter connects the two halves of 9/56, which are 9/112 and 65/112, so we can read off the kneading sequence:

$$K(9/56) = AAB\,\overline{AAA}.$$

Example 4. Let $\theta = 4/15$, which is periodic of period 4. The kneading sequence is then $K(4/15) = \overline{AAB{*}_2}$.

Exercise. Check that, if the angle θ is periodic under angle doubling, the kneading sequence $K(\theta)$ is periodic, too. If the angle θ is strictly preperiodic, then the kneading sequence $K(\theta)$ will also be strictly preperiodic, such that the number of steps before the period is equal. The length of the periodic part of the kneading sequence divides that of the angle.

Remark. Example 3 above shows that, for preperiodic angles, the periodic part of the kneading sequence may in fact be strictly shorter than the periodic part of the binary expansion of the angle. Irrational angles (whose binary expansions are not eventually periodic) may or may not have periodic kneading sequences.

3. Spiders and the Spider Map

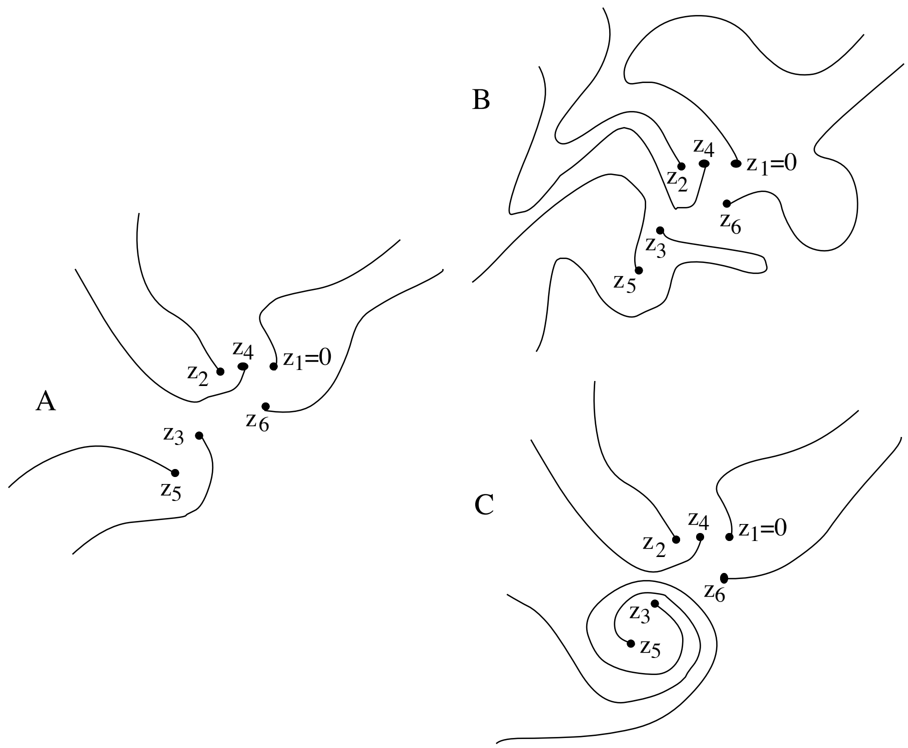
Define the space $\mathcal{S}_\theta$ of θ-spiders to be the quotient of the space

$$\mathcal{S}_\theta^0 = \left\{ \begin{array}{c} \varphi : \mathbb{S}_\theta \to \overline{\mathbb{C}} \mid \varphi(\infty) = \infty, \varphi(x_1) = 0, \varphi \text{ injective, continuous} \\ \text{and respects the circular order at } \infty. \end{array} \right\}$$

by the equivalence relation where two θ-spiders φ_0, φ_1 are equivalent if there exists a continuous 1-parameter family φ_t of θ-spiders connecting them such that for every $j \geq 2$ the ratio $\varphi_t(x_j)/\varphi_t(x_2)$ is constant as a function of t.

Our equivalence relation is really generated by two different equivalence relations: moving the legs with endpoints fixed, and scaling. Note that the standard spider $\mathbb{S}_\theta$ is not in the spider space, because the endpoint $\varphi(x_1)$ is not at the origin.

In the following figure, all three spiders are elements of $\mathcal{S}^0_{9/56}$; while A and B are equivalent, C is different (the legs to z_3 and z_5 cannot be untangled without moving the endpoints).



Now the principal actor of this paper can be introduced: the spider mapping

$$\sigma_\theta : \mathcal{S}_\theta \to \mathcal{S}_\theta.$$

Let $\varphi \in \mathcal{S}_\theta^0$, and call $z_j = \varphi(x_j)$ and γ_j the leg leading to z_j; we will add a tilde to the corresponding parts of $\tilde{\varphi} = \sigma_\theta(\varphi)$. Let P be the polynomial

$$P(z) = z_2 \left(1 + \frac{z}{2}\right)^2.$$

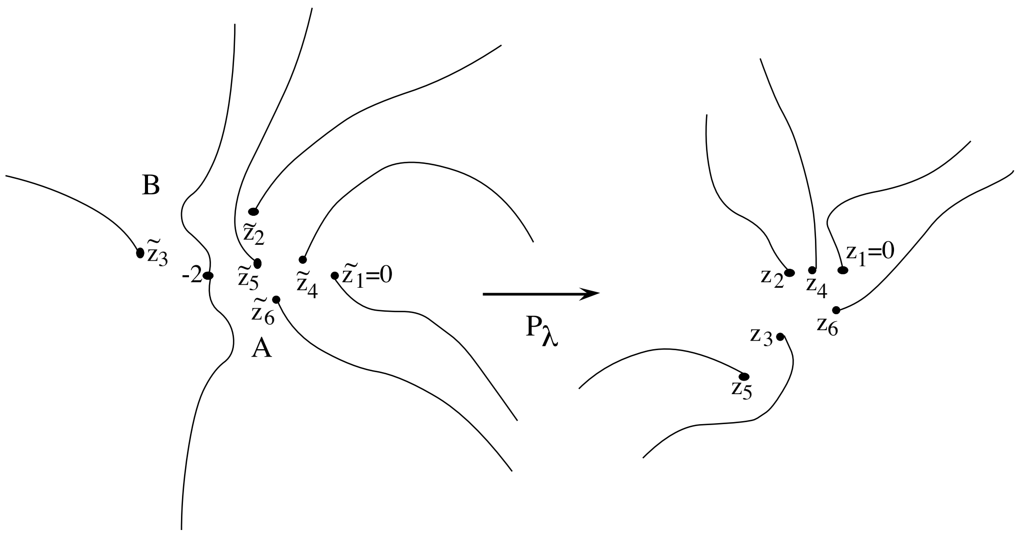
The set $P^{-1}(\gamma_1)$ is a curve cutting the plane into two parts, because γ_1 is a curve going from ∞ to 0 (the critical value of P), so the two lifts of γ meet at the critical point -2. Label A and B the two sectors of the plane obtained, so that 0 is in A.

Now define $\tilde{z}_j$ to be the point in $P^{-1}(z_{j+1})$ which is in the sector specified by the jth term of the kneading sequence $K(\theta)$, and let $\tilde{\gamma}_j$ be the component of $P^{-1}(\gamma_{j+1})$ which has $\tilde{z}_j$ as an endpoint.

Exercise. Check that we obtain exactly n disjoint legs again, which have the same circular order near ∞ as before.

Exercise. Verify that the equivalence class of $\tilde{\varphi}$ depends only on the equivalence class of φ, so that the spider map is well defined as a map from $\mathcal{S}_\theta$ to itself.

The following picture illustrates this procedure for $\theta = 9/56$. On the right is the spider φ, on the left the spider $\tilde{\varphi}$, where we have chosen the inverse image in the sector A or B according to the kneading data. Note that both $\tilde{z}_3$ and $\tilde{z}_6$ are inverse images of z_4, since $z_7 = z_4$.



If the angle θ is periodic, so that the kneading sequence contains a symbol $*_i$, the last endpoint $\tilde{z}_n$ is the only inverse image -2 of the critical value $z_{n+1} = z_1 = 0$, and the leg $\tilde{\gamma}_n$ is one of the two inverse images of γ_1 which form the boundary of the two sectors A and B. Using the same circular order as in the definition of the kneading sequence, we can label these two inverse images $*_1$ and $*_2$ and proceed analogously.

The spider space as a complex manifold.

Note that the spider space contains both analytic and combinatorial information: the ratios of the endpoints are analytic functions on the spider space, while the legs supply combinatorial information, since only their homotopy class is defined. This is made precise in Proposition 3.1.

The mapping $\mathcal{S}_\theta^0 \to \mathbb{C}^{n-2}$ given by

$$\varphi \mapsto \begin{bmatrix} \varphi(x_3)/\varphi(x_2) \\ \vdots \\ \varphi(x_n)/\varphi(x_2) \end{bmatrix}$$

induces a mapping $\pi_\theta : \mathcal{S}_\theta \to \mathbb{C}^{n-2}$, the image of which is the subset $U_n \subset \mathbb{C}^{n-2}$ consisting of points whose coordinates are different from $0, 1$ and from each other.

Proposition 3.1. *The mapping $\pi_\theta : \mathcal{S}_\theta \to U_n$ is a universal covering mapping.*

In particular, $\mathcal{S}_\theta$ is a complex manifold of dimension $n - 2$.

Sketch of Proof. Let $\mathbf{w} = (w_3, \dots, w_n) \in U_n$, and choose open disks $D_3, \dots, D_n$ around $w_3, \dots, w_n$ having disjoint closures and not containing 0 or 1, so that

$$\mathbf{D} = D_3 \times D_4 \times \cdots \times D_n \subset U_n.$$

For any $\varphi \in \mathcal{S}_\theta$ with $\pi_\theta(\varphi) = \mathbf{w}$, we need to construct a continuous section s of π_θ over $\mathbf{D}$.

It is easy but cumbersome to check that φ can be modified within its equivalence class so that $\varphi(x_2) = 1$ and so that for each i, the ith leg does not enter D_j for any $j \neq i$, intersects ∂D_i in precisely one point ζ_i and is straight within D_i. If $\mathbf{w}' = (w'_3, \dots, w'_n) \in \mathbf{D}$, we will set $s(\mathbf{w}')$ to be the spider with endpoints the w'_i, and with legs coinciding with those of φ outside of the D_i, and joining ζ_i to w'_i by the line segment within D_i.

Finally, to see that this covering space is universal, we need to show that $\mathcal{S}_\theta$ is simply connected. Choose a circle $\varphi_t, t \in S^1$ in $\mathcal{S}_\theta$, and a big circle containing all the endpoints of all φ_t. Pull the endpoints back along their legs until they reach the last intersection of the leg with the circle. You then have a collection of points on the circle with the correct circular order; these can be moved into standard position without collisions. □

The knowledgeable reader will see that we have provided an alternative proof that Teichmüller spaces of genus 0 are contractible. This result is known in general as Teichmüller's Theorem.

4. Fixed Points of the Spider Map

The main thing to realize is that if the spider mapping has a fixed point φ, and if we set $\lambda = \varphi(x_2)$, then the polynomial

$$P_\lambda(z) = \lambda \left(1 + \frac{z}{2}\right)^2$$

does have combinatorics which reflect the digits of θ written in base 2. The orbit of the critical point -2 is

$$-2 \mapsto 0 = z_1 = \tilde{z}_1 \mapsto z_2 = \tilde{z}_2 \mapsto z_3 = \tilde{z}_3 \mapsto z_4 \dots,$$

so it repeats in exactly the same way as the angles $2^j\theta$, i.e., as the digits of θ written in base 2.

Of course, it does rather better than just that: the orbit of the critical point has the same circular order as the points $2^j\theta$ on the circle $\mathbb{R}/\mathbb{Z}$. This requires a bit of amplification: after all, these are just points in $\mathbb{C}$, and don't a priori have a circular order. What does have a circular order are the legs, but they are not obviously related to the polynomial P_λ and its filled-in Julia set K_λ. The connection comes from the following statement:

Proposition 4.1. *Suppose that θ is preperiodic but not periodic under angle doubling, and that the spider map σ_θ has a fixed point φ_θ. Let $\lambda = \varphi_\theta(x_2)$. Then, for every $\theta_j = 2^{j-1}\theta$, the external ray of K_λ at angle θ_j lands at $z_j = P_\lambda^{j-1}(0)$. The union of these rays is (equivalent to) the fixed spider φ_θ.*

The proof of this statement consists of considering the legs not as mere homotopy classes, but as geometric curves. The spider map is still defined, and under iteration, arbitrary legs will approach external rays. The details (of a more general result) can be found in [BFH, Thm I].

We see that a better way of saying that the orbit of the critical point reflects the digits of θ is to say that there is an external ray landing at the critical value, the orbit of which reflects the digits of θ.

Remark. We will see in the next section that something analogous is true even if σ_θ does not have a fixed point in $\mathcal{S}_\theta$.

To give the analogous statement when θ is periodic under angle doubling, we need a bit of terminology.

Let P be a quadratic polynomial with superattracting cycle $z_0, z_1, \dots, z_{n-1}, z_n = z_0$, where z_0 is critical, and let $U_j \subset \mathring{K}_\lambda$ be the component containing z_j. There is a unique *internal parametrization* $\psi_j : \overline{\mathbb{D}} \to \overline{U_j}$ conjugating $z \mapsto z^2$ to $P^{\circ n} : \overline{U_j} \to \overline{U_j}$. We call $\psi_j(1)$ the root of U_j: the roots form a repelling cycle of period dividing n.

Proposition 4.2. *Suppose that θ is periodic of period n under angle doubling, and that φ_θ is a fixed point of the spider map σ_θ; let $\lambda = \varphi_\theta(x_2)$. Then the polynomial P_λ has a superattracting cycle of period n, and the external ray of K_λ at angle θ_j lands at the root of U_j. The j-th leg of φ_θ is homotopic rel the critical orbit to the union of the external ray at angle θ_j and the "internal ray" $\psi_j([0,1])$ joining z_j to the root of U_j.*

Remark. As we will see in Section 7, the spider map always has a unique fixed point in the periodic case, but the union of external and internal rays above may (just barely) fail to be the fixed spider. If the period of the roots strictly divides n, each of these points will be used by several legs, so the injectivity condition is violated. This can be cured by an arbitrarily small homotopy: the legs "touch" but do not "cross".

5. Does a Fixed Point of the Spider Mapping exist?

The statement below on the convergence of the spider map is a special case of a deep theorem due to Thurston (see [DH2]). Its proof is quite difficult indeed. We will give a proof when θ is periodic under angle doubling, but even for quadratic spiders, when θ is preperiodic but not periodic under angle doubling, we know of no essential simplification of Thurston's proof.

Theorem 5.1. *For rational θ, let $\varphi \in \mathcal{S}_\theta$, and set*

$$\varphi^{(1)} = \sigma_\theta(\varphi),\ \varphi^{(2)} = \sigma_\theta(\varphi^{(1)}), \dots .$$

Further, set $z_j^{(n)} = \varphi^{(n)}(x_j)$. Then for each j the sequence

$$z_j, z_j^{(1)}, \dots, z_j^{(n)}, \dots$$

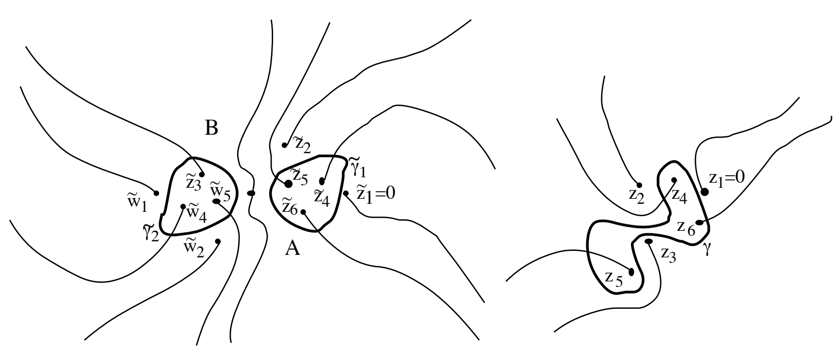
converges. Moreover, the ith and jth sequence will have the same limit if and only if the kneading sequence of θ from the ith position on is the same as from the jth position on, i.e., if $k_\theta(2^{i-1}\theta) = k_\theta(2^{j-1}\theta)$.

Notice that we were careful not to say that the sequence of spiders $\varphi^{(n)}$ converges in $\mathcal{S}_\theta$; recall that spiders are required to be injective, and the limits of such sequences may fail to be injective; the second part of the statement says that this occurs exactly if the periodic part of the kneading sequence repeats with a period which strictly divides that of the angle θ.

6. An example of a Thurston Obstruction

In this section we will show with an example how the existence of systems of simple closed curves with certain properties can prevent the spider map from having a fixed point. Our example will show a special case of a Thurston obstruction, which consists of a single simple closed curve. At the end we give as exercises to find Thurston obstructions consisting of two or more curves.

Let us consider again our example $\theta = 9/56$. In the following picture, you see on the right a spider, and on the left its full inverse image, including both the $\tilde{z}_i$, and the other unused inverse images $\tilde{w}_i$. We have also drawn in, on the right, a simple closed curve γ which surrounds z_4, z_5, z_6, the homotopy class of which is completely specified by requiring that it not intersect the legs to the points not surrounded, and that it intersect the legs to the points surrounded exactly once.



Its complete inverse image $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ is drawn in on the left. To check that this is really so, note first that γ did not intersect the leg to z_1, so its inverse image will have one component in sector A and one in B. Moreover, all inverse images of z_4, z_5, z_6 must be surrounded by one of the two components, but no other inverse image of any z_i. Inverse images of legs to points not surrounded must not be intersected, however. This specifies the homotopy classes of the components of the inverse images.

The important thing to notice is that $\tilde{\gamma}_1$ is in the same homotopy class as γ.

Now consider the lengths of the curves γ and $\tilde{\gamma}_1$ in the hyperbolic metric of

$$\mathbb{C} - \{z_1, \dots, z_6\} \quad \text{and} \quad \mathbb{C} - \{\tilde{z}_1, \dots, \tilde{z}_6\}$$

and denote them by $l_\varphi(\gamma)$ and $l_{\tilde{\varphi}}(\tilde{\gamma}_1)$.

Lemma 6.1. *We have the inequality*

$$l_{\tilde{\varphi}}(\tilde{\gamma}_1) < l_\varphi(\gamma).$$

In particular, $\sigma_{9/56}$ cannot have a fixed point in the spider space.

Proof. The mapping P_λ is a covering map, hence an infinitesimal isometry in the Poincaré metric, if considered as a mapping

$$\mathbb{C} - P_\lambda^{-1}\{z_1, \dots, z_6\} \to \mathbb{C} - \{z_1, \dots, z_6\}.$$

So in the metric of $\mathbb{C} - P_\lambda^{-1}\{z_1, \dots, z_6\}$, i.e., in the metric of the complement of all the points drawn, $\tilde{\gamma}_1$ has the same length as γ in $\mathbb{C} - \{z_1, \dots, z_6\}$.

The inclusion

$$\mathbb{C} - P_\lambda^{-1}\{z_1, \dots, z_6\} \to \mathbb{C} - \{\tilde{z}_1, \dots, \tilde{z}_6\}$$

is strictly contracting in the Poincaré metric (like all strict inclusions of Riemann surfaces), so that the length of $\tilde{\gamma}_1$ is strictly smaller than the length of γ.

The statement about the spider map follows: if $\sigma_{9/56}$ had a fixed point, let γ' be the geodesic in the homotopy class of γ on $\mathbb{C} - \{z_1, \dots, z_6\}$. Then the length of the component $\tilde{\gamma}_1$ of its inverse must be strictly shorter on $\mathbb{C} - \{\tilde{z}_1, \dots, \tilde{z}_6\}$, and the geodesic $\tilde{\gamma}_1'$ in the homotopy class of $\tilde{\gamma}_1$ on $\mathbb{C} - \{\tilde{z}_1, \dots, \tilde{z}_6\}$ is shorter yet. But there is only one geodesic in a given homotopy class, so $\tilde{\gamma}_1' = \gamma'$. This is a contradiction. □

The reader should observe that the existence of the curve γ, for which one component of the inverse image is homotopic to the curve itself, has everything to do with the fact that the 9/56-itineraries of $2^{j-1}9/56$ coincide for $j = 4, 5, 6$. A similar construction will be possible whenever the symbols in the kneading sequence repeat with period strictly smaller than the period of the angle itself.

Exercise. Show that for the angle $\theta = 27/60$, there is a system of two simple closed curves in the complement of the endpoints, such that one component of the inverse image of each is homotopic to the other. Why does this prevent the map $\sigma_{27/60}$ from having a fixed point?

Exercise. Find an angle θ such that there is a system of 3 curves behaving as above.

7. A Proof of Convergence when θ is Periodic

In this section we will prove the convergence of the spider algorithm in the case when θ is periodic. This is a special case of Thurston's theorem, but it seems worthwhile to give the proof anyway, as it is really very much simpler than the general case.

The proof consists of two parts: a compactness statement, which replaces a much more difficult argument, and a contraction statement, which is somewhat simpler than the more general proof.

Before starting, we will slightly modify our definition of S_θ: we will consider only spiders φ satisfying $\varphi(x_n) = -2$. Note that any spider φ with the previous definition is equivalent to a unique one satisfying this condition, namely

$$\frac{-2}{\varphi(x_n)}\varphi,$$

so this requirement does not change the spider space. Moreover, the image of the spider map consists entirely of spiders fulfilling this requirement. We will freely write $x_0 = x_n$, keeping in mind that indices are only defined modulo n.

The compactness statement.

Our compactness statement will say that a certain subset of $\mathcal{S}_\theta$, in which the endpoints are bounded away from each other and infinity, is invariant under the spider map.

The image of this subset under π_θ is a compact subset of U_n, and it follows, since $\varphi(x_n) = -2$, that the *endpoints* z_j $(j = 1, \dots, n)$ belong to a compact subset of $\mathbb{C}^n - \Delta$, where Δ is the set where any two coordinates are equal. The set of spiders belonging to this invariant subset is an infinite cover of the space of endpoints and is *not* compact.

The statement is a bit convoluted; the proof should explain why.

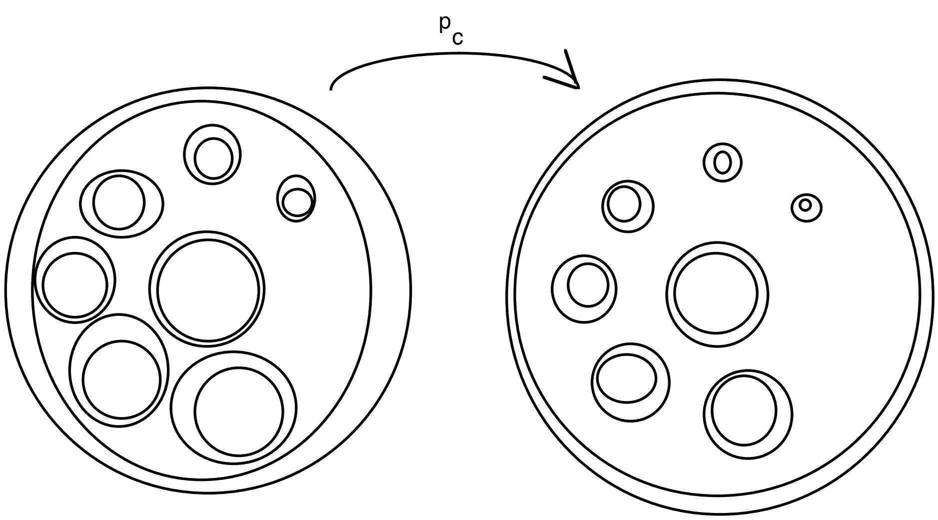
Proposition 7.1. *Choose ε satisfying $0 < \varepsilon < 4^{2-n}$ and let $\mathcal{S}_{\theta,\varepsilon}$ be the set of θ-spiders φ such that the discs of radius $4^{j-n}\varepsilon/|z_2|$ around z_j and the disc of radius $\varepsilon/|z_2|$ around $z_0 = z_n = -2$ are disjoint, and all z_j are contained in a disc around -2 of radius $8/|z_2|$. Then $\mathcal{S}_{\theta,\varepsilon}$ is invariant under σ_θ.*

Both statement and proof of this proposition will become much simpler if we choose the right coordinate system. The polynomial $P_\lambda(z) = \lambda(1 + z/2)^2$ is conjugate to $p_c(w) = w^2 + c$ by an affine map ψ if and only if $c = \lambda/2$ (and $\lambda \neq 0$). Specifically, for $w = \psi(z) = (z + 2)\lambda/4$, we have $p_c = \psi \circ P_\lambda \circ \psi^{-1}$. The critical point is now $w_0 = \psi(-2) = 0$ and the critical value is $w_1 = \psi(0) = c$.

Let $\mathbb{D}_2$ be the disc around the origin of radius 2. Then it is easy to verify that a spider φ is in $\mathcal{S}_{\theta,\varepsilon}$ if and only if the points $w_j = \psi(z_j)$ are contained in $\mathbb{D}_2$ and are surrounded by disjoint discs of radius $r_j = 4^{j-1-n}\varepsilon$, for $j = 1, \ldots , n$ (remember $\lambda = z_2$).

The result now follows from applying ψ^{-1} to the discs provided by the following lemma. Note that this lemma does not require the right choice of inverse images to be taken.

Lemma 7.2. *Choose ε as above and suppose that $w_0 = 0$, $w_1 = c$, $\ldots$, w_{n-1}, $w_n = w_0$ are contained in $\mathbb{D}_2$ such that the discs of radius $r_j = 4^{j-1-n}\varepsilon$ around w_j are disjoint for $j = 1, \ldots , n$. For each $j = 0, \ldots , n-1$, let $\tilde{w}_j$ be an element of $p_c^{-1}(w_{j+1})$ (which forces $\tilde{w}_0 = 0$), and set $\tilde{w}_n = 0$ also. Then all these $\tilde{w}_j$ will be contained in $\mathbb{D}_2$, and the discs of radius r_j around them will again be disjoint for $j = 1, \ldots , n$.*



Proof of Lemma 7.2. In order to show that each inverse image $\tilde{w}_j = \sqrt{w_{j+1} - c}$ is contained in $\mathbb{D}_2$, it suffices to estimate $|w_{j+1} - c| < |w_{j+1}| + |c| < 4$.

To exclude the possibility that points get too close together, we first note that the inverse image of the disc around $w_1 = c$ of radius $r_1 = 4^{-n}\varepsilon$ is a disc around $\tilde{w}_0 = 0$ of radius $4^{-n/2}\sqrt{\varepsilon}$ and hence contains the disc $D_{r_0}(0)$ of radius $r_0 = r_n = 4^{-1}\varepsilon < 4^{-1}4^{(2-n)/2}\sqrt{\varepsilon} = 4^{-n/2}\sqrt{\varepsilon}$.

Note that $|p_c'| < 4$ on $\mathbb{D}_2$. It follows that if $\tilde{w} \in \mathbb{D}_2$ is contained in a disc around some $\tilde{w}_j$ of radius r_j $(1 \le j \le n-1)$, then $w := p_c(\tilde{w})$ is contained in a disc around $w_{j+1} = p_c(\tilde{w}_j)$ of radius $4r_j = r_{j+1}$.

Now, if $D_{r_i}(\tilde{w}_i)$ and $D_{r_j}(\tilde{w}_j)$ had non-empty intersection, we could choose a point $\tilde{w}$ in their intersection which was also contained in $\mathbb{D}_2$. But the image point $w = p_c(\tilde{w})$ would then be contained in the given discs around the image points $p_c(\tilde{w}_i)$ and $p_c(\tilde{w}_j)$, contradicting the hypotheses. □

The contraction statement.

We wish to prove that the spider map is strictly contracting. This requires a metric on the space of spiders, and defining it is not quite straightforward.

We will define an infinitesimal metric, which will be given by a norm (which is *not* an inner product) on the tangent spaces. This kind of structure is called a Finsler metric; such structures carry a great deal of geometry. Many of the deepest results about Teichmüller spaces are obtained by looking really carefully at the unit balls for such metrics.

Let θ be periodic of period n. The tangent space to $\mathcal{S}_\theta$ at a spider φ with endpoints $Z_\varphi = \{z_0 = -2, z_1 = 0, z_2, \ldots, z_{n-1}\}$ is the space

$$T_\varphi \mathcal{S}_\theta = \{\underline{\xi}\} := \left\{ \left(\vec{\xi}_2, \ldots \vec{\xi}_{n-1}\right) \middle| \vec{\xi}_j = \xi_j \ \partial/\partial z \in T_{z_j}\overline{\mathbb{C}} \right\}.$$

This follows from the fact that the mapping which associates to a spider its endpoints is a covering map, so the tangent space to the spider space is simply the tangent space to $\mathbb{C}^{n-2}$.

Remark. Although we will be working on $\mathbb{C}$, where there is a global chart, we will keep tangent vectors distinct from complex numbers, by writing

$$\vec{\xi} = \xi \frac{\partial}{\partial z}.$$

Confusing numbers and tangent vectors can lead to conceptual difficulties.

The cotangent space $T^*_\varphi \mathcal{S}_\theta$ also has a very nice description: it is the space of polynomials of degree at most $n-3$. This is not an honest way of saying things: the cotangent space is really the space Q_φ of meromorphic quadratic differentials on $\overline{\mathbb{C}}$ with poles only on $Z_\varphi \cup \{\infty\}$ and all poles at most simple. All such quadratic differentials can be written

$$q = \frac{p(z)}{\prod_{j=0}^{n-1}(z - z_j)} dz^2 \tag{1}$$

with p a polynomial of degree at most $n-3$. The reason for this restriction is that the denominator has a zero of order n at ∞, the quadratic differential dz^2 has a pole of order 4, so that if $\deg p \le n-3$, then q has at most a simple pole at ∞.

The fact that Q_φ is the cotangent space is really a special case of the Serre Duality Theorem. In this case, the duality can be made quite explicit.

Proposition 7.3. *The pairing*

$$\langle \underline{\xi}, q\rangle = \sum_{j=2}^{n-1} Res_{z_j} \vec{\xi}_j q = \sum_{j=2}^{n-1} \xi_j \frac{p(z_j)}{\prod_{i\neq j}(z_j - z_i)}$$

induces a duality between $T_\varphi \mathcal{S}_\theta$ and Q_φ.

Remark. The product of a quadratic differential and a vector field is naturally a differential 1-form, as is suggested by the equation

$$\big(q(z)dz^2\big)\left(\xi(z)\frac{\partial}{\partial z}\right) = \xi(z)q(z)dz.$$

Thus the residue in the statement is really the residue of a 1-form; this is the kind of residue which is studied in elementary complex analysis.

Proof. It is clear that we have written a pairing, which induces a mapping of each space into the dual of the other. Think of it as a mapping

$$T_\varphi \mathcal{S}_\theta \to (Q_\varphi)^* \ ;$$

we will show it is injective. If $\underline{\xi} = (\vec{\xi}_2, \ldots, \vec{\xi}_{n-1}) \neq 0$, there exists j such that $\xi_j \neq 0$. Then $\underline{\xi}$ pairs non-trivially with the quadratic differential

$$\frac{dz^2}{z(z+2)(z-z_j)}$$

which is of the form (1) with

$$p(z) = \prod_{i=2,\ i\neq j}^{n-1} (z - z_i).$$

Since $T_\varphi \mathcal{S}_\theta$ and Q_φ both have dimension $n-2$, this proves the Proposition. $\square$

If $q = q(z)dz^2$ is a quadratic differential, then $|q| := |q(z)|dx\,dy$ is a positive measure in a natural way. Moreover, simple poles are integrable:

$$\int_{\mathbb{D}} \left|\frac{dz^2}{z}\right| = \int_{\mathbb{D}} \frac{1}{|z|} dx\,dy = 2\pi$$

is finite. Thus we can define the L^1 norm on Q_φ by the formula

$$\|q\| = \int_{\mathbb{C}} |q|.$$

Exercise. Check directly that the condition $\deg p \le n-3$ is equivalent to

$$\int_{\mathbb{C}} \left| \frac{p(z)}{\prod_{j=0}^{n-1}(z-z_j)} \right| dx\, dy < \infty.$$

Note that the condition φ injective, which keeps the z_j distinct, is essential for this statement to be true: double poles are *not* integrable.

We are finally in a position to define our metric on the spider space: it is the metric given by the norm on the tangent spaces dual to the L^1 norm above on the cotangent spaces. Recall that if E is a Banach space (in our case finite dimensional) with norm $\| \ \|_E$, then the dual space E^* (the space of continuous linear functionals $E \to \mathbb{C}$) naturally carries the norm

$$\|\alpha\|_{E^*} = \sup_{x \in E - \{0\}} \frac{|\alpha(x)|}{\|x\|_E}.$$

Proposition 7.4. *The spider mapping is strictly contracting, i.e.,*

$$\|d_\varphi \sigma_\theta\| < 1$$

for all θ-spiders φ.

Moreover, the norm of the derivative depends only on the endpoints of φ and $\tilde\varphi$, and not on the combinatorial information contained in the legs.

We first need to define the operator

$$(P_\lambda)_* : Q_{\tilde\varphi} \to Q_\varphi.$$

Choose a point $z \in \mathbb{C}$. Suppose $z \ne 0$, and choose a simply connected neighborhood U of z which does not include 0, and let ζ be a local coordinate in U. Since U contains no critical value of P_λ, $P_\lambda^{-1}(U)$ consists of two components U_1, U_2, in which the lifts ζ_1, ζ_2 of ζ are local coordinates. Any $q \in Q_{\tilde\varphi}$ (in fact, any quadratic differential in the "domain" Riemann surface) can be written $q_i(\zeta_i)d\zeta_i^2$ on U_i. We define

$$(P_\lambda)_* q = (q_1(\zeta) + q_2(\zeta))d\zeta^2 \tag{2}$$

on U. This defines $(P_\lambda)_* q$ everywhere but at the critical value 0. It is clear that $(P_\lambda)_* q$ is holomorphic, except for possible simple poles at the images of the poles of

q under P_λ, and at 0 where *a priori* we only know that it has an isolated singularity. Lemma 7.5 implies that even at 0 it has at worst a simple pole.

Now the proposition follows from the following two lemmas.

Lemma 7.5. *The mapping*

$$(P_\lambda)_* : Q_{\tilde{\varphi}} \to Q_\varphi.$$

satisfies $\|(P_\lambda)_*\| < 1$.

Lemma 7.6. *The mapping*

$$(P_\lambda)_* : Q_{\tilde{\varphi}} \to Q_\varphi.$$

is the transpose (also known as adjoint) of $d_\varphi \sigma_\theta$.

Proof of Lemma 7.5. Use the notation of (2) and before. For every U which does not contain the critical value, we have

$$\int_U |(P_\lambda)_* q| \leq \int_U |q_1| + \int_U |q_2| = \int_{P_\lambda^{-1}(U)} |q|$$

by the triangle inequality; it follows easily that $\|(P_\lambda)_*\| \leq 1$. If $\|(P_\lambda)_*\| = 1$, there exists $q \in Q_{\tilde{\varphi}}$ such that

$$\|(P_\lambda)_* q\| = \|q\|, \tag{3}$$

since the spaces are finite dimensional. Where can the poles of such a q be? Every z_j with $j \neq 1$ has two inverse images, $\tilde{z}_{j-1}$ and another point $\tilde{w}_{j-1}$ where q cannot have a pole. Choose a neighborhood U of z_j for $j \neq 1$; equality (3) says that nothing is lost in the triangle inequality, i.e. that $q_1 = Cq_2$ for some $C \geq 0$. But one of the U_i contains $\tilde{w}_{j-1}$, hence q_1 has no pole there, and so q has no pole at $\tilde{z}_{j-1}$.

The upshot of this analysis is that q satisfying equation (3) can have poles only at -2 and ∞; but a quadratic differential always has at least 4 poles. Thus the case $\|(P_\lambda)_*\| = 1$ is excluded. $\square$

Proof of Lemma 7.6. It is not hard to differentiate σ_θ; if you find the formulas cumbersome, look for an explanation below. Suppose $\sigma_\theta(\varphi) = \tilde{\varphi}$; the endpoints of φ and $\tilde{\varphi}$ are called z_j and $\tilde{z}_j$, respectively, for $j = 0, \ldots, n-1$ as usual. If $d_\varphi \sigma_\theta(\underline{\xi}) = \underline{\tilde{\xi}}$, then differentiating the relation

$$P_{z_2}(\tilde{z}_j) = z_{j+1}$$

yields

$$\xi_2 \left(1 + \frac{\tilde{z}_j}{2}\right)^2 + z_2 \left(1 + \frac{\tilde{z}_j}{2}\right) \tilde{\xi}_j = \xi_{j+1}.$$

Solving for $\tilde{\xi}_j$, we find

$$\tilde{\xi}_j = \left(\frac{1}{z_2(1 + \tilde{z}_j/2)}\right) \xi_{j+1} - \left(\frac{1 + \tilde{z}_j/2}{z_2}\right) \xi_2. \tag{4}$$

Now we define

$$\mu_j := \left(\frac{1}{z_2(1 + \tilde{z}_j/2)}\right) \xi_{j+1}$$

so that

$$\vec{\mu}_j := \mu_j \frac{\partial}{\partial z} = \left(d_{\tilde{z}_j} P_{z_2}\right)^{-1} \vec{\xi}_{j+1}.$$

Moreover, let

$$\vec{\mu} := \mu(\tilde{z}) \frac{\partial}{\partial \tilde{z}} = \left(\frac{1 + \tilde{z}/2}{z_2}\right) \xi_2 \frac{\partial}{\partial \tilde{z}}$$

be the global vector field on $\overline{\mathbb{C}}$ which vanishes at -2 and ∞, and the value of which at $0 = \tilde{z}_1$ is

$$\left(d_{\tilde{z}_j} P_{z_2}\right)^{-1} \xi_2 \frac{\partial}{\partial \tilde{z}} = \left(\frac{1}{z_2}\right) \xi_2 \frac{\partial}{\partial \tilde{z}}.$$

Using this notation, equation (4) becomes

$$\tilde{\xi}_j = \mu_j - \mu(\tilde{z}_j).$$

A word of explanation of what these formulae mean: one might well expect that the derivative of σ_θ would simply take the tangent vectors $\vec{\xi}_{j+1}$, and lift each one to a vector at $\tilde{z}_j$ by the derivative of P_λ. This construction yields the tangent vectors $\vec{\mu}_j$, but it isn't quite the required derivative. In particular, $\vec{\mu}_1$ may well fail to vanish, even though $\tilde{z}_1 = 0$ is constant. The correct derivative is given by subtracting from each $\vec{\mu}_j$ the value at z_j of the unique global vector field which vanishes at -2 and ∞, and which is equal to $\vec{\mu}_1$ at 0.

The formula we need to verify is that

$$\langle \underline{\tilde{\xi}}, q \rangle = \langle \underline{\xi}, (P_\lambda)_* q \rangle$$

for any $q \in Q_{\tilde{\varphi}}$ and any $\underline{\xi} \in T_\varphi \mathcal{S}_\theta$.

By the definition of the pairing, we have

$$\begin{aligned} \langle \underline{\tilde{\xi}}, q \rangle &= \sum_{j=2}^{n-1} \mathrm{Res}_{\tilde{z}_j} \vec{\tilde{\xi}}_j q \\ &= \sum_{j=1}^{n-1} \mathrm{Res}_{\tilde{z}_j} \vec{\mu}_j q - \sum_{j=1}^{n-1} \mathrm{Res}_{\tilde{z}_j} \vec{\mu}(z_j) q. \end{aligned} \tag{5}$$

Note that the terms corresponding to $j = 1$ in the two sums cancel. The last sum in equation (5) vanishes by the Cauchy integral formula, since $\vec{\mu}q$ is a global 1-form on the Riemann sphere, with poles only at the $\tilde{z}_j$ for $j = 1, \ldots, n-1$ (and not at -2 and ∞, since $\vec{\mu}$ vanishes there). Moreover, $\mu_{n-1} = 0$ since it is the pull-back of the tangent vector to $z_0 = -2$ which is fixed. So altogether we get

$$\langle \underline{\tilde{\xi}}, q \rangle = \sum_{j=1}^{n-2} \mathrm{Res}_{\tilde{z}_j} \vec{\mu}_j q \ .$$

On the other hand, by the definition of the push-forward, we have

$$\begin{aligned} \langle \underline{\xi}, (P_\lambda)_* q \rangle &= \sum_{j=2}^{n-1} \mathrm{Res}_{z_j} \vec{\xi}_j \, (P_\lambda)_* \, q \\ &= \sum_{j=1}^{n-2} \mathrm{Res}_{\tilde{z}_j} \vec{\mu}_j q + \sum_{j=1}^{n-2} \mathrm{Res}_{\tilde{w}_j} \vec{\nu}_j q \end{aligned} \tag{6}$$

where $\tilde{w}_j$ is the inverse image of z_{j+1} which is not $\tilde{z}_j$, and

$$\nu_j = \left(d_{\tilde{w}_j} P_{z_2}\right)^{-1} \xi_{j+1}$$

is the lift of ξ_{j+1} there. The switch of indices from $2, \ldots, n-1$ to $1, \ldots, n-2$ comes from the labeling of inverse images. The second sum in equation (6) vanishes because q does not have poles at the $\tilde{w}_j$. Putting together equations (5) and (6) (and remembering that the last sums in both of them vanish) proves Lemma 7.6.

Proof of Thurston's Theorem. We are now in a position to prove the main result:

Theorem 7.7. *If θ is periodic under angle doubling, there is a unique fixed point φ_θ of σ_θ in $\mathcal{S}_\theta$, and all of $\mathcal{S}_\theta$ is attracted to it under interation of σ_θ.*

Proof. Recall the space $\mathcal{S}_{\theta,\varepsilon}$ from Proposition 7.1. We will require the following statement.

Lemma 7.8. *For any two spiders $\varphi_0, \varphi_1 \in \mathcal{S}_{\theta,\varepsilon}$, there exists $\delta > 0$ and a path γ joining φ_0, φ_1 contained in $\mathcal{S}_{\theta,\delta}$.*

In Proposition 3.1, we showed much more than connectivity of the spider space, we showed contractibility. Recall that this required drawing a large circle, and pulling the legs back to their last intersection with this circle. A similar proof gives Lemma 7.8: move to the w-plane, as in the proof of Lemma 7.2, and repeat the argument, but with the circle of radius 2. You then will obtain a path connecting the two spiders, with all endpoints staying in the disc of radius 2. The endpoints will stay some distance apart, and it is easy to choose δ in terms of that distance.

To continue the proof of Thurston's theorem, choose $\varphi_0 \in \mathcal{S}_{\theta,\varepsilon}$, set $\varphi_1 = \sigma_\theta(\varphi_0)$, and find δ and a path γ_0 connecting φ_0 to φ_1 in $\mathcal{S}_{\theta,\delta}$, which we may suppose continuously differentiable, hence of finite length L. Set $\varphi_m = \sigma_\theta^m(\varphi_0)$, and $\gamma_m = \sigma_\theta^m(\gamma_0)$ for $m \geq 2$. All of these φ_m and γ_m are contained in $\mathcal{S}_{\theta,\delta}$, so Proposition 7.4 says that there is a constant $C_\delta < 1$ such that the length of γ_m is at most $C_\delta^m L$. In particular, the sequence φ_m is a Cauchy sequence.

This is not quite enough to know that it converges: we would need to know that $\mathcal{S}_\theta$ is complete, which is true but rather harder to see than anything we have used so far.

Instead, we argue as follows. The endpoints $z_{j,m}$ of φ_m certainly converge, say to y_j, where the y_j are distinct, and with distances bounded below since all φ_m belong to $\mathcal{S}_{\theta,\delta}$ for a fixed δ. It is now easy to see that if the φ_m do not converge, there exist distinct θ-spiders with these same endpoints y_j which are arbitrarily close together. But to move from one δ-spider to another with the same endpoints, at least one endpoint will have to leave its disc. The length of such a path is certainly bounded below by a positive constant which depends only on the y_j. $\square$

8. Extensions to Higher Degrees

Everything we have done extends to the family of polynomials $P_{d,\lambda}(z) = \lambda(1 + z/d)^d$. The standard θ-spider is now defined using the mapping $\mathbb{T} \to \mathbb{T}$ given by $t \mapsto dt$, rather than angle-doubling. The kneading sequence is now a sequence of sectors limited by the d angles $\theta/d, (\theta+1)/d, \ldots, (\theta+d-1)/d$.

We can define the space $\mathcal{S}_{d,\theta}$ exactly as for degree 2, and the spider map

$$\sigma_{d,\theta} : \mathcal{S}_{d,\theta} \to \mathcal{S}_{d,\theta}$$

is now easy to define, once we realize that the inverse image of the leg to $z_1 = 0$ consists of d rays landing at $-d$, cutting the plane into d sectors, which each contain exactly one inverse image of every point not on the leg of z_1.

We can even extend the spider mapping to the family of exponentials when the "angle" is preperiodic; the d-th root which appears in $\sigma_{d,\theta}$ becomes a logarithm. Thurston's proof for the convergence of the algorithm does not carry over to exponentials.

The polynomials $P_{d,\lambda}(z) = \lambda(1 + z/d)^d$ are special: they have only one critical point. The spider algorithm can be adapted to polynomials with arbitrarily many critical points, all of the orbits of which are finite. The underlying mathematics is done in [BFH] for the preperiodic case, and in [P] in general. To give details would take us too far afield.

The "elementary" proof of convergence is a bit delicate to generalize to this more general situation. The proof of Proposition 7.1 will generalize to arbitrary polynomials for which every critical point lands on a cycle containing a critical point. This is a substantial simplification of Thurston's proof in that case. The statement of Proposition 7.4 is true in full generality, but the proof we have given can only be extended to rational functions with at most three critical values.

9. A Comment on Implementation

Any one thinking of implementing the spider algorithm should pause when notions like "continuous curve", and "choose the inverse image on this side of the continuous curve … " appear. It is notoriously difficult to simulate the continuous on the computer; in particular, the idea of encoding a leg by a sequence of pixels is a bit repelling.

Fortunately, it is not necessary. In fact, the algorithm can be encoded in a remarkably clean way.

We will suppose that spiders are encoded as endpoints $z_1 = 0, z_2, \dots, z_n$ (floating-point complex numbers) and legs (polygonal lines in the plane, going from z_j to a circle of sufficiently large radius).

There are two essential steps in the spider algorithm: computing the inverse images $\tilde{z}_j$ in the appropriate sectors, and computing the new legs. Let us take these in order.

Computing the New Endpoints. Among the inverse images $P_{d,z_2}^{-1}(z_{j+1})$ of z_{j+1}, let $\tilde{w}_j$ be the one closest to 0. What sector does it belong to? If we can answer that, then we can find the inverse image which belongs to the required sector. It would be quite hard to answer this question using the description given in Section 3, because that used the inverse images of the leg leading to z_1, and we don't have these inverse images available.

We claim that $\tilde{w}_j$ is in the sector numbered (mod d) by the algebraic intersection number $I(\gamma_1, [z_2, z_{j+1}])$ of the leg γ_1 to z_1 and of the segment $[z_2, z_{j+1}]$, because the lift of this segment starting at 0 leads to $\tilde{w}_j$. More precisely, orient γ_1 so that it goes from z_1 to ∞, and orient $[z_2, z_{j+1}]$ so that it goes from z_2 to z_{j+1}. Then this algebraic intersection number is the sum over all intersections of the numbers $+1$ if the leg crosses the segment from left to right and -1 if it crosses from right to left.

The number $I(\gamma_1, [z_2, z_{j+1}])$ is the sum, over all the oriented line segments s_j of the leg, of the intersections $s_j \cap [z_2, z_{j+1}]$. Thus we see that for this part of the program, the central subroutine is the one which, given two line segments defined by their endpoints, tells whether they intersect, and if so, with which orientation. This subroutine uses about half the computer time, and it pays to do it efficiently.

Lifting the Legs. The inverse image of a polygonal leg is a curve formed of segments of curves which look like hyperbolas (and are hyperbolas for $d = 2$). We need to replace these by polygonal curves in the same homotopy classes.

It may not be correct to simply join the inverse images of the vertices. If an endpoint $\tilde{z}_j$ (which we have already found) is inside the convex hull of a segment of "hyperbola", then replacing the segment of "hyperbola" by the segment of straight line would change the homotopy class of the leg.

So one must discover whether that is the case or not; the computer uses the other half of its time here. It requires three checks:

(1) that $\tilde{z}_j$ is on the same side as $-d$ of the line,
(2) that it is inside the "hyperbola", which means that z_{j+1} and 0 are on opposite sides of the line carrying the segment of which we are computing an inverse image, and finally
(3) that seen from $-d$, $\tilde{z}_j$ and the endpoints of the segment of hyperbola make an angle smaller than π/d.

Each one of these checks is verifying one simple inequality (and it really pays to encode it economically).

Of course, one needs to decide what to do if a segment of hyperbola does "enclose" an endpoint, which fortunately doesn't happen often. Recursively add points cutting the hyperbola into smaller pieces until they don't. This may mean that a leg is encoded by a lot of segments, and it is best to prune before continuing.

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