MODULAR FORMS AND SOME CASES OF THE INVERSE GALOIS PROBLEM

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Abstract. We prove new cases of the inverse Galois problem by considering the residual Galois representations arising from a fixed newform. Specific choices of weight 3 newforms will show that there are Galois extensions of $\mathbb{Q}$ with Galois group $\operatorname{PSL}_2(\mathbb{F}_p)$ for all primes p and $\operatorname{PSL}_2(\mathbb{F}_{p^3})$ for all odd primes $p \equiv \pm 2, \pm 3, \pm 4, \pm 6 \pmod{13}$.

1. Introduction

The Inverse Galois Problem asks whether every finite group is isomorphic to the Galois group of some extension of $\mathbb{Q}$. There has been much work on using modular forms to realize explicit simple groups of the form $\operatorname{PSL}_2(\mathbb{F}_{\ell^r})$ as Galois groups of extensions of $\mathbb{Q}$, cf. [Rib75],[RV95], [DV00], [Die01], [Die08]. For example, [DV00, §3.2] shows that $\operatorname{PSL}_2(\mathbb{F}_{\ell^2})$ occurs as a Galois group of an extension of $\mathbb{Q}$ for all primes ℓ in a explicit set of density $1 - 1/2^{10}$ (and for primes $\ell \leq 5000000$). Also it is shown in [DV00] that $\operatorname{PSL}_2(\mathbb{F}_{\ell^4})$ occurs as a Galois group of an extension of $\mathbb{Q}$ when $\ell \equiv 2, 3 \pmod 5$ or $\ell \equiv \pm 3, \pm 5, \pm 6, \pm 7 \pmod{17}$.

The goal of this paper is to try to realize more groups of the form $\operatorname{PSL}_2(\mathbb{F}_{\ell^r})$ for *odd* r. We will achieve this by working with newforms of odd weight; the papers mentioned above focus on even weight modular forms (usual weight 2). We will give background and describe the general situation in §1.1. In §1.2 and §1.3, we will use specific newforms of weight 3 to realize many groups of the form $\operatorname{PSL}_2(\mathbb{F}_{\ell^r})$ with r equal to 1 and 3, respectively.

Throughout the paper, we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and define the group $G := \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For a ring R, we let $\operatorname{PSL}_2(R)$ and $\operatorname{PGL}_2(R)$ be the quotient of $\operatorname{SL}_2(R)$ and $\operatorname{GL}_2(R)$, respectively, by its subgroup of scalar matrices (in particular, this notation may disagree with the R-points of the corresponding group scheme PSL_2 or PGL_2).

1.1. **General results.** Fix a non-CM newform $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ of weight $k > 1$ on $\Gamma_1(N)$, where the a_n are complex numbers and $q = e^{2\pi i \tau}$ with τ a variable of the complex upper-half plane. Let $\varepsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be the nebentypus of f.

Let E be the subfield of $\mathbb{C}$ generated by the coefficients a_n; it is also generated by the coefficients a_p with primes $p \nmid N$. The field E is a number field and all the a_n are known to lie in its ring of integers $\mathcal{O}$. The image of ε lies in $E^\times$. Let K be the subfield of E generated by the algebraic integers $r_p := a_p^2/\varepsilon(p)$ for primes $p \nmid N$; denote its ring of integer by R.

Take any non-zero prime ideal Λ of $\mathcal{O}$ and denote by $\ell = \ell(\Lambda)$ the rational prime lying under Λ. Let E_Λ and $\mathcal{O}_\Lambda$ be the completions of E and $\mathcal{O}$, respectively, at Λ. From Deligne [Del71], we know that there is a continuous representation

$$\rho_\Lambda\colon G \to \operatorname{GL}_2(\mathcal{O}_\Lambda)$$

such that for each prime $p \nmid N\ell$, the representation ρ_Λ is unramified at p and satisfies

$$\operatorname{tr}(\rho_\Lambda(\operatorname{Frob}_p)) = a_p \quad \text{and} \quad \det(\rho_\Lambda(\operatorname{Frob}_p)) = \varepsilon(p)p^{k-1}. \tag{1.1}$$

The representation ρ_Λ is uniquely determined by the conditions (1.1) up to conjugation by an element of $\operatorname{GL}_2(E_\Lambda)$. By composing ρ_Λ with the natural projection arising from the reduction map

$\mathcal{O}_\Lambda \to \mathbb{F}_\Lambda := \mathcal{O}/\Lambda$, we obtain a representation

$$\bar{\rho}_\Lambda \colon G \to \mathrm{GL}_2(\mathbb{F}_\Lambda).$$

Composing $\bar{\rho}_\Lambda$ with the natural quotient map $\mathrm{GL}_2(\mathbb{F}_\Lambda) \to \mathrm{PGL}_2(\mathbb{F}_\Lambda)$, we obtain a homomorphism

$$\bar{\rho}_\Lambda^{\mathrm{proj}} \colon G \to \mathrm{PGL}_2(\mathbb{F}_\Lambda).$$

Define the field $\mathbb{F}_\lambda := R/\lambda$, where $\lambda := \Lambda \cap R$. There are natural injective homomorphisms $\mathrm{PSL}_2(\mathbb{F}_\lambda) \hookrightarrow \mathrm{PGL}_2(\mathbb{F}_\lambda) \hookrightarrow \mathrm{PGL}_2(\mathbb{F}_\Lambda)$ and $\mathrm{PSL}_2(\mathbb{F}_\Lambda) \hookrightarrow \mathrm{PGL}_2(\mathbb{F}_\Lambda)$ that we shall view as inclusions.

The main task of this paper is to describe the group $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ for all Λ outside of some *explicit* set. The following theorem of Ribet gives two possibilities for $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ for all but finitely many Λ; we will give a proof of Theorem 1.1 in §4 that allows one to compute such a set S.

Theorem 1.1 (Ribet)**.** *There is a finite set S of non-zero prime ideals of R such that if Λ is a non-zero prime ideal of $\mathcal{O}$ with $\lambda := R \cap \Lambda \notin S$, then the group $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to either $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$.*

Proof. As noted in §3 of [DW11], this is an easy consequence of [Rib85]. □

We now explain how to distinguish the two possibilities from Theorem 1.1. Let $L \subseteq \mathbb{C}$ be the extension of K generated by the square roots of the values $r_p = a_p^2/\varepsilon(p)$ with $p \nmid N$; it is a finite extension of K (moreover, it is contained in a finite cyclotomic extension of E).

Theorem 1.2. *Let Λ be a non-zero prime ideal of $\mathcal{O}$ such that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$, where $\lambda = \Lambda \cap R$. After conjugating $\bar{\rho}_\Lambda$, we may assume that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) \subseteq \mathrm{PGL}_2(\mathbb{F}_\lambda)$. Let ℓ be the rational prime lying under Λ.*

- *(i) If k is odd, then $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) = \mathrm{PSL}_2(\mathbb{F}_\lambda)$ if and only if λ splits completely in L.*
- *(ii) If k is even and $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is even, then $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) = \mathrm{PSL}_2(\mathbb{F}_\lambda)$ if and only if λ splits completely in L.*
- *(iii) If k is even, $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is odd, and $\ell \nmid N$, then $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) = \mathrm{PGL}_2(\mathbb{F}_\lambda)$.*

Remark 1.3. From Theorem 1.2, we see that it is more challenging to produce Galois extensions of $\mathbb{Q}$ with Galois group $\mathrm{PSL}_2(\mathbb{F}_{\ell^r})$ with odd r if we focus solely on newforms with k even. However, it is still possible to obtain such groups in the excluded case $\ell | N$.

1.2. **An example realizing the groups PSL$_2(\mathbb{F}_\ell)$.** We now give an example that realizes the simple groups $\mathrm{PSL}_2(\mathbb{F}_\ell)$ as Galois groups of an extension of $\mathbb{Q}$ for all primes $\ell \geq 7$. Let $f = \sum_{n=1}^\infty a_n q^n$ be a non-CM newform of weight 3, level $N = 27$ and nebentypus $\varepsilon(a) = \left(\frac{-3}{a}\right)$. We can choose f so that[1]

$$f = q + 3iq^2 - 5q^4 - 3iq^5 + 5q^7 - 3iq^8 + 9q^{10} - 15iq^{11} - 10q^{13} + \cdots;$$

the other possibility for f is its complex conjugate $\sum_n \bar{a}_n q^n$.

The subfield E of $\mathbb{C}$ generated by the coefficients a_n is $\mathbb{Q}(i)$. Take any prime $p \neq 3$. We will see that $\bar{a}_p = \varepsilon(p)^{-1} a_p$. Therefore, a_p or ia_p belongs to $\mathbb{Z}$ when $\varepsilon(p)$ is 1 or -1, respectively, and hence $r_p = a_p^2/\varepsilon(p)$ is a square in $\mathbb{Z}$. Therefore, $L = K = \mathbb{Q}$.

In §6.1, we shall verify that Theorem 1.1 holds with $S = \{2, 3, 5\}$. Take any prime $\ell \geq 7$ and prime $\Lambda \subseteq \mathbb{Z}[i]$ dividing ℓ. Theorem 1.2 with $L = K = \mathbb{Q}$ implies that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_\ell)$. The following theorem is now an immediate consequence (it is easy to prove directly for the group $\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$).

[1] More explicitly, take $f = \frac{i}{2} g\theta_0 - \frac{1+i}{2} g\theta_1 + \frac{3}{2} g\theta_2$, where $g := q\prod_{n\geq 1}(1-q^{3n})^2(1-q^{9n})^2$ and $\theta_j := \sum_{x,y\in\mathbb{Z}} q^{3^j(x^2+xy+y^2)}$, cf. [Ser87, p. 228].

Theorem 1.4. *For each prime $\ell \geq 5$, there is a Galois extension $K/\mathbb{Q}$ such that $\operatorname{Gal}(K/\mathbb{Q})$ is isomorphic to the simple group $\operatorname{PSL}_2(\mathbb{F}_\ell)$.* □

Remark 1.5.

(i) In §5.5 of [Ser87], J-P. Serre describes the image of $\bar{\rho}_{(7)}$ and proves that it gives rise to a $\operatorname{PSL}_2(\mathbb{F}_7)$-extension of $\mathbb{Q}$, however, he does not consider the image modulo other primes. Note that Serre was actually giving an example of his conjecture, so he started with the $\operatorname{PSL}_2(\mathbb{F}_7)$-extension and then found the newform f.

(ii) Theorem 1.4 was first proved by the author in [Zyw14] by considering the Galois action on the second étale cohomology of a specific surface. One can show that the Galois extensions of [Zyw14] could also be constructed by first starting with an appropriate newform of weight 3 and level 32.

1.3. **Another example.** We now give an example with $K \neq \mathbb{Q}$. Additional details will be provided in §6.2. Let $f = \sum_n a_n q^n$ be a non-CM newform of weight 3, level $N = 160$ and nebentypus $\varepsilon(a) = \left(\frac{-5}{a}\right)$.

Take E, K, L, R and $\mathcal{O}$ as in §1.1. We will see in §6.2 that $E = K(i)$ and that K is the unique cubic field in $\mathbb{Q}(\zeta_{13})$. We will also observe that $L = K$.

Take any odd prime ℓ congruent to ± 2, ± 3, ± 4 or ± 6 modulo 13. Let Λ be any prime ideal of $\mathcal{O}$ dividing ℓ and set $\lambda = \Lambda \cap R$. The assumption on ℓ modulo 13 implies that $\lambda = \ell R$ and that $\mathbb{F}_\lambda \cong \mathbb{F}_{\ell^3}$. In §6.2, we shall compute a set S as in Theorem 1.1 which does not contain λ. Theorem 1.2 with $L = K$ implies that $\bar{\rho}_\Lambda^{\text{proj}}(G)$ is isomorphic to $\operatorname{PSL}_2(\mathbb{F}_\lambda) \cong \operatorname{PSL}_2(\mathbb{F}_{\ell^3})$. The following is an immediate consequence.

Theorem 1.6. *If ℓ is an odd prime congruent to ± 2, ± 3, ± 4 or ± 6 modulo 13, then the simple group $\operatorname{PSL}_2(\mathbb{F}_{\ell^3})$ occurs as the Galois group of an extension of $\mathbb{Q}$.*

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2. The fields K and L

Take a newform f with notation and assumptions as in §1.1.

2.1. **The field K.** Let Γ be the set of automorphisms γ of the field E for which there is a primitive Dirichlet character χ_γ that satisfies

$$\gamma(a_p) = \chi_\gamma(p) a_p \tag{2.1}$$

for all primes $p \nmid N$. The set of primes p with $a_p \neq 0$ has density 1 since f is non-CM, so the image of χ_γ lies in $E^\times$ and the character χ_γ is uniquely determined from γ.

Define M to be N or $4N$ if N is odd or even, respectively. The conductor of χ_γ divides M, cf. [Mom81, Remark 1.6]. Moreover, there is a quadratic Dirichlet character α with conductor dividing M and an integer i such that χ_γ is the primitive character coming from $\alpha\varepsilon^i$, cf. [Mom81, Lemma 1.5(i)].

For each prime $p \nmid N$, we have $\bar{a}_p = \varepsilon(p)^{-1} a_p$, cf. [Rib77, p. 21], so complex conjugation induces an automorphism γ of E and χ_γ is the primitive character coming from ε. In particular, $\Gamma \neq 1$ if ε is non-trivial.

Remark 2.1. More generally, we could have instead considered an embedding $\gamma\colon E \to \mathbb{C}$ and a Dirichlet character χ_γ such that (2.1) holds for all sufficiently large primes p. This gives the same twists, since $\gamma(E) = E$ and the character χ_γ is unramified at primes $p \nmid N$, cf. [Mom81, Remark 1.3].

The set Γ is in fact an abelian subgroup of $\mathrm{Aut}(E)$, cf. [Mom81, Lemma 1.5(ii)]. Denote by E^Γ the fixed field of E by Γ.

Lemma 2.2.

(i) We have $K = E^\Gamma$ and hence $\mathrm{Gal}(E/K) = \Gamma$.
(ii) There is a prime $p \nmid N$ such that $K = \mathbb{Q}(r_p)$.

Proof. Take any $p \nmid N$. For each $\gamma \in \Gamma$, we have

$$\gamma(r_p) = \gamma(a_p^2)/\gamma(\varepsilon(p)) = \chi_\gamma(p)^2 a_p^2/\gamma(\varepsilon(p)) = a_p^2/\varepsilon(p) = r_p,$$

where we have used that $\chi_\gamma(p)^2 = \gamma(\varepsilon(p))/\varepsilon(p)$, cf. [Mom81, proof of Lemma 1.5(ii)]. This shows that r_p belong in E^Γ and hence $K \subseteq E^\Gamma$ since $p \nmid N$ was arbitrary. To complete the proof of the lemma, it thus suffices to show that $E^\Gamma = \mathbb{Q}(r_p)$ for some prime $p \nmid N$.

For $\gamma \in \Gamma$, let $\widetilde{\chi}_\gamma \colon G \to \mathbb{C}^\times$ be the continuous character such that $\widetilde{\chi}_\gamma(\mathrm{Frob}_p) = \chi_\gamma(p)$ for all $p \nmid N$. Define the group $H = \bigcap_{\gamma\in\Gamma} \ker \widetilde{\chi}_\gamma$; it is an open normal subgroup of G with G/H is abelian. Let $\mathcal{K}$ be the subfield of $\overline{\mathbb{Q}}$ fixed by H; it is a finite abelian extension of $\mathbb{Q}$.

Fix a prime ℓ and a prime ideal $\Lambda|\ell$ of $\mathcal{O}$. In the proof of Theorem 3.1 of [Rib85], Ribet proved that $E^\Gamma = \mathbb{Q}(a_v^2)$ for a positive density set of finite place $v \nmid N\ell$ of $\mathcal{K}$, where $a_v := \mathrm{tr}(\rho_\Lambda(\mathrm{Frob}_v))$. There is thus a finite place $v \nmid N\ell$ of $\mathcal{K}$ of degree 1 such that $E^\Gamma = \mathbb{Q}(a_v^2)$. We have $a_v = a_p$, where p is the rational prime that v divides, so $E^\Gamma = \mathbb{Q}(a_p^2)$. Since v has degree 1 and $\mathcal{K}/\mathbb{Q}$ is abelian, the prime p must split completely in $\mathcal{K}$ and hence $\chi_\gamma(p) = 1$ for all $\gamma \in \Gamma$; in particular, $\varepsilon(p) = 1$. Therefore, $E^\Gamma = \mathbb{Q}(r_p)$. ☐

2.2. **The field L.** Recall that we defined L to be the extension of K in $\mathbb{C}$ obtained by adjoining the square root of $r_p = a_p^2/\varepsilon(p)$ for all $p \nmid N$. The following allows one to find a finite set of generators for the extension L/K and gives a way to check the criterion of Theorem 1.2.

Lemma 2.3.

(i) Choose primes $p_1, \dots, p_m \nmid N$ that generate the group $(\mathbb{Z}/M\mathbb{Z})^\times$ and satisfy $r_{p_i} \neq 0$ for all $1 \le i \le m$. Then $L = K(\sqrt{r_{p_1}}, \dots, \sqrt{r_{p_m}})$.
(ii) Take any non-zero prime ideal λ of R that does not divide 2. Let $p_1, \dots, p_m$ be primes as in (i). Then the following are equivalent:
  (a) λ splits completely in L,
  (b) for all $p \nmid N$, r_p is a square in K_λ,
  (c) for all $1 \le i \le m$, r_{p_i} is a square in K_λ.

Proof. Take any prime $p \nmid N$. To prove part (i), it suffices to show that $\sqrt{r_p}$ belongs to the field $L' := K(\sqrt{r_{p_1}}, \dots, \sqrt{r_{p_m}})$. This is obvious if $r_p = 0$, so assume that $r_p \neq 0$. Since the p_i generate $(\mathbb{Z}/M\mathbb{Z})^\times$ by assumption, there are integers $e_i \ge 0$ such that $p \equiv p_1^{e_1} \cdots p_m^{e_m} \pmod M$. Take any $\gamma \in \Gamma$. Using that the conductor of χ_γ divides M and (2.1), we have

$$\gamma\Big(\frac{a_p}{\prod_i a_{p_i}^{e_i}}\Big) = \frac{\chi_\gamma(p)}{\chi_\gamma(\prod_i p_i^{e_i})} \cdot \frac{a_p}{\prod_i a_{p_i}^{e_i}} = \frac{\chi_\gamma(p)}{\chi_\gamma(p)} \cdot \frac{a_p}{\prod_i a_{p_i}^{e_i}} = \frac{a_p}{\prod_i a_{p_i}^{e_i}},$$

Since $E^\Gamma = K$ by Lemma 2.2(i), the value $a_p/\prod_i a_{p_i}^{e_i}$ belongs to K; it is non-zero since $r_p \neq 0$ and $r_{p_i} \neq 0$. We have $\varepsilon(p) = \prod_i \varepsilon(p_i)^{e_i}$ since the conductor of ε divides M. Therefore,

$$\frac{r_p}{\prod_i r_{p_i}^{e_i}} = \frac{a_p^2}{\prod_i (a_{p_i}^2)^{e_i}} = \left(\frac{a_p}{\prod_i a_{p_i}^{e_i}}\right)^2 \in (K^\times)^2.$$

This shows that $\sqrt{r_p}$ is contained in L' as desired. This proves (i); part (ii) is an easy consequence of (i). ☐

Remark 2.4. Finding primes p_i as in Lemma 2.3(i) is straightforward since $r_p \neq 0$ for all p outside a set of density 0 (and the primes representing each class $a \in (\mathbb{Z}/M\mathbb{Z})^\times$ have positive density). Lemma 2.3(ii) gives a straightforward way to check if λ splits completely in L. Let e_i be the λ-adic valuation of r_{p_i} and let π be a uniformizer of K_λ; then r_{p_i} is a square in K_λ if and only if e is even and the image of r_{p_i}/π^{e_i} in $\mathbb{F}_\lambda$ is a square.

3. Proof of Theorem 1.2

We may assume that $\bar\rho_\Lambda^{\mathrm{proj}}(G)$ is $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$. For any $n \geq 1$, the group $\mathrm{GL}_2(\mathbb{F}_{2^n})$ is generated by $\mathrm{SL}_2(\mathbb{F}_{2^n})$ and its scalar matrices, so $\mathrm{PSL}_2(\mathbb{F}_{2^n}) = \mathrm{PGL}_2(\mathbb{F}_{2^n})$. The theorem is thus trivial when $\ell = 2$, so we may assume that ℓ is odd.

Take any $\alpha \in \mathrm{PGL}_2(\mathbb{F}_\lambda) \subseteq \mathrm{PGL}_2(\mathbb{F}_\Lambda)$ and choose any matrix $A \in \mathrm{GL}_2(\mathbb{F}_\Lambda)$ whose image in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ is α. The value $\mathrm{tr}(A)^2/\det(A)$ does not depend on the choice of A and lies in $\mathbb{F}_\lambda$ (since we can choose A in $\mathrm{GL}_2(\mathbb{F}_\lambda)$); by abuse of notation, we denote this common value by $\mathrm{tr}(\alpha)^2/\det(\alpha)$.

Lemma 3.1. *Suppose that $p \nmid N\ell$ is a prime for which $r_p \not\equiv 0 \pmod{\lambda}$. Then $\bar\rho_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p)$ is contained in $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ if and only if the image of $a_p^2/(\varepsilon(p)p^{k-1}) = r_p/p^{k-1}$ in $\mathbb{F}_\lambda^\times$ is a square.*

Proof. Define $A := \bar\rho_\Lambda(\mathrm{Frob}_p)$ and $\alpha := \bar\rho_\Lambda(\mathrm{Frob}_p)$; the image of A in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ is α. The value $\xi_p := \mathrm{tr}(\alpha)^2/\det(\alpha) = \mathrm{tr}(A)^2/\det(A)$ agrees with the image of $a_p^2/(\varepsilon(p)p^{k-1}) = r_p/p^{k-1}$ in $\mathbb{F}_\Lambda$. Since $r_p \in R$ is non-zero modulo λ by assumption, the value ξ_p lies in $\mathbb{F}_\lambda^\times$. Fix a matrix $A_0 \in \mathrm{GL}_2(\mathbb{F}_\lambda)$ whose image in $\mathrm{PGL}_2(\mathbb{F}_\lambda)$ is α; we have $\xi_p = \mathrm{tr}(A_0)^2/\det(A_0)$. Since $\xi_p \neq 0$, we find that ξ_p and $\det(A_0)$ lies in the same coset in $\mathbb{F}_\lambda^\times/(\mathbb{F}_\lambda^\times)^2$.

The determinant gives rise to a homomorphism $d\colon \mathrm{PGL}_2(\mathbb{F}_\lambda) \to \mathbb{F}_\lambda^\times/(\mathbb{F}_\lambda^\times)^2$ whose kernel is $\mathrm{PSL}_2(\mathbb{F}_\lambda)$. Define the character

$$\xi\colon G \xrightarrow{\bar\rho_\Lambda^{\mathrm{proj}}} \mathrm{PGL}_2(\mathbb{F}_\lambda) \xrightarrow{d} \mathbb{F}_\lambda^\times/(\mathbb{F}_\lambda^\times)^2.$$

We have $\xi(\mathrm{Frob}_p) = \det(A_0)\cdot(\mathbb{F}_\lambda^\times)^2 = \xi_p \cdot (\mathbb{F}_\lambda^\times)^2$. So $\xi(\mathrm{Frob}_p) = 1$, equivalently $\bar\rho_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p) \in \mathrm{PSL}_2(\mathbb{F}_\lambda)$, if and only if $\xi_p \in \mathbb{F}_\lambda^\times$ is a square. □

Let M be the integer from §2.1.

Lemma 3.2. *For each $a \in (\mathbb{Z}/M\ell\mathbb{Z})^\times$, there is a prime $p \equiv a \pmod{M\ell}$ such that $r_p \not\equiv 0 \pmod{\lambda}$.*

Proof. Set $H = \bar\rho_\Lambda^{\mathrm{proj}}(G)$; it is $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$ by assumption. Let H' be the commutator subgroup of H. We claim that for each coset κ of H' in H, there exists an $\alpha \in \kappa$ with $\mathrm{tr}(\alpha)^2/\det(\alpha) \neq 0$. If $H' = \mathrm{PSL}_2(\mathbb{F}_\lambda)$, then the claim is easy; note that for any $t \in \mathbb{F}_\lambda$ and $d \in \mathbb{F}_\lambda^\times$, there is a matrix in $\mathrm{GL}_2(\mathbb{F}_\lambda)$ with trace t and determinant d. When $\#\mathbb{F}_\lambda \neq 3$, the group $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ is non-abelian and simple, so $H' = \mathrm{PSL}_2(\mathbb{F}_\lambda)$. When $\#\mathbb{F}_\lambda = 3$ and $H = \mathrm{PGL}_2(\mathbb{F}_\lambda)$, we have $H' = \mathrm{PSL}_2(\mathbb{F}_\lambda)$. It thus suffices to prove the claim in the case where $\mathbb{F}_\lambda = \mathbb{F}_3$ and $H = \mathrm{PSL}_2(\mathbb{F}_3)$. In this case, H' is the unique subgroup of H of index 3 and the cosets of H/H' are represented by $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ with $b \in \mathbb{F}_3$. The claim is now immediate in this remaining case.

Let $\chi\colon G \twoheadrightarrow (\mathbb{Z}/M\ell\mathbb{Z})^\times$ be the cyclotomic character that satisfies $\chi(\mathrm{Frob}_p) \equiv p \pmod{M\ell}$ for all $p \nmid M\ell$. The set $\bar\rho_\Lambda(\chi^{-1}(a))$ is thus the union of cosets of H' in H. By the claim above, there exists an $\alpha \in \bar\rho_\Lambda^{\mathrm{proj}}(\chi^{-1}(a))$ with $\mathrm{tr}(\alpha)^2/\det(\alpha) \neq 0$. By the Chebotarev density theorem, there is a prime $p \nmid M\ell$ satisfying $p \equiv a \pmod{M\ell}$ and $\bar\rho_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p) = \alpha$. The lemma follows since r_p/p^{k-1} modulo λ agrees with $\mathrm{tr}(\alpha)^2/\det(\alpha) \neq 0$. □

Case 1: *Assume that k is odd or $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is even.*

First suppose that $\bar\rho_\Lambda^{\mathrm{proj}}(G) = \mathrm{PSL}_2(\mathbb{F}_\lambda)$. By Lemma 3.2, there are primes $p_1, \ldots, p_m \nmid N\ell$ that generate the group $(\mathbb{Z}/M\mathbb{Z})^\times$ and satisfy $r_{p_i} \not\equiv 0 \pmod{\lambda}$ for all $1 \leq i \leq m$. By Lemma 3.1 and the assumption $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) = \mathrm{PSL}_2(\mathbb{F}_\lambda)$, the image of $r_{p_i}/{p_i}^{k-1}$ in $\mathbb{F}_\lambda$ is a non-zero square for all $1 \le i \le m$. For each $1 \le i \le m$, the assumption that k is odd or $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is even implies that p_i^{k-1} is a square in $\mathbb{F}_\lambda$ and hence the image of r_{p_i} in $\mathbb{F}_\lambda$ is a non-zero square. Since $\lambda \nmid 2$, we deduce that each r_{p_i} is a square in K_λ. By Lemma 2.3(ii), the prime λ splits completely in L.

Now suppose that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) = \mathrm{PGL}_2(\mathbb{F}_\lambda)$. There exists an element $\alpha \in \mathrm{PGL}_2(\mathbb{F}_\lambda) - \mathrm{PSL}_2(\mathbb{F}_\lambda)$ with $\mathrm{tr}(\alpha)^2/\det(\alpha) \ne 0$. By the Chebotarev density theorem, there is a prime $p \nmid N\ell$ such that $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p) = \alpha$. We have $r_p \equiv \mathrm{tr}(\alpha)^2/\det(\alpha) \not\equiv 0 \pmod{\lambda}$. Since $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p) \notin \mathrm{PSL}_2(\mathbb{F}_\lambda)$, Lemma 3.1 implies that the image of r_p/p^{k-1} in $\mathbb{F}_\lambda$ is not a square. Since k is odd or $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is even, the image of r_p in $\mathbb{F}_\lambda$ is not a square. Therefore, r_p is not a square in K_λ. By Lemma 2.3(ii), we deduce that λ does not split completely in L.

Case 2: *Assume that k is even, $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is odd, and $\ell \nmid N$.*

Since $\ell \nmid N$, there is an integer $a \in \mathbb{Z}$ such that $a \equiv 1 \pmod{M}$ and a is not a square modulo ℓ. By Lemma 3.2, there is a prime $p \equiv a \pmod{M\ell}$ such that $r_p \not\equiv 0 \pmod{\lambda}$.

We claim that $a_p \in R$ and $\varepsilon(p) = 1$. With notation as in §2.1, take any $\gamma \in \Gamma$. Since the conductor of χ_γ divides M and $p \equiv 1 \pmod{M}$, we have $\gamma(a_p) = \chi_\gamma(p) a_p = a_p$. Since $\gamma \in \Gamma$ was arbitrary, we have $a_p \in K$ by Lemma 2.2. Therefore, $a_p \in R$ since it is an algebraic integer. We have $\varepsilon(p) = 1$ since $p \equiv 1 \pmod{N}$.

Since $a_p \in R$ and $r_p \not\equiv 0 \pmod{\lambda}$, the image of a_p^2 in $\mathbb{F}_\lambda$ is a non-zero square. Since k is even, p^k is a square in $\mathbb{F}_\lambda$. Since p is not a square modulo ℓ and $[\mathbb{F}_\lambda : \mathbb{F}_\ell]$ is odd, the prime p is not a square in $\mathbb{F}_\lambda$. So the image of

$$a_p^2/(\varepsilon(p)p^{k-1}) = p \cdot a_p^2/p^k$$

in $\mathbb{F}_\lambda$ is not a square. Lemma 3.1 implies that $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p) \notin \mathrm{PSL}_2(\mathbb{F}_\lambda)$. Therefore, $\bar{\rho}_\Lambda^{\mathrm{proj}}(G) = \mathrm{PGL}_2(\mathbb{F}_\lambda)$.

4. An effective version of Theorem 1.1

Take a newform f with notation and assumptions as in §1.1. Let λ be a non-zero prime ideal of R and let ℓ be the prime lying under λ. Let k_λ be the subfield of $\mathbb{F}_\lambda$ generated by the image of r_p modulo λ with primes $p \nmid N\ell$. Take any prime ideal Λ of $\mathcal{O}$ that divides λ.

In this section, we describe how to compute an explicit finite set S of prime ideals of R as in Theorem 1.1. First some simple definitions:

- Let $\mathbb{F}$ be an extension of $\mathbb{F}_\Lambda$ of degree $\gcd(2, \ell)$.
- Let $e_0 = 0$ if $\ell \ge k-1$ and $\ell \nmid N$, and $e_0 = \ell - 2$ otherwise.
- Let $e_1 = 0$ if N is odd, and $e_1 = 1$ otherwise.
- Let $e_2 = 0$ if $\ell \ge 2k$, and $e_2 = 1$ otherwise.
- Define $\mathcal{M} = 4^{e_1} \ell^{e_2} \prod_{p|N} p$.

We will prove the following in §5.

Theorem 4.1. *Suppose that all the following conditions hold:*

(a) *For every integer $0 \le j \le e_0$ and character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}^\times$, there is a prime $p \nmid N\ell$ such that $\chi(p)p^j \in \mathbb{F}$ is not a root of the polynomial $x^2 - a_p x + \varepsilon(p)p^{k-1} \in \mathbb{F}_\Lambda[x]$.*

(b) *For every non-trivial character $\chi\colon (\mathbb{Z}/\mathcal{M}\mathbb{Z})^\times \to \{\pm 1\}$, there is a prime $p \nmid N\ell$ such that $\chi(p) = -1$ and $r_p \not\equiv 0 \pmod{\lambda}$.*

(c) *If $\#k_\lambda \notin \{4, 5\}$, then at least one of the following hold:*
   - *$\ell > 5k - 4$ and $\ell \nmid N$,*
   - *$\ell \equiv 0, \pm 1 \pmod{5}$ and $\#k_\lambda \ne \ell$,*
   - *$\ell \equiv \pm 2 \pmod{5}$ and $\#k_\lambda \ne \ell^2$,*

- *there is a prime $p \nmid N\ell$ such that the image of $a_p^2/(\varepsilon(p)p^{k-1})$ in $\mathbb{F}_\lambda$ is not equal to 0, 1 and 4, and is not a root of $x^2 - 3x + 1$.*

(d) If $\#k_\lambda \notin \{3, 5, 7\}$, then at least one of the following hold:
- *$\ell > 4k - 3$ and $\ell \nmid N$,*
- *$\#k_\lambda \neq \ell$,*
- *there is a prime $p \nmid N\ell$ such that the image of $a_p^2/(\varepsilon(p)p^{k-1})$ in $\mathbb{F}_\lambda$ is not equal to 0, 1, 2 and 4.*

(e) If $\#k_\lambda \in \{5, 7\}$, then for every non-trivial character $\chi\colon (\mathbb{Z}/4^{e_1}\ell N\mathbb{Z})^\times \to \{\pm 1\}$ there is a prime $p \nmid N\ell$ such that $\chi(p) = 1$ and $a_p^2/(\varepsilon(p)p^{k-1}) \equiv 2 \pmod{\lambda}$.

Then the group $\bar\rho_\Lambda^{\mathrm{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(k_\lambda)$ or $\mathrm{PGL}_2(k_\lambda)$.

Remark 4.2. Note that the above conditions simplify greatly if one also assumes that $\ell \nmid N$ and $\ell > 5k - 4$.

Though we will not prove it, Theorem 4.1 has been stated so that all the conditions (a)–(e) hold if and only if $\bar\rho_\Lambda^{\mathrm{proj}}(G)$ is conjugate to $\mathrm{PSL}_2(k_\lambda)$ or $\mathrm{PGL}_2(k_\lambda)$. In particular, after considering enough primes p, one will obtain the minimal set S of Theorem 1.1 (one could use an effective version of Chebotarev density to make this a legitimate algorithm).

Let us now describe how to compute a set of exceptional primes as in Theorem 1.1. Define $M = N$ if N is odd and $M = 4N$ otherwise. Set $\mathcal{M}' := 4^{e_1} \prod_{p|N} p$. We first choose some primes:

- Let $q_1, \ldots, q_n$ be primes congruent to 1 modulo N.
- Let $p_1, \ldots, p_m \nmid N$ be primes with $r_{p_i} \neq 0$ such that for every non-trivial character $\chi\colon (\mathbb{Z}/\mathcal{M}'\mathbb{Z})^\times \to \{\pm 1\}$, we have $\chi(p_i) = -1$ for some $1 \leq i \leq m$.
- Let $p_0 \nmid N$ be a prime such that $\mathbb{Q}(r_{p_0}) = K$.

That such primes $p_1, \ldots, p_m$ exist is clear since the set of primes p with $r_p \neq 0$ has density 1. That such a prime q exists follows from Lemma 2.2 (the set of such q actually has positive density). Define the ring $R' := \mathbb{Z}[a_q^2/\varepsilon(q)]$; it is an order in R.

Define S to be the set of non-zero primes λ of R, dividing a rational prime ℓ, that satisfy one of the following conditions:

- $\ell \leq 5k - 4$ or $\ell \leq 7$,
- $\ell | N$,
- for all $1 \leq i \leq n$, we have $\ell = q_i$ or $r_{q_i} \equiv (1 + q_i^{k-1})^2 \pmod{\lambda}$,
- for some $1 \leq i \leq m$, we have $\ell = p_i$ or $r_{p_i} \equiv 0 \pmod{\lambda}$,
- $\ell = q$ or ℓ divides $[R : R']$.

Note that the set S is *finite* (the only part that is not immediate is that $r_{q_i} - (1 + q_i^{k-1})^2 \neq 0$; this follows from Deligne's bound $|r_{q_i}| = |a_{q_i}| \leq 2q_i^{(k-1)/2}$ and $k > 1$). The following is our effective version of Theorem 1.1.

Theorem 4.3. *Take any non-zero prime ideal $\lambda \notin S$ of R and let Λ be any prime of $\mathcal{O}$ dividing λ. Then the group $\bar\rho_\Lambda^{\mathrm{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to either $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$.*

Proof. Let ℓ be the rational prime lying under λ. We shall verify the conditions of Theorem 4.1.

We first show that condition (a) of Theorem 4.1 holds. Take any integer $0 \leq j \leq e_0$ and character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}^\times = \mathbb{F}_\Lambda^\times$. We have $\ell > 5k - 4 > k - 1$ and $\ell \nmid N$ since $\lambda \notin S$, so $e_0 = 0$ and hence $j = 0$. Take any $1 \leq i \leq n$. Since $q_i \equiv 1 \pmod{N}$ and $j = 0$, we have $\chi(q_i)q_i^j = 1$ and $\varepsilon(q_i) = 1$. Since $\lambda \notin S$, we also have $q_i \nmid N\ell$ ($q_i \nmid N$ is immediate from the congruence imposed on q_i). If $\chi(q_i)q_i^j = 1$ was a root of $x^2 - a_{q_i}x + \varepsilon(q_i)q_i^{k-1}$ in $\mathbb{F}_\Lambda[x]$, then we would have $a_{q_i} \equiv 1 + q_i^{k-1} \pmod{\Lambda}$; squaring and using that $\varepsilon(q_i) = 1$, we deduce that $r_{q_i} \equiv (1 + q_i^{k-1})^2 \pmod{\lambda}$. Since

$\lambda \notin S$, we have $r_{q_i} \not\equiv (1+q_i^{k-1})^2 \pmod{\lambda}$ for some $1 \le i \le n$ and hence $\chi(q_i)q_i^j$ is not a root of $x^2 - a_{q_i}x + \varepsilon(q_i)q_i^{k-1}$.

We now show that condition (b) of Theorem 4.1 holds. We have $e_2 = 0$ since $\lambda \notin S$, and hence $\mathcal{M}' = \mathcal{M}$. Take any non-trivial character $\chi\colon (\mathbb{Z}/\mathcal{M}\mathbb{Z})^\times \to \{\pm 1\}$. By our choice of primes $p_1, \dots, p_m$, we have $\chi(p_i) = -1$ for some $1 \le i \le m$. The prime p_i does not divide $N\ell$ (that $p_i \neq \ell$ follows since $\lambda \notin S$). Since $\lambda \notin S$, we have $r_{p_i} \not\equiv 0 \pmod{\lambda}$.

Since $\lambda \notin S$, the prime $\ell \nmid N$ is greater that 7, $4k-3$ and $5k-4$. Conditions (c), (d) and (e) of Theorem 4.1 all hold.

Theorem 4.1 now implies that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to either $\mathrm{PSL}_2(k_\lambda)$ or $\mathrm{PGL}_2(k_\lambda)$. It remains to prove that $k_\lambda = \mathbb{F}_\lambda$. We have $q \neq \ell$ since $\lambda \notin S$. The image of the reduction map $R' \to \mathbb{F}_\lambda$ thus lies in k_λ. We have $\ell \nmid [R : R']$ since $\lambda \notin S$, so the map $R' \to \mathbb{F}_\lambda$ is surjective. Therefore, $k_\lambda = \mathbb{F}_\lambda$. $\square$

5. Proof of Theorem 4.1

5.1. **Some group theory.** Fix a prime ℓ and an integer $r \ge 1$.

A Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$ is a subgroup conjugate to the subgroup of upper triangular matrices.

A split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$ is a subgroup conjugate to the subgroup of diagonal matrices. A non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$ is a subgroup that is cyclic of order $(\ell^r)^2 - 1$. Fix a Cartan subgroup $\mathcal{C}$ of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$. Let $\mathcal{N}$ be the normalizer of $\mathcal{C}$ in $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$. One can show that $[\mathcal{N} : \mathcal{C}] = 2$ and that $\mathrm{tr}(g) = 0$ and g^2 is scalar for all $g \in \mathcal{N} - \mathcal{C}$.

Lemma 5.1. *Fix a prime ℓ and an integer $r \ge 1$. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$ and let $\overline{G}$ be its image in $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$. Then at least one of the following hold:*

(1) G is contained in a Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$,
(2) G is contained in the normalizer of a Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_{\ell^r})$,
(3) $\overline{G}$ is isomorphic to $\mathfrak{A}_4$,
(4) $\overline{G}$ is isomorphic to $\mathfrak{S}_4$,
(5) $\overline{G}$ is isomorphic to $\mathfrak{A}_5$,
(6) $\overline{G}$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ or $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for some integer s dividing r.

Proof. This can be deduced directly from a theorem of Dickson, cf. [Hup67, Satz 8.27], which will give the finite subgroups of $\mathrm{PSL}_2(\overline{\mathbb{F}}_\ell) = \mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$. The finite subgroups of $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$ have been worked out in [Fab12]. $\square$

Lemma 5.2. *Fix a prime ℓ and an integer $r \ge 1$. Take a matrix $A \in \mathrm{GL}_2(\mathbb{F}_{\ell^r})$ and let m be its order in $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$.*

(i) Suppose that $\ell \nmid m$. If m is 1, 2, 3 or 4, then $\mathrm{tr}(A)^2/\det(A)$ is 4, 0, 1 or 2, respectively. If $m = 5$, then $\mathrm{tr}(A)^2/\det(A)$ is a root of $x^2 - 3x + 1$.
(ii) If $\ell | m$, then $\mathrm{tr}(A)^2/\det(A) = 4$.

Proof. The quantity $\mathrm{tr}(A)^2/\det(A)$ does not change if we replace A by a scalar multiple or by a conjugate in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$. If $\ell \nmid m$, then we may thus assume that $A = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\zeta \in \overline{\mathbb{F}}_\ell$ has order m. We have $\mathrm{tr}(A)^2/\det(A) = \zeta + \zeta^{-1} + 2$, which is 4, 0, 1 or 2 when m is 1, 2, 3 or 4, respectively. If $m = 5$, then $\zeta + \zeta^{-1} + 2$ is a root of $x^2 - 3x + 1$. If $\ell | m$, then after conjugating and scaling, we may assume that $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and hence $\mathrm{tr}(A)^2/\det(A) = 4$. $\square$

5.2. **Image of inertia at ℓ.** Fix an inertia subgroup $\mathcal{I}_\ell$ of $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for the prime ℓ; it is uniquely defined up to conjugacy. The following gives important information concerning the representation $\bar{\rho}_\Lambda|_{\mathcal{I}_\ell}$ for large ℓ. Let $\chi_\ell\colon G \twoheadrightarrow \mathbb{F}_\ell^\times$ be the character such that for each prime $p \nmid \ell$, χ_ℓ is unramified at p and $\chi_\ell(\mathrm{Frob}_p) \equiv p \pmod{\ell}$.

Lemma 5.3. *Fix a prime $\ell \geq k-1$ for which $\ell \nmid 2N$. Let Λ be a prime ideal of $\mathcal{O}$ dividing ℓ and set $\lambda = \Lambda \cap R$.*

(i) Suppose that $r_\ell \not\equiv 0 \pmod{\lambda}$. After conjugating $\bar{\rho}_\Lambda$ by a matrix in $\mathrm{GL}_2(\mathbb{F}_\Lambda)$, we have

$$\bar{\rho}_\Lambda|_{\mathcal{I}_\ell} = \begin{pmatrix} \chi_\ell^{k-1}|_{\mathcal{I}_\ell} & * \\ 0 & 1 \end{pmatrix}$$

In particular, $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathcal{I}_\ell)$ contains a cyclic group of order $(\ell-1)/\gcd(\ell-1, k-1)$.

(ii) Suppose that $r_\ell \equiv 0 \pmod{\lambda}$. Then $\bar{\rho}_\Lambda|_{\mathcal{I}_\ell}$ is absolutely irreducible and $\bar{\rho}_\Lambda(\mathcal{I}_\ell)$ is cyclic. Furthermore, the group $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathcal{I}_\ell)$ is cyclic of order $(\ell+1)/\gcd(\ell+1, k-1)$.

Proof. Parts (i) and (ii) follow from Theorems 2.5 and Theorem 2.6, respectively, of [Edi92]; they are theorems of Deligne and Fontaine, respectively. We have used that $r_\ell = a_\ell^2/\varepsilon(\ell) \in R$ is congruent to 0 modulo λ if and only if $a_\ell \in \mathcal{O}$ is congruent to 0 modulo Λ. ◻

5.3. **Borel case.** Suppose that $\bar{\rho}_\Lambda(G)$ is a reducible subgroup of $\mathrm{GL}_2(\mathbb{F})$. There are thus characters $\psi_1, \psi_2 \colon G \to \mathbb{F}^\times$ such that after conjugating the $\mathbb{F}$-representation $G \xrightarrow{\bar{\rho}_\Lambda} \mathrm{GL}_2(\mathbb{F}_\Lambda) \subseteq \mathrm{GL}_2(\mathbb{F})$, we have

$$\bar{\rho}_\Lambda = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

The characters ψ_1 and ψ_2 are unramified at each prime $p \nmid N\ell$ since $\bar{\rho}_\Lambda$ is unramified at such primes.

Lemma 5.4. *For each $i \in \{1,2\}$, there is a unique integer $0 \leq m_i < \ell - 1$ such that $\psi_i \chi_\ell^{-m_i} \colon G \to \mathbb{F}^\times$ is unramified at all primes $p \nmid N$. If $\ell \geq k-1$ and $\ell \nmid N$, then m_1 or m_2 is 0.*

Proof. The existence and uniqueness of m_i is an easy consequence of class field theory for $\mathbb{Q}_\ell$. A choice of embedding $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_\ell$ induces an injective homomorphism $G_{\mathbb{Q}_\ell} := \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \hookrightarrow G$. Let $\mathbb{Q}_\ell^{\mathrm{ab}}$ be the maximal abelian extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Restricting ψ_i to $G_{\mathbb{Q}_\ell}$, we obtain a representation $\psi_i \colon G_{\mathbb{Q}_\ell}^{\mathrm{ab}} := \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ab}}/\mathbb{Q}_\ell) \to \mathbb{F}^\times$. By local class field, the inertia subgroup $\mathcal{I}$ of $G_{\mathbb{Q}_\ell}^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}_\ell^\times$. Since ℓ does not divide the cardinality of $\mathbb{F}^\times$, we find that $\psi_i|_{\mathcal{I}}$ factors through a group isomorphic to $\mathbb{F}_\ell^\times$. The character $\psi_i|_{\mathcal{I}}$ must agree with a power of $\chi_\ell|_{\mathcal{I}}$ since $\chi_\ell \colon G_{\mathbb{Q}_\ell} \to \mathbb{F}_\ell^\times$ satisfies $\chi_\ell(\mathcal{I}) = \mathbb{F}_\ell^\times$ and $\mathbb{F}_\ell^\times$ is cyclic.

The second part of the lemma follows immediately from Lemma 5.3. ◻

Take any $i \in \{1,2\}$. By Lemma 5.4, there is a unique $0 \leq m_i < \ell - 1$ such that the character

$$\tilde{\psi}_i := \psi_i \chi_\ell^{-m_i} \colon G \to \mathbb{F}^\times$$

is unramified at ℓ and at all primes $p \nmid N$. There is a character $\chi_i \colon (\mathbb{Z}/N_i\mathbb{Z})^\times \to \mathbb{F}^\times$ with $N_i \geq 1$ dividing some power of N and $\ell \nmid N_i$ such that $\tilde{\psi}_i(\mathrm{Frob}_p) = \chi_i(p)$ for all $p \nmid N\ell$. We may assume that χ_i is taken so that N_i is minimal.

Lemma 5.5. *The integer N_i divides N.*

Proof. We first recall the notion of an Artin conductor. Consider a representation $\rho \colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$, where V is a finite dimensional $\mathbb{F}$-vector space. Take any prime $p \neq \ell$. A choice of embedding $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$ induces an injective homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow G$. Choose any finite Galois extension $L/\mathbb{Q}_p$ for which $\rho(\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)) = \{I\}$. For each $i \geq 0$, let H_i be the i-th ramification subgroup of $\mathrm{Gal}(L/\mathbb{Q}_p)$ with respect to the lower numbering. Define the integer

$$f_p(\rho) = \sum_{i \geq 0} [H_0 : H_i]^{-1} \cdot \dim_{\mathbb{F}} V/V^{H_i}.$$

The Artin conductor of ρ is the integer $N(\rho) := \prod_{p \neq \ell} p^{f_p(\rho)}$.

Using that the character $\tilde{\psi}_i\colon G \to \mathbb{F}^\times$ is unramified at ℓ, one can verify that $N(\tilde{\psi}_i) = N_i$. Consider our representation $\bar{\rho}_\Lambda\colon G \to \mathrm{GL}_2(\mathbb{F})$. For a fixed prime $p \neq \ell$, take L and H_i as above. The semisimplification of $\bar{\rho}_\Lambda$ is $V_1 \oplus V_2$, where V_i is the one dimensional representation given by ψ_i. We have $f_p(\psi_1) + f_p(\psi_2) \leq f_p(\bar{\rho}_\Lambda)$ since $\dim_{\mathbb{F}} V^{H_i} \leq \dim_{\mathbb{F}} V_1^{H_i} + \dim_{\mathbb{F}} V_2^{H_i}$. By using this for all $p \neq \ell$, we deduce that $N(\psi_1)N(\psi_2) = N_1N_2$ divides $N(\bar{\rho}_\Lambda)$. The lemma follows since $N(\bar{\rho}_\Lambda)$ divides N, cf. [Liv89, Prop. 0.1]. $\square$

Fix an $i \in \{1, 2\}$; if $\ell \geq k-1$ and $\ell \nmid N$, then we may suppose that $m_i = 0$ by Lemma 5.4. Since the conductor of χ_i divides N by Lemma 5.5, assumption (a) implies that there is a prime $p \nmid N\ell$ for which $\chi_i(p)p^{m_i} \in \mathbb{F}$ is not a root of $x^2 - a_px + \varepsilon(p)p^{k-1} \in \mathbb{F}[x]$. However, this is a contradiction since

$$\chi_i(p)p^{m_i} = \tilde{\psi}_i(\mathrm{Frob}_p)\chi_\ell(\mathrm{Frob}_p)^{m_i} = \psi_i(\mathrm{Frob}_p)$$

is a root of $x^2 - a_px + \varepsilon(p)p^{k-1}$.

Therefore, the $\mathbb{F}$-representation $\bar{\rho}_\Lambda$ is irreducible. In particular, $\bar{\rho}_\Lambda(G)$ is not contained in a Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$.

5.4. **Cartan case.**

Lemma 5.6. *The group $\bar{\rho}_\Lambda(G)$ is not contained in a Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$.*

Proof. Suppose that $\bar{\rho}_\Lambda(G)$ is contained in a Cartan subgroup $\mathcal{C}$ of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$. If $\ell = 2$, then $\mathcal{C}$ is reducible as a subgroup of $\mathrm{GL}_2(\mathbb{F})$ since $\mathbb{F}/\mathbb{F}_\Lambda$ is a quadratic extension. However, we saw in §5.3 that $\bar{\rho}_\Lambda(G) \subseteq \mathcal{C}$ is an irreducible subgroup of $\mathrm{GL}_2(\mathbb{F})$. Therefore, ℓ is odd. If $\mathcal{C}$ is split, then $\bar{\rho}_\Lambda(G)$ is a reducible subgroup of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$. This was ruled out in §5.3, so $\mathcal{C}$ must be a non-split Cartan subgroup with ℓ odd.

Recall that the representation $\bar{\rho}_\Lambda$ is *odd*, i.e., if $c \in G$ is an element corresponding to complex conjugation under some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, then $\det(\bar{\rho}_\Lambda(c)) = -1$. Therefore, $\bar{\rho}_\Lambda(c)$ has order 2 and determinant $-1 \neq 1$ (this last inequality uses that ℓ is odd). A non-split Cartan subgroup $\mathcal{C}$ of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$ is cyclic and hence $-I$ is the unique element of $\mathcal{C}$ of order 2. Since $\det(-I) = 1$, we find that $\bar{\rho}_\Lambda(c)$ does not belong to $\mathcal{C}$; this gives the desired contradiction. $\square$

5.5. **Normalizer of a Cartan case.** Suppose that $\bar{\rho}_\Lambda(G)$ is contained in the normalizer $\mathcal{N}$ of a Cartan subgroup $\mathcal{C}$ of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$. The group $\mathcal{C}$ has index 2 in $\mathcal{N}$, so we obtain a character

$$\beta_\Lambda\colon G \xrightarrow{\bar{\rho}_\Lambda} \mathcal{N} \to \mathcal{N}/\mathcal{C} \cong \{\pm 1\}.$$

The character β_Λ is non-trivial since $\bar{\rho}_\Lambda(G) \not\subseteq \mathcal{C}$ by Lemma 5.6.

Lemma 5.7. *The character β_Λ is unramified at all primes $p \nmid N\ell$. If $\ell \geq 2k$ and $\ell \nmid N$, then the character β_Λ is also unramified at ℓ.*

Proof. The character β_Λ is unramified at each prime $p \nmid N\ell$ since $\bar{\rho}_\Lambda$ is unramified at such primes. Now suppose that $\ell \geq 2k$ and $\ell \nmid N$. We have $\ell > 2$, so $\ell \nmid |\mathcal{N}|$ and hence Lemma 5.3 implies that $\bar{\rho}_\Lambda(\mathcal{I}_\ell)$ is cyclic. Moreover, Lemma 5.3 implies that $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathcal{I}_\ell)$ is cyclic of order $d \geq (\ell-1)/(k-1)$. Our assumption $\ell \geq 2k$ ensures that $d > 2$.

Now take a generator g of $\bar{\rho}_\Lambda(\mathcal{I}_\ell)$. Suppose that β_Λ is ramified at ℓ and hence g belongs to $\mathcal{N} - \mathcal{C}$. The condition $g \in \mathcal{N} - \mathcal{C}$ implies that g^2 is a scalar matrix and hence $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathcal{I}_\ell)$ is a group of order 1 or 2. This contradicts $d > 2$, so β_Λ is unramified at ℓ. $\square$

Let χ be the primitive Dirichlet character that satisfies $\beta_\Lambda(\mathrm{Frob}_p) = \chi(p)$ for all primes $p \nmid N\ell$. Since β_Λ is a quadratic character, Lemma 5.7 implies that the conductor of χ divides $\mathcal{M}$. The character χ is non-trivial since β_Λ is non-trivial. Assumption (b) implies that there is a prime $p \nmid N\ell$ satisfying $\chi(p) = -1$ and $r_p \not\equiv 0 \pmod{\lambda}$. We thus have $g \in \mathcal{N} - \mathcal{C}$ and $\mathrm{tr}(g) \neq 0$, where $g := \bar{\rho}_\Lambda(\mathrm{Frob}_p) \in \mathcal{N}$. However, this contradicts that $\mathrm{tr}(A) = 0$ for all $A \in \mathcal{N} - \mathcal{C}$.

Therefore, the image of $\bar{\rho}_\Lambda$ does not lie in the normalizer of a Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$.

5.6. **$\mathfrak{A}_5$ case.** Assume that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is isomorphic to $\mathfrak{A}_5$ with $\#k_\lambda \notin \{4,5\}$.

The image of $r_p/p^{k-1} = a_p^2/(\varepsilon(p)p^{k-1})$ in $\mathbb{F}_\lambda$ is equal to $\mathrm{tr}(A)^2/\det(A)$ with $A = \bar{\rho}_\Lambda(\mathrm{Frob}_p)$. Every element of $\mathfrak{A}_5$ has order 1, 2, 3 or 5, so Lemma 5.2 implies that the image of r_p/p^{k-1} in $\mathbb{F}_\lambda$ is 0, 1, 4 or is a root of x^2-3x+1 for all $p \nmid N\ell$. If $\lambda | 5$, then Lemma 5.2 implies that $k_\lambda = \mathbb{F}_5$ which is excluded by our assumption on k_λ. So $\lambda \nmid 5$ and Lemma 5.2 ensures that k_λ is the splitting field of x^2-3x+1 over $\mathbb{F}_\ell$. So k_λ is $\mathbb{F}_\ell$ if $\ell \equiv \pm 1 \pmod 5$ and $\mathbb{F}_{\ell^2}$ if $\ell \equiv \pm 2 \pmod 5$.

From assumption (c), we find that $\ell > 5k-4$ and $\ell \nmid N$. By Lemma 5.3, the group $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ contains an element of order at least $(\ell-1)/(k-1) > ((5k-4)-1)/(k-1) = 5$. This is a contradiction since $\mathfrak{A}_5$ has no elements with order greater than 5.

5.7. **$\mathfrak{A}_4$ and $\mathfrak{S}_4$ cases.** Suppose that $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is isomorphic to $\mathfrak{A}_4$ or $\mathfrak{S}_4$ with $\#k_\lambda \neq 3$.

First suppose that $\#k_\lambda \notin \{5,7\}$. The image of $r_p/p^{k-1} = a_p^2/(\varepsilon(p)p^{k-1})$ in $\mathbb{F}_\lambda$ is equal to $\mathrm{tr}(A)^2/\det(A)$ with $A = \bar{\rho}_\Lambda(\mathrm{Frob}_p)$. Since every element of $\mathfrak{S}_4$ has order at most 4, Lemma 5.2 implies that r_p/p^{k-1} is congruent to 0, 1, 2 or 4 modulo λ for all primes $p \nmid N\ell$. In particular, $k_\lambda = \mathbb{F}_\ell$. By assumption (d), we must have $\ell > 4k-3$ and $\ell \nmid N$. By Lemma 5.3, the group $\bar{\rho}_\Lambda(G)$ contains an element of order at least $(\ell-1)/(k-1) > ((4k-3)-1)/(k-1) = 4$. This is a contradiction since $\mathfrak{S}_4$ has no elements with order greater than 4.

Now suppose that $\#k_\lambda \in \{5,7\}$. By assumption (e), with any χ, there is a prime $p \nmid N\ell$ such that $a_p^2/(\varepsilon(p)p^{k-1}) \equiv 2 \pmod \lambda$. The element $g := \bar{\rho}_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p)$ has order 1, 2, 3 or 4. By Lemma 5.2, we deduce that g has order 4. Since $\mathfrak{A}_4$ has no elements of order 4, we deduce that $H := \bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is isomorphic to $\mathfrak{S}_4$. Let H' be the unique index 2 subgroup of H; it is isomorphic to $\mathfrak{A}_4$. Define the character

$$\beta\colon G \xrightarrow{\bar{\rho}_\Lambda^{\mathrm{proj}}} H \to H/H' \cong \{\pm 1\}.$$

The quadratic character β corresponds to a Dirichlet character χ whose conductor divides $4^e \ell N$. By assumption (e), there is a prime $p \nmid N\ell$ such that $\chi(p) = 1$ and $a_p^2/(\varepsilon(p)p^{k-1}) \equiv 2 \pmod \lambda$. Since $\beta(\mathrm{Frob}_p) = \chi(p) = 1$, we have $\bar{\rho}_\Lambda^{\mathrm{proj}}(\mathrm{Frob}_p) \in H'$. Since $H' \cong \mathfrak{A}_4$, Lemma 5.2 implies that the image of $a_p^2/(\varepsilon(p)p^{k-1})$ in $\mathbb{F}_\lambda$ is 0, 1 or 4. This contradicts $a_p^2/(\varepsilon(p)p^{k-1}) \equiv 2 \pmod \lambda$.

Therefore, the image of $\bar{\rho}_\Lambda^{\mathrm{proj}}$ is not isomorphic to either of the groups $\mathfrak{A}_4$ or $\mathfrak{S}_4$.

5.8. **End of proof.** In §5.3, we saw that $\bar{\rho}_\Lambda(G)$ is not contained in a Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$. In §5.5, we saw that $\bar{\rho}_\Lambda(G)$ is not contained in the normalizer of a Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\Lambda)$.

In §5.6, we showed that if $\#k_\lambda \notin \{4,5\}$, then $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is not isomorphic to $\mathfrak{A}_5$. We want to exclude the cases $\#k_\lambda \in \{4,5\}$ since $\mathrm{PSL}_2(\mathbb{F}_4)$ and $\mathrm{PSL}_2(\mathbb{F}_5)$ are both isomorphic to $\mathfrak{A}_5$.

In §5.7, we showed that if $\#k_\lambda \neq 3$, then $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is not isomorphic to $\mathfrak{A}_4$ and not isomorphic to $\mathfrak{S}_4$. We want to exclude the case $\#k_\lambda = 3$ since $\mathrm{PSL}_2(\mathbb{F}_3)$ and $\mathrm{PGL}_2(\mathbb{F}_3)$ are isomorphic to $\mathfrak{A}_4$ and $\mathfrak{S}_4$, respectively.

By Lemma 5.1, the group $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ must be conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(\mathbb{F}')$ or $\mathrm{PGL}_2(\mathbb{F}')$, where $\mathbb{F}'$ is a subfield of $\mathbb{F}_\Lambda$. One can then show that $\mathbb{F}'$ is the subfield of $\mathbb{F}_\Lambda$ generated by the set $\{\mathrm{tr}(A)^2/\det(A) : A \in \bar{\rho}_\Lambda(G)\}$. By the Chebotarev density theorem, the field $\mathbb{F}'$ is the subfield of $\mathbb{F}_\Lambda$ generated by the images of $a_p^2/(\varepsilon(p)p^{k-1}) = r_p/p^{k-1}$ with $p \nmid N\ell$. Therefore, $\mathbb{F}' = k_\lambda$ and hence $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(k_\lambda)$ or $\mathrm{PGL}_2(k_\lambda)$.

6. Examples

6.1. **Example from §1.2.** Let f be the newform from §1.2. We have $E = \mathbb{Q}(i)$. We know that $\Gamma \neq 1$ since ε is non-trivial. Therefore, $\Gamma = \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ and $K = E^\Gamma$ equals $\mathbb{Q}$. So Γ is generated by complex conjugation and we have $\bar{a}_p = \varepsilon(p)^{-1}a_p$ for $p \nmid N$. As noted in §1.2, this implies that r_p is a square in $\mathbb{Z}$ for all $p \nmid N$ and hence L equals $K = \mathbb{Q}$. Fix a prime $\ell = \lambda$ and a prime ideal $\Lambda|\ell$ of $\mathcal{O} = \mathbb{Z}[i]$.

Set $q_1 = 109$ and $q_2 = 379$; they are primes that are congruent to 1 modulo 27. Set $p_1 = 5$, we have $\chi(p_1) = -1$, where χ is the unique non-trivial character $(\mathbb{Z}/3\mathbb{Z})^\times \to \{\pm 1\}$. Set $q = 5$; the field $\mathbb{Q}(r_q)$ equals $K = \mathbb{Q}$ and hence $\mathbb{Z}[r_q] = \mathbb{Z}$.

One can verify that $a_{109} = 164$, $a_{379} = 704$ and $a_5 = -3i$, so $r_{109} = 164^2$, $r_{379} = 704^2$ and $r_5 = 3^2$. We have

$$r_{109} - (1+109^2)^2 = -2^2 \cdot 3^3 \cdot 7 \cdot 19 \cdot 31 \cdot 317 \quad \text{and} \quad r_{379} - (1+379^2)^2 = -2^2 \cdot 3^3 \cdot 2647 \cdot 72173. \tag{6.1}$$

So if $\ell \geq 11$, then there is an $i \in \{1, 2\}$ such that $\ell \neq q_i$ and $r_{q_i} \not\equiv (1 + q_i^2)^2 \pmod{\ell}$.

Let S be the set from §4 with the above choice of q_1, q_2, p_1 and q. We find that $S = \{2, 3, 5, 7, 11\}$. Theorem 4.3 implies that $\bar{\rho}_\Lambda^{\text{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(\mathbb{F}_\ell)$ when $\ell > 11$.

Now take $\ell \in \{7, 11\}$. Choose a prime ideal Λ of $\mathcal{O}$ dividing ℓ. We have $e_0 = e_1 = e_2 = 0$ and $\mathcal{M} = 3$. The subfield k_ℓ generated over $\mathbb{F}_\ell$ by the images of r_p modulo ℓ with $p \nmid N\ell$ is of course $\mathbb{F}_\ell$ (since the r_p are rational integers). We now verify the conditions of Theorem 4.1.

We first check condition (a). Suppose there is a character $\chi\colon (\mathbb{Z}/27\mathbb{Z})^\times \to \mathbb{F}_\ell^\times$ such that $\chi(q_2)$ is a root of $x^2 - a_{q_2}x + \varepsilon(q_2)q_2^2$ modulo ℓ. Since $q_2 \equiv 1 \pmod{27}$ and $a_{q_2} = 704$, we find that 1 is a root of $x^2 - a_{q_2}x + q_2^2 \in \mathbb{F}_\ell[x]$. Therefore, $a_{q_2} \equiv 1 + q_2^2 \pmod{\ell}$ and hence $r_{q_2}^2 = a_{q_2}^2 \equiv (1 + q_2^2)^2 \pmod{\ell}$. Since $\ell \in \{7, 11\}$, this contradicts (6.1). This proves that condition (a) holds.

We now check condition (b). Let $\chi\colon (\mathbb{Z}/3\mathbb{Z})^\times \to \{\pm 1\}$ be the non-trivial character. We have $\chi(5) = -1$ and $r_5 = 9 \not\equiv 0 \pmod{\ell}$. Therefore, (b) holds.

We now check condition (c). If $\ell = 7$, we have $\ell \equiv 2 \pmod 5$ and $\#k_\ell = \ell \neq \ell^2$, so condition (c) holds. Take $\ell = 11$. We have $a_5^2/(\varepsilon(5)5^2) = 9/5^2 \equiv 3 \pmod{11}$, which verifies (c).

Condition (d) holds since $\#k_\ell = 5$ if $\ell = 7$, and $\ell > 4k - 3 = 9$ and $\ell \nmid N$ if $\ell = 11$.

Finally we explain why condition (e) holds when $\ell = 7$. Let $\chi\colon (\mathbb{Z}/7 \cdot 27\mathbb{Z})^\times \to \{\pm 1\}$ be any non-trivial character. A quick computation shows that there is a prime $p \in \{13, 37, 41\}$ such that $\chi(p) = 1$ and that $a_p^2/(\varepsilon(p)p^2) \equiv 2 \pmod 7$.

Theorem 4.1 implies that $\bar{\rho}_\Lambda^{\text{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(\mathbb{F}_\ell)$ or $\mathrm{PGL}_2(\mathbb{F}_\ell)$. Since $L = K$, the group $\bar{\rho}_\Lambda^{\text{proj}}(G)$ isomorphic to $\mathrm{PSL}_2(\mathbb{F}_\ell)$ by Theorem 1.2(i).

6.2. **Example from §1.3.** Let f be a newform as in §1.3; we have $k = 3$ and $N = 160$. The `Magma` code below verifies that f is uniquely determined up to replacing by a quadratic twist and then a Galois conjugate. So the group $\bar{\rho}_\Lambda^{\text{proj}}(G)$, up to isomorphism, does not depend on the choice of f.

```
eps:=[c: c in Elements(DirichletGroup(160)) | Order(c) eq 2 and Conductor(c) eq 20][1];
M:=ModularSymbols(eps,3);
F:=NewformDecomposition(NewSubspace(CuspidalSubspace(M)));
#F eq 2;  _,chi:=IsTwist(F[1],F[2],5);  Order(chi) eq 2;
```

Define $b = \zeta_{13}^1 + \zeta_{13}^5 + \zeta_{13}^8 + \zeta_{13}^{12}$, where ζ_{13} is a primitive 13-th root of unity in $\overline{\mathbb{Q}}$ (note that $\{1, 5, 8, 12\}$ is the unique index 3 subgroup of $\mathbb{F}_{13}^\times$). The characteristic polynomial of b is $x^3 + x^2 - 4x + 1$ and hence $\mathbb{Q}(b)$ is the unique cubic extension of $\mathbb{Q}$ in $\mathbb{Q}(\zeta_{13})$. The code below shows that the coefficient field E is equal to $\mathbb{Q}(b, i)$ (it is a degree 6 extension of $\mathbb{Q}$ that contains roots of $x^3 + x^2 - 4x + 1$ and $x^2 + 1$).

```
f:=qEigenform(F[1],2001);
a:=[Coefficient(f,n): n in [1..2000]];
E:=AbsoluteField(Parent(a[1]));
Pol<x>:=PolynomialRing(E);
Degree(E) eq 6 and HasRoot(x^3+x^2-4*x+1) and HasRoot(x^2+1);
```

Fix notation as in §2.1. We have $\Gamma \neq 1$ since ε is non-trivial. The character χ_γ^2 is trivial for $\gamma \in \Gamma$ (since χ_γ is always a quadratic character times some power of ε). Therefore, Γ is a 2-group. The field $K = E^\Gamma$ is thus $\mathbb{Q}(b)$ which is the unique cubic extension of $\mathbb{Q}$ in E. Therefore, $r_p = a_p^2/\varepsilon(p)$ lies in $K = \mathbb{Q}(b)$ for all $p \nmid N$.

The code below verifies that r_3, r_7 and r_{11} are squares in K that do not lie in $\mathbb{Q}$ (and in particular, are non-zero). Since 3, 7 and 11 generate the group $(\mathbb{Z}/40\mathbb{Z})^\times$, we deduce from Lemma 2.3 that the field $L = K(\{\sqrt{r_p} : p \nmid N\})$ is equal to K.

```
_,b:=HasRoot(x^3+x^2-4*x+1);  K:=sub<E|b>;
r3:=K!(a[3]^2/eps(3)); r7:=K!(a[7]^2/eps(7)); r11:=K!(a[11]^2/eps(11));
IsSquare(r3) and IsSquare(r7) and IsSquare(r11);
r3 notin Rationals() and r7 notin Rationals() and r11 notin Rationals();
```

Let $N_{K/\mathbb{Q}} \colon K \to \mathbb{Q}$ be the norm map. The following code verifies that $N_{K/\mathbb{Q}}(r_3) = 2^6$, $N_{K/\mathbb{Q}}(r_7) = 2^6$, $N_{K/\mathbb{Q}}(r_{11}) = 2^{12}5^4$, $N_{K/\mathbb{Q}}(r_{13}) = 2^{12}13^2$, $N_{K/\mathbb{Q}}(r_{17}) = 2^{18}5^2$, and that

$$\gcd\Big(641 \cdot N_{K/\mathbb{Q}}(r_{641} - (1+642^2)^2),\ 1061 \cdot N_{K/\mathbb{Q}}(r_{1061} - (1+1061^2)^2)\Big) = 2^{12}. \tag{6.2}$$

```
r13:=K!(a[13]^2/eps(13)); r17:=K!(a[17]^2/eps(17));
Norm(r3) eq 2^6; Norm(r7) eq 2^6; Norm(r11) eq 2^12*5^4;
Norm(r13) eq 2^12*13^2; Norm(r17) eq 2^18*5^2;
r641:=K!(a[641]^2/eps(641));   r1061:=K!(a[1061]^2/eps(1061));
n1:=Integers()!Norm(r641-(1+641^2)^2);   n2:=Integers()!Norm(r1061-(1+1061^2)^2);
GCD(641*n1,1061*n2) eq 2^12;
```

Set $q_1 = 641$ and $q_2 = 1061$; they are primes congruent to 1 modulo 160. Let λ be a prime ideal of R dividing a rational prime $\ell > 3$. From (6.2), we find that $\ell \neq q_i$ and $r_{q_i} \not\equiv (1+q_i^2)^2 \pmod{\lambda}$ for some $i \in \{1,2\}$ (otherwise λ would divide 2).

Set $p_1 = 3$, $p_2 = 7$ and $p_3 = 11$. For each non-trivial quadratic characters $\chi \colon (\mathbb{Z}/40\mathbb{Z})^\times \to \{\pm 1\}$, we have $\chi(p_i) = -1$ for some prime $i \in \{1,2,3\}$ (since 3, 7 and 11 generate the group $(\mathbb{Z}/40\mathbb{Z})^\times$). From the computed values of $N_{K/\mathbb{Q}}(r_p)$, we find that $r_{p_i} \not\equiv 0 \pmod{\lambda}$ for all $i \in \{1,2,3\}$ and all non-zero prime ideals $\lambda \nmid N$ of R.

Set $q = 3$. We have noted that $r_q \in K - \mathbb{Q}$, so $K = \mathbb{Q}(r_q)$. The index of the order $\mathbb{Z}[r_q]$ in R is a power of 2 since $N_{K/\mathbb{Q}}(q)$ is a power of 2.

Let S be the set from §4 with the above choice of q_1, q_2, p_1, p_2, p_3 and q. The above computations show that S consists of the prime ideals λ of R that divide a prime $\ell \leq 11$.

Now let ℓ be an odd prime congruent to ± 2, ± 3, ± 4 or ± 6 modulo 13. Since K is the unique cubic extension in $\mathbb{Q}(\zeta_{13})$, we find that the ideal $\lambda := \ell R$ is prime in R and that $\mathbb{F}_\lambda \cong \mathbb{F}_{\ell^3}$. The above computations show that $\lambda \notin S$ when $\ell \notin \{3,7,11\}$. Theorem 4.3 implies that if $\ell \notin \{3,7,11\}$, then $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$, where Λ is a prime ideal of $\mathcal{O}$ dividing λ. So if $\ell \notin \{3,7,11\}$, the group $\bar{\rho}_\Lambda^{\mathrm{proj}}(G)$ isomorphic to $\mathrm{PSL}_2(\mathbb{F}_\lambda) \cong \mathrm{PSL}_2(\mathbb{F}_{\ell^3})$ by Theorem 1.2(i) and the equality $L = K$.

Now take $\lambda = \ell R$ with $\ell \in \{3,7,11\}$; it is a prime ideal. Choose a prime ideal Λ of $\mathcal{O}$ dividing λ. We noted above that $\mathbb{Z}[r_3]$ is an order in R with index a power of 2; the same argument shows that this also holds for the order $\mathbb{Z}[r_7]$. Therefore, the field k_λ generated over $\mathbb{F}_\ell$ by the images of r_p modulo λ with $p \nmid N\ell$ is equal to $\mathbb{F}_\lambda$. Since $\#\mathbb{F}_\lambda = \ell^3$, we find that conditions (c), (d) and (e) of Theorem 4.1 hold.

We now show that condition (a) of Theorem 4.1 holds for our fixed Λ. We have $e_0 = 0$, so take any character $\chi \colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_\Lambda^\times$. We claim that $\chi(q_i) \in \mathbb{F}_\Lambda$ is not a root of $x^2 - a_{q_i}x + \varepsilon(q_i)q_i^2$ for some $i \in \{1,2\}$. Since $q_i \equiv 1 \pmod{N}$, the claim is equivalent to showing that $a_{q_i} \not\equiv 1 + q_i^2$

(mod Λ) for some $i \in \{1,2\}$. So we need to prove that $r_{q_i} \equiv (1+q_i^2)^2 \pmod{\lambda}$ for some $i \in \{1,2\}$; this is clear since otherwise ℓ divides the quantity (6.2). This completes our verification of (a).

We now show that condition (b) of Theorem 4.1 holds. We have $r_p \not\equiv 0 \pmod{\lambda}$ for all primes $p \in \{3,7,11,13,17\}$; this is a consequence of $N_{K/\mathbb{Q}}(r_p) \not\equiv 0 \pmod{\ell}$. We have $\mathcal{M} = 120$ if $\ell = 3$ and $\mathcal{M} = 40$ otherwise. Condition (b) holds since $(\mathbb{Z}/\mathcal{M}\mathbb{Z})^\times$ is generated by the primes $p \in \{3,7,11,13,17\}$ for which $p \nmid \mathcal{M}\ell$.

Theorem 4.1 implies that $\bar{\rho}_\Lambda^{\text{proj}}(G)$ is conjugate in $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ to $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$. Since $L = K$, the group $\bar{\rho}_\Lambda^{\text{proj}}(G)$ isomorphic to $\mathrm{PSL}_2(\mathbb{F}_\lambda) \cong \mathrm{PSL}_2(\mathbb{F}_{\ell^3})$ by Theorem 1.2(i).

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