Linked Solenoid Mappings and the Non-Transversality Locus Invariant

JOHN H. HUBBARD & RALPH W. OBERSTE-VORTH

1. INTRODUCTION

We will study in this paper diffeomorphisms $f : S^3 \to S^3$ with two invariant linked solenoids Σ^+, Σ^-, one attracting and one repelling. We will call such mappings *linked solenoid* mappings. Such mappings arise when studying Hénon mappings in $\mathbb{C}^2$, and have been studied in [HO1], [HPV], [BS], and [Bu].

More precisely, a linked solenoid mapping is one for which the 3-sphere S^3 can be cut into two linked unknotted solid tori T^+, T^-, such that $f : T^+ \to T^+$ and $f^{-1} : T^{-1} \to T^{-1}$ are conjugate to the standard maps (see below) giving rise to solenoids, as first studied in [vD] and [V]. These maps are structurally stable, and hence our linked solenoid maps will be structurally stable on their non-wandering sets. But they are not structurally stable.

In Section 3 of this paper we will define a conjugacy invariant $\mathrm{ntl}(f) \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ of linked solenoid mappings mapping. We will compute $\mathrm{ntl}(f)$ for some particular linked solenoid maps in Section 4, we will show it can take on an infinite-dimensional set of values in Section 5, and in Section 6 we will show that in an open set of solenoidal mappings it is a complete invariant and classifies these mappings up to topological conjugacy.

There are many things about this invariant which we don't know; we present some of these in Section 7.

We thank Thierry Bousch, whose criticism of [HO1] prompted part of this paper.

2. LINKED SOLENOID MAPPINGS

Let $\mathbf{T} = S^1 \times D$ be the solid torus, where S^1 and D are the unit circle and the closed unit disc in $\mathbb{C}$, respectively. A mapping $f : \mathbf{T} \to \mathbf{T}$ will be called *solenoidal of degree* m if it is conjugate to a mapping of the form

$$\sigma_k : (\zeta, z) \mapsto \left(\zeta^m\,,\, \frac{1}{2}\zeta + \varepsilon z \zeta^{k-m+1}\right),$$

where ε is chosen small enough for the map to be injective.

In [HO1] we prove that k is a conjugacy invariant of such mappings and that, under appropriate hyperbolicity conditions, it is the only invariant. Moreover, we prove that if $\mathbf{T}$ is embedded in S^3 in the standard way, then σ_k extends to an orientation-preserving homeomorphism $\tilde{\sigma}_k : S^3 \to S^3$ if and only if $k = 0$; in that case, $\tilde{\sigma}_k^{-1}$ is itself a solenoidal mapping from $T^- = S^3 \setminus \mathring{\mathbf{T}}$ into itself.

We will call a homeomorphism $f : S^3 \to S^3$ a *linked solenoid mapping of degree m* if there is an embedded torus $T \subset S^3$ cutting S^3 into two solid tori, T^+ and T^-, such that both

$$f : T^+ \to T^+ \quad \text{and} \quad f^{-1} : T^- \to T^-$$

are solenoidal of degree m. We will call

$$\Sigma^+ = \bigcap_{n \geq 0} f^n(T^+) \quad \text{and} \quad \Sigma^- = \bigcap_{n \geq 0} f^{-n}(T^-)$$

the attracting and repelling solenoids.

3. THE INVARIANT ntl(f)

Since the restriction of f to T^+ and of f^{-1} to T^- are solenoidal, there exist (see [HO1]) continuous mappings $\pi^\pm : T^\pm \to \mathbb{R}/2\pi\mathbb{Z}$ making T^+ and T^- into bundles of disks over the circle, and such that the diagrams

$$\text{(3.1)} \qquad \begin{array}{ccc} T^+ & \xrightarrow{f} & T^+ \\ {\scriptstyle \pi^+}\downarrow & & \downarrow{\scriptstyle \pi^+} \\ \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow[t \mapsto mt]{} & \mathbb{R}/2\pi\mathbb{Z} \end{array} \quad , \quad \begin{array}{ccc} T^- & \xrightarrow{f^{-1}} & T^- \\ {\scriptstyle \pi^-}\downarrow & & \downarrow{\scriptstyle \pi^-} \\ \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow[t \mapsto mt]{} & \mathbb{R}/2\pi\mathbb{Z} \end{array}$$

commute.

Remark 3.1. The mappings $\pi^\pm$ are usually not differentiable, however smooth f might be. Indeed, the solenoids $\Sigma^\pm$ contain lots of periodic points. Let $p \in \Sigma^+$ be a point such that $f^i(p) = p$. Then the unstable manifold of p is the leaf of the solenoid through p, tangent at p to the eigenspace of $Df^i(p)$ with eigenvalue $\lambda > 1$. But there is no reason to expect $\lambda = m^i$, and this would have to happen if π^+ were differentiable.

But we will use in a crucial way the fact that the fibers of $\pi^\pm$ are C^r surfaces if f is of class C^r, and that they depend continuously on f in the C^r topology if f varies continuously in the C^r topology. This follows from the fact that these fibers are the stable manifolds of points of the solenoids $\Sigma^\pm$, which are hyperbolic sets ([Ru, 15.2], [HiPu]).

The *angle mappings* $\pi^{\pm}$ are only unique up to addition of a multiple of $2\pi/(m-1)$; to lift this ambiguity we define the *normalized angle mappings* $\Phi^{\pm} = (m-1)\pi^{\pm}$.

Define the tori $T_k = f^k(T)$ for all $k \in \mathbb{Z}$; further set $T_k^{\pm} = f^k(T^{\pm})$, so that $T_k^{\pm}$ is the closure of the component of $S^3 - T_k$ which contains $\Sigma^{\pm}$. Clearly $f : T_k^+ \to T_k^+$ and $f^{-1} : T_k^- \to T_k^-$ are solenoidal, since $f^{\circ k}$ conjugates them to our original maps $f : T^+ \to T^+$ and $f^{-1} : T^- \to T^-$. Call the associated angle mappings $\pi_k^{\pm} = \pi^{\pm} \circ f^{-k} : T_k^{\pm} \to \mathbb{R}/\mathbb{Z}$, and the normalized angle mappings $\Phi_k^{\pm}$. These mappings satisfy

$$
\begin{aligned}
\Phi_k^+ &= m^{\ell-k}\Phi_\ell^+ \quad \text{on } T_\ell^+ \text{ when } \ell \geq k,\\
\Phi_k^- &= m^{k-\ell}\Phi_\ell^- \quad \text{on } T_\ell^+ \text{ when } \ell \leq k.
\end{aligned}
\tag{3.2}
$$

Let U_k be the region between T_k and T_{k+1}:

$$U_k = S^3 \setminus (\mathring{T}_{k+1}^+ \cup \mathring{T}_k^-) = T_k^+ \cap T_{k+1}^-.$$

The mappings Φ_k^+ and Φ_{k+1}^- are both defined in U_k. Because they are not differentiable, we cannot quite speak of (Φ_k^+, Φ_{k+1}^-) being or not being a submersion; we will say that it is a topological submersion at $x \in U_k$ if there exists another function h defined near x such that $(\Phi_k^+, \Phi_{k+1}^-, h)$ are local coordinates near x.

Lemma 3.2. *The map* $(\Phi_k^+, \Phi_{k+1}^-) : U_k \to (\mathbb{R}/2\pi\mathbb{Z})^2$ *is not a topological submersion.*

Proof. If the map $(\Phi_k^+, \Phi_{k+1}^-) : U_k \to (\mathbb{R}/2\pi\mathbb{Z})^2$ were a topological submersion, it would represent U_k as a locally trivial fiber bundle over the torus. Since U_k is connected, all the fibers would have to be homeomorphic 1-dimensional manifolds with boundary, hence intervals, since the boundary of U_k is non-empty. If we collapse these fibers to points, we manufacture a finite covering space of the torus with the homotopy type of U_k. All finite covers of the torus are themselves tori, so U_k would have the homotopy type of the torus, and its fundamental group would be a free abelian group on two generators.

But that is not the case. The angle mapping π_k^+ makes U_k into a bundle of $m+1$ times punctured spheres over the circle, and the long homotopy exact sequence of this fibration gives the short exact sequence

$$\{1\} \to F_m \to \pi_1(U_k) \to \mathbb{Z} \to \{1\},$$

where F_m is the free group on m generators. In particular, $\pi_1(U_k)$ is highly non-abelian. ◻

Thus the fibers of Φ_k^+ and Φ_{k+1}^- cannot be transverse to each other in U_k. Define $X_k \subset U_k$ to be the locus where these two foliations are not transverse, and define $\mathrm{ntl}_k(f) \subset \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ to be the image of X_k by the mapping (Φ_k^+, Φ_{k+1}^-).

Lemma 3.3. *The invariants* $\mathrm{ntl}_k(f)$ *are all equal.*

Proof. Clearly $f(X_k) = X_{k+1}$. We need to check that

$$(\Phi^+_{k+1}(f(x)),\Phi^-_{k+2}(f(x))) = (\Phi^+_k(x),\Phi^-_{k+1}(x)).$$

For the first coordinate, using (3.1) and (3.2), we get

$$\Phi^+_{k+1}(f(x)) = \frac{1}{m}\Phi^+_k(f(x)) = \frac{1}{m}\cdot m\cdot\Phi^+_k(x),$$

and for the second we have

$$\Phi^-_{k+2}(f(x)) = m\Phi^-_{k+1}(f(x)) = m\cdot\frac{1}{m}\cdot\Phi^-_{k+1}(x). \qquad \square$$

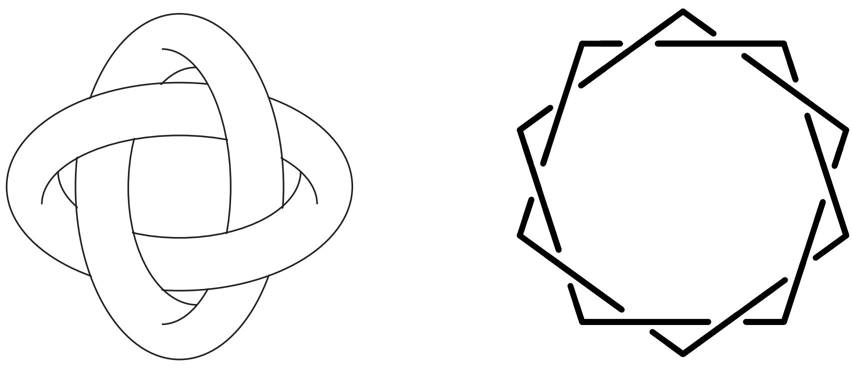

FIGURE 1. The standard linked tori, when $m = 2$ and $m = 5$.

4. THE STANDARD LINKED SOLENOID MAPPINGS

The standard linked-solenoid mappings, which we will construct below, result from an almost explicit construction. It was sketched in [HO1], but the description given there is not sufficiently precise to determine the non-transversality invariant; here we will be more careful.

Step 1. The functions p_0, p_1, ***their arguments, and the rotations*** R_m. Define $\arg : \mathbb{C}^* \to \mathbb{R}/2\pi\mathbb{Z}$ by the formula $\arg z = \mathrm{Im}\log(z/|z|)$, so that $\mathrm{d}\arg z = \mathrm{Im}(\mathrm{d}z/z)$.

Consider the functions $p_0, p_1 : \mathbb{C}^2 \to \mathbb{C}$ given by

$$p_0\begin{pmatrix}x\\y\end{pmatrix} = y - x^m \quad\text{and}\quad p_1\begin{pmatrix}x\\y\end{pmatrix} = y + x^m,$$

and their arguments $\vartheta_0 = \arg p_0$ and $\vartheta_1 = \arg p_1$, respectively defined on the complements of the curves $C_i = S^3 \cap \{p_i = 0\}$.

The rotation $R_m : S^3 \to S^3$ around the y-axis by π/m, given by

$$R_m \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{i\pi/m} x \\ y \end{pmatrix},$$

will play an important role below. Note that

$$p_1 \left(R_m \begin{pmatrix} x \\ y \end{pmatrix} \right) = y + (e^{\pi i/m} x)^m = p_0 \begin{pmatrix} x \\ y \end{pmatrix},$$

$$p_0 \left(R_m \begin{pmatrix} x \\ y \end{pmatrix} \right) = y - (e^{\pi i/m} x)^m = p_1 \begin{pmatrix} x \\ y \end{pmatrix},$$

so that

$$\vartheta_1 \left(R_m \begin{pmatrix} x \\ y \end{pmatrix} \right) = \vartheta_0 \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \vartheta_0 \left(R_m \begin{pmatrix} x \\ y \end{pmatrix} \right) = \vartheta_1 \begin{pmatrix} x \\ y \end{pmatrix}.$$

Lemma 4.1. *The map $(\vartheta_0, \vartheta_1) : S^3 - (C_0 \cup C_1) \to \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ is a submersion, except on the intersection of the y-axis with S^3.*

Proof. At a point $\binom{x}{y}$ where our map is not a submersion, we have $\mathrm{d}\,\vartheta_0 = \lambda\,\mathrm{d}\,\vartheta_1$ for some real number λ, i.e.,

$$\mathrm{Im}\,\frac{\mathrm{d}\,p_0(v)}{p_0} = \lambda\,\mathrm{Im}\,\frac{\mathrm{d}\,p_1(v)}{p_1}$$

for all vectors $v \in \mathbb{C}^2$. Replacing v by iv, we see that the real parts must be equal also, so

$$\frac{\mathrm{d}\,p_0(v)}{p_0} = \lambda \frac{\mathrm{d}_1(v)}{p_1}, \quad \text{i.e.,} \quad \frac{1}{y - x^m}[-mx^{m-1}, 1] = \frac{\lambda}{y + x^m}[mx^{m-1}, 1].$$

The second entry gives $\lambda = (y + x^m)/(y - x^m)$, and then the first entry becomes $x^{m-1} = -x^{m-1}$, i.e., $x = 0$. □

Step 2. The solid tori T_0^- and T_1^+. Consider the solid tori

$$T_0^- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in S^3 : \left| p_0 \begin{pmatrix} x \\ y \end{pmatrix} \right| \leq \rho \right\},$$

$$T_1^+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in S^3 : \left| p_1 \begin{pmatrix} x \\ y \end{pmatrix} \right| \leq \rho \right\},$$

for some $\rho > 0$, sufficiently small so that these solid tori are disjoint. Their "core curves" are the curves $C_0 = \{p_0 = 0\} \cap S^3$, $C_1 = \{p_1 = 0\} \cap S^3$ parametrized respectively by

$$\zeta \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} \zeta \\ \zeta^m \end{bmatrix} \quad \text{and} \quad \zeta \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} \zeta \\ -\zeta^m \end{bmatrix},$$

where ζ is a complex number with $|\zeta| = 1$. These are two disjoint unknotted circles in S^3 which link in the simplest fashion with linking number m, hence the tori are also unknotted and linked with linking number m.

We will set T_0^+ (resp. T_1^-) to be the closure of the complement of T_0^- (resp. T_1^+).

Lemma 4.2. *There exist unique functions*

$$\varphi_0 : T_0^- \to \mathbb{R}/2\pi\mathbb{Z} \quad \text{and} \quad \varphi_1 : T_1^+ \to \mathbb{R}/2\pi\mathbb{Z}$$

such that

$$m\varphi_0 = \vartheta_1 \quad \text{and} \quad m\varphi_1 = \vartheta_0$$

and such that

$$\varphi_0\begin{pmatrix} \zeta \\ \zeta^m \end{pmatrix} = \arg\zeta \quad \text{and} \quad \varphi_1\begin{pmatrix} \zeta \\ -\zeta^m \end{pmatrix} = \arg\zeta.$$

Proof. This is standard covering space theory: lifting $\vartheta_0 : T_1^+ \to \mathbb{R}/2\pi\mathbb{Z}$ to the m-fold cover of $\mathbb{R}/2\pi\mathbb{Z}$. Clearly

$$\vartheta_0\begin{pmatrix} \zeta \\ -\zeta_m \end{pmatrix} = \arg\zeta^m = m\arg\zeta.$$

Thus the lift exists on C_1; since the inclusion $C_1 \hookrightarrow T_1^+$ is a homotopy equivalence, the lift exists on T_1^+, and is specified by its value ζ on C_1. The result about ϑ_1 is similar. □

Lemma 4.3. *The maps $T_0^-, T_1^+ \to D_\rho \times \mathbb{R}/2\pi\mathbb{Z}$ given by (p_0, φ_0) and (p_1, φ_1) respectively, are homeomorphisms.*

Proof. Lemma 4.1 says that these mappings are local homeomorphisms. They are evidently proper, hence they are covering maps. But they are homeomorphisms on C_0 and C_1 respectively, so they are of degree 1, hence homeomorphisms. □

Unfortunately, we do not know a nice way to parametrize the complementary solid tori, but the following result gives enough information for our purposes.

Lemma 4.4. *The maps $\vartheta_0 : T_0^+ \to \mathbb{R}/2\pi\mathbb{Z}$ and $\vartheta_1 : T_1^- \to \mathbb{R}/2\pi\mathbb{Z}$ fiber the solid tori T_0^+ and T_1^- over the circle, with fibers diffeomorphic to a disc.*

Proof. This is essentially the trivial case of the Milnor fibration theorem, see [Mil, Lemma 2.13]. ◻

Step 3. The reflection $\rho_{\alpha,\beta}$. It follows from Lemma 4.3 that the torus T_1 is parametrized by $\vartheta_1 = \arg p_1$ and φ_1. Our reflection $\rho_{\alpha,\beta}$ will be a mapping $S^3 \to S^3$ which will map T_1 to itself, and the outside T_1^- to the inside T_1^+, and the inside to the outside.

It will be convenient to denote by $D_0^-(\omega) \subset T_0^-$ the disc $\varphi_0 = \omega$, by $D_0^+(\omega) \subset T_0^+$ the disc $\vartheta_0 = \omega$, by $D_1^+(\omega) \subset T_1^+$ the disc $\varphi_1 = \omega$, and by $D_1^-(\omega) \subset T_1^-$ the disc $\vartheta_1 = \omega$.

Lemma 4.5. *There exists a diffeomorphism $\rho_{\alpha,\beta} : S^3 \to S^3$ such that*

(i) $\rho_{\alpha,\beta}(T_1^-) = (T_1^+)$ *and* $\rho_{\alpha,\beta}(T_1^+) = (T_1^-)$*;*
(ii) *The restriction of $\rho_{\alpha,\beta}$ to T_1 is the mapping $(\vartheta_1, \varphi_1) \mapsto (\varphi_1 + \alpha, \vartheta_1 + \beta)$;*
(iii) *The mapping $\rho_{\alpha,\beta}$ maps the disc $D_1^-(\omega)$ to the disc $D_1^+(\omega + \beta)$ and the disc $D_1^+(\omega)$ to the disc $D_1^-(\omega + \alpha)$, and is strictly contracting.*

Of course, part (ii) is part of part (iii).

Proof. In the absence of an explicit parametrization of T_1^-, we resort to bigger guns. Given two smooth bundles

$$p_A : A \to \mathbb{R}/2\pi Z \quad \text{and} \quad p_B : B \to \mathbb{R}/2\pi Z$$

of closed disks over the circle, and a bundle diffeomorphism $\varphi : \partial A \to \partial B$, there always exists an extension $\tilde{\varphi} : A \to B$ which is a bundle diffeomorphism. Indeed, such bundles are classified by their monodromy, which is an element of the group of isotopy classes of diffeomorphisms of the disk which coincide with the identity on the boundary, and this group is trivial.

In our case, the two bundles are

$$\vartheta_1 : T_1^- \to \mathbb{R}/2\pi Z \quad \text{and} \quad \varphi_1 - \beta : T_1^+ \to \mathbb{R}/2\pi\mathbb{Z},$$

and the boundary map is $(\vartheta_1, \varphi_1) \mapsto (\varphi_1 + \alpha, \vartheta_1 + \beta)$. ◻

Step 4. The linked solenoid mapping. We can now define our mapping. Define

$$f_{\alpha,\beta} : S^3 \to S^3 \quad \text{by} \quad f_{\alpha,\beta} = \rho_{\alpha,\beta} \circ R_m.$$

Lemma 4.6. *The mapping $f_{\alpha,\beta}$ is a linked solenoid map of degree m.*

Proof. Notice that the disc $D_0^+(\omega)$ maps as follows:

$$D_0^+(\omega) \xrightarrow{R_m} D_1^-(\omega) \xrightarrow{\rho_{\alpha,\beta}} D_1^+(\omega+\beta) \subset D_0^+(m\omega + m\beta).$$

That means that if we set $\pi_0^+ = \vartheta_0 + m\beta/(m-1)$, the diagram

$$\begin{tikzcd} T_0^+ \arrow[r, "f"] \arrow[d, "\pi_0^+"'] & T_0^+ \arrow[d, "\pi_0^+"] \\ \mathbb{R}/2\pi\mathbb{Z} \arrow[r, "t\mapsto dt"'] & \mathbb{R}/2\pi\mathbb{Z} \end{tikzcd}$$

commutes, for any choice of $\beta/(m-1)$. Since R_m is an isometry and $\rho_{\alpha,\beta}$ is contracting on the discs $D_0^+(\omega)$, the result follows from [HO1, Proposition 3.11. □

Step 5. The computation of Φ_0^+ ***and*** Φ_1^-***, and*** $\mathrm{ntl}(f)$**.** We have just computed π_0^+, so $\Phi_0^+ = (m-1)\pi_0^+ = (m-1)\vartheta_0 + m\beta$. A similar calculation shows that under $f_{\alpha,\beta}^{-1} = R_m^{-1} \circ \rho_{\alpha,\beta}^{-1}$, the disc $D_1^-(\omega)$ maps as follows:

$$D_1^-(\omega) \xrightarrow{\rho_{\alpha,\beta}^{-1}} D_1^+(\omega-\alpha) \xrightarrow{R_m^{-1}} D_0^-(\omega-\alpha) \subset D_1^-(m\omega - m\alpha),$$

which leads to $\pi_1^- = \vartheta_1 - \alpha/(m-1)$, and finally $\Phi_1^- = (m-1)\vartheta_1 - m\alpha$.

We can now read off the non-transversality invariant of our map $f_{\alpha,\beta}$.

Theorem 4.7. *The non-transversality invariant of* $f_{\alpha,\beta}$ *is the subset*

$$\mathrm{ntl}(f_{\alpha,\beta}) = \{(u,v) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \mid u - v = m(\alpha+\beta)\}.$$

Proof. We need to apply (Φ_0^+, Φ_1^-) to the circle $x = 0$, to find

$$\mathrm{ntl}(f) = \{((m-1)\zeta + m\beta, (m-1)\zeta - m\alpha) \mid \zeta \in \mathbb{C},\ |\zeta| = 1\}.$$ □

So any translate of the diagonal is the non-transversality invariant of some linked solenoid mapping.

5. VARYING THE INVARIANT

The examples above suggest that the non-transversality invariant might be rather rigid; we will now show that any small perturbation of a translate of the diagonal is still $\mathrm{ntl}(g)$ for some linked solenoid mapping $g : S^3 \to S^3$.

Recall that U_0 is the region between T_0 and T_1. In U_0 choose small disjoint neighborhoods $V = V_0 \cup V_1$ of $\partial U_0 = T_0 \cup T_1$ and W of the y-axis (which is really a circle in S^3).

Lemma 5.1. *For any diffeomorphism $\eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ sufficiently close to the identity in the C^1 topology, there exists a diffeomorphism $h : U_0 \to U_0$ which preserves V and W, and which*
- *is the identity on V,*
- *maps $D_1^-(\omega) \cap W$ to $D_1^-(\eta(\omega)) \cap W$, and*
- *sends each $D_1^-(\omega_1)$ to a disc which is transverse to all discs $D_0^+(\omega_0)$ except on the y-axis.*

Proof. Our map is already defined in W and V. These maps can be patched together by a partition of unity, and the patched map will be close to the identity in the C^1 topology if η is. The result follows, since a C^1-small perturbation of compact transverse manifolds is still transverse. □

Now choose some $f_{(\alpha,\beta)}$, which we will simply call f, and consider $g = h \circ f$.

Theorem 5.2. *We have*

$$\mathrm{ntl}(g) = \{(u,v) \mid \eta(v) = u - (\alpha+\beta)\}.$$

Proof. On T_0^+, we have $\Phi_0^+[g] = \Phi_0^+[f]$, since $f = g$ on T_0^+. On T_1^-, we have $\Phi_1^-[g] = \Phi_1^-[f] \circ h^{-1}$. Indeed, we have $\Phi_0^-[g] = \Phi_0^-[f]$ since $f^{-1} = g^{-1}$ on T_0^-, and

$$\Phi_1^-[g] = m\Phi_0^-[g] \circ g^{-1} = m\Phi_0^-[f] \circ f^{-1} \circ h^{-1} = \Phi_1^-[f] \circ h^{-1}.$$

The locus of non-transversality $X(g) \cap U_0$ is precisely the y-axis by construction, so we need to evaluate

$$\begin{aligned}\left(\Phi_0^+[g]\begin{pmatrix}0\\ \zeta\end{pmatrix}, \Phi_1^-[g]\begin{pmatrix}0\\ \zeta\end{pmatrix}\right) &= \left(\Phi_0^+[f]\begin{pmatrix}0\\ \zeta\end{pmatrix}, \Phi_1^-[f]\left(h^{-1}\begin{pmatrix}0\\ \zeta\end{pmatrix}\right)\right)\\ &= \left(\vartheta_0\begin{pmatrix}0\\ \zeta\end{pmatrix}+\beta, \vartheta_1\left(\eta^{-1}\begin{pmatrix}0\\ \zeta\end{pmatrix}\right)-\alpha\right).\end{aligned}$$

In particular,

$$\mathrm{ntl}(g) = \{(u,v) \mid \eta(v) = u - (\alpha+\beta)\}.$$ □

6. When the invariant non-transversality locus(f) determines the conjugacy class of f

For a standard linked solenoid mappings of degree m, the fibers of Φ^+ and Φ^- have contacts like the contact of the surface of equation $z = \mathrm{Re}(x+iy)^m$ with the surface $z = 0$. Such a contact is unstable if $m > 2$, and there is no chance that $\mathrm{ntl}(g)$ determines the conjugacy class of g for g in a neighborhood of such a linked solenoid map. In this section, we will restrict ourselves to linked solenoid mappings of degree $m = 2$. In that case, we will show that the ntl-invariant determines the conjugacy class in a neighborhood of the standard mappings.

Notation. *As we will be considering all the constructions of Section* 3 *for each of several mappings, we will use the same letters to denote objects as there, but with the mapping in square brackets, as in* $T_1[f]$, $\Phi_0^+[f]$, *etc.*

Choose a particular $f_{\alpha,\beta}$, which will remain fixed for the remainder of the discussion, and which we denote simply by f.

Theorem 6.1. *There exists a* C^2*-neighborhood* $\mathcal{F}$ *of* f *such that if* $g_1, g_2 \in \mathcal{F}$ *and* $\operatorname{ntl}(g_1) = \operatorname{ntl}(g_2)$, *then* g_1 *is topologically conjugate to* g_2.

Remark. This result is presumably false if we use a C^1 neighborhood instead of a C^2 neighborhood. Indeed, a C^1 perturbation of a function with a non-degenerate critical point can have arbitrarily complicated critical points. For instance, the functions

$$f_\varepsilon(x) = \begin{cases} (x-\varepsilon)^2 & \text{if } x \ge \varepsilon, \\ 0 & \text{if } |x| \le \varepsilon, \\ (x+\varepsilon)^2 & \text{if } x \le -\varepsilon \end{cases}$$

depend continuously on ε in the C^1 topology.

As we saw in Remark 3.1, the mappings $\Phi^\pm[g]$ are usually not even of class C^1. We will use in a crucial way that their fibers are of class C^2 if g is of class C^2, and depend continuously on g in the C^2 topology (both for g and for the fiber).

Proof. The proof is quite long, and we will break it up into 5 steps.

Note first that if $\mathcal{F}$ is sufficiently small, then g_1 and g_2 are linked solenoid mappings. In fact, we can use for all three mappings the same torus T_0: for g sufficiently close to f, the mappings $g : T_0^+ \to T_0^+$ and the mappings $g^{-1} : T_0^- \to T_0^-$ will be solenoidal. Moreover, in degree 2 the mappings $\pi_0^\pm[g] = \Phi_0^\pm[g]$ are unique.

Step 1: The non-transversality locus of a perturbation of $f_{(\alpha,\beta)}$.

Lemma 6.2. *If* $\mathcal{F}$ *is sufficiently small, then for all* $g \in \mathcal{F}$, *the non-transversality locus of* g *in* U_0 *is a simple closed curve, parametrized both by* $\pi_0^+[g]$ *and by* $\pi_1^-[g]$.

Remark. The proof is not terribly difficult, but it may seem more complicated than it needs to be: the result appears to follow immediately from the C^2 stability of non-degenerate critical points. However, that theory does not apply here: the functions $\Phi_0^+[g]$ and $\Phi_1^-[g]$ are not of class C^1, never mind C^2, which is what we would need. So we are forced into a more ad-hoc argument.

Proof. In a neighborhood $|x| \le \varepsilon$ of $X[f]$, each leaf $\Phi_0^+[f] = r$ and $\Phi_1^-[f] = s$ represents $\arg y$ as a C^2 function of x. Let us call these functions $\sigma_r[f]$ and $\tau_s[f]$. The same is true if we perturb f (even in the C^2 topology), giving functions $\sigma_r[g]$ and $\tau_s[g]$, which are C^2, and C^2-close to $\sigma_r[f]$ and $\tau_s[f]$ respectively.

In particular, since the Hessian of $\sigma_r[f] - \tau_s[f]$ is non-degenerate, the same is true for g, and $\sigma_r[g] - \tau_s[g]$ has exactly one critical point, of signature $(1, 1)$.

Now consider how the graphs of $\tau_s[g]$ intersect the graph of $\sigma_r[g]$, for fixed r as s varies. These intersections "foliate" the graph of $\sigma_r[g]$, with leaves which are arcs, except for finitely many crosses. Moreover, near the boundary of the graph of $\sigma_r[g]$, this foliation should be topologically the same as that for f, i.e., should be transverse to the boundary except at 4 points. A standard degree argument shows that the number of crosses inside the disk is determined by the foliation near the boundary; since this is 1 for f, it is also 1 for g.

Thus for each $s \in \mathbb{R}/2\pi\mathbb{Z}$, the graph of σ_r is tangent to the graph of τ_s for exactly one s. This proves the lemma. □

Step 2: General observations.

(a) The continuous maps $\Phi_0^+[g]$ and $\Phi_1^-[g]$ have fibers which are smooth manifolds depending continuously, in the C^2 topology, on $g \in \mathcal{F}$ (which also has the C^2 topology). In a neighborhood of T_0, these fibers and T_0 are transverse when $g = f$, hence remain transverse for $g \in \mathcal{F}$ if $\mathcal{F}$ is chosen sufficiently small (here the C^1 topology would do as well).

Since $(\Phi_0^+[f], \Phi_1^-[f]) : T_0 \to (\mathbb{R}/2\pi\mathbb{Z})^2$ is a double covering, the mappings $(\Phi_0^+[g], \Phi_1^-[g]) : T_0 \to (\mathbb{R}/2\pi\mathbb{Z})^2$ are also double covering maps for $g \in \mathcal{F}$. In particular, there exists a unique homeomorphism $\varphi_0 : T_0 \to T_0$ close to the identity such that

$$\Phi_0^+[g_1] = \Phi_0^+[g_2] \circ \varphi_0 \quad \text{and} \quad \Phi_1^-[g_1] = \Phi_1^-[g_2] \circ \varphi_0 \tag{6.1}$$

on T_0.

(b) The same is true for T_1: the maps $(\Phi_0^+[g], \Phi_1^-[g])$ are covering maps $T_1[g] \to (\mathbb{R}/\mathbb{Z})^2$, so there exists a unique homeomorphism $\varphi_1 : T_1[g_1] \to T_1[g_2]$ close to the identity and such that

$$\Phi_1^{\pm}[g_1] = \Phi_1^{\pm}[g_2] \circ \varphi.$$

We now have

$$\Phi_0^+[g_1] = \Phi_0^+[g_2] \circ \varphi_1 \quad \text{and} \quad \Phi_1^-[g_1] = \Phi_1^-[g_2] \circ \varphi_1$$

on T_1.

(c) Call $\pi[g] = (\Phi_0^+[g], \Phi_1^-[g]) : U_0[g] \to (\mathbb{R}/2\pi Z)^2$, and set $S[g] = \pi[g]^{-1}\,\mathrm{ntl}(g)$. For the standard linked solenoid mapping f, the region $U_0[f] \setminus S[f]$ fibers over $\mathbb{R}/2\pi\mathbb{Z})^2 - \mathrm{ntl}(f)$, and the fibers are unions of two intervals, each connecting T_0 to $T_1[f]$. All these properties are stable under small perturbations (as we saw in Lemma 6.2), so they will still be true for g_1 and g_2.

Note that the connecting arcs foliating $U_0[g_i]$ are differentiable, and of finite length; so they can be parametrized by arc length.

Parts (a) and (b) above define φ on $T_0 \cup T_1[g_1]$. Most of the work involves the extension of φ to a homeomorphism $U_0[g_1] \to U_0[g_2]$, so that

$$\Phi_1^{\pm}[g_1] = \Phi_1^{\pm}[g_2] \circ \varphi.$$

Step 3: Adjusting the lengths. It is easy to write a homeomorphism between two closed metric arcs, if we know which ends are supposed to correspond: use the map which is affine with respect to arc length. This is canonical, so it also gives a homeomorphism of bundles of closed metric arcs (again, if the homeomorphism is imposed on the ends).

This construction gives a bundle homeomorphism $\varphi : U_0[g_1] - S[g_1] \to U_0[g_2] - S[g_2]$, i.e., a homeomorphism such that $\pi[g_2] \circ \varphi = \pi[g_1]$, and which coincides with φ_0 on T_0 and with φ_1 on $T_1[g_1]$.

This is almost what we want; the problem is that φ will not in general extend continuously to $S[g_1]$. Indeed, suppose the 4 segments (arms) emanating from some point in $X_0[g_1]$ have lengths $\ell_1, \ell_2, \ell_3, \ell_4$, with the first and third leading to T_0, and the second and fourth leading to T_1. Suppose the corresponding segments in $X_0[g_2]$ have lengths $\ell_1', \ell_2', \ell_3', \ell_4'$. Then as we approach the point of the first arm of $S[g_1]$ a distance $t \le \ell_1$ from T_0 from both sides, the images will be respectively the points at distance

$$t\frac{\ell_1' + \ell_2'}{\ell_1 + \ell_2} \quad \text{and} \quad t\frac{\ell_1' + \ell_4'}{\ell_1 + \ell_4}$$

from T_0 in $U_0[g_2]$, and these points will not in general coincide.

This problem will disappear if $\ell_1, \ell_2, \ell_3, \ell_4$ all have the same length, and so do $\ell_1', \ell_2', \ell_3', \ell_4'$. Since there is nothing special about the standard metric of S^3, we will adjust it so these lengths are equal.

Lemma 6.3. *For $g \in \mathcal{F}$, there exists a continuous Riemannian metric ρ on $U_0[g]$ such that for that metric, the arms of the crosses forming $S[g]$ all have length* 1.

Proof. This is more or less obvious, but we give a proof, since we don't see how to tie it to some generality. Denote by $|dx|$ the standard Riemann metric of S^3. Choose a number L greater than the lengths of all the arms, and a continuous positive function u on $U_0[g]$ which vanishes on $X(g)$ and whose support intersects all arms of crosses.

For $p \in S[g] - X(g)$, let $\ell_i(p)$ be the length of the arm through p for the standard metric, and let $\eta(p) > 0$ be the integral of u over the arm through p with respect to $|dx|$. For the metric on $S[g]$ given by

$$v(p)|dx| = \frac{1}{L}\left(1 + \frac{L - \ell_i(p)}{\eta(p)}u(p)\right)|dx|,$$

the arms all have length 1. Now use the Tietze extension theorem to extend v as a positive function to all of $U_0[g]$, and set $\rho = v|dx|$. □

Using metrics ρ_1 and ρ_2 on $U_0[g_1]$ and $U_0[g_2]$ respectively, we obtain the following result.

Lemma 6.4. *The homeomorphism $U_0[g_1] - S[g_1] \to U_0[g_2] - S[g_2]$, which extends φ_0 and maps points a distance t along the fibers of $\pi[g_1]$ to points the same distance from T_0 along fibers of $\pi[g_2]$, extends uniquely to a fiber homeomorphism $\varphi : U_0[g_1] \to U_0[g_2]$.*

Step 4: Extending φ to $S^3 - (\Sigma^+ \cup \Sigma^-)$. This is now straightforward: for any $x \in S^3 - (\Sigma^+ \cup \Sigma^-)$, there exists $n \in \mathbb{Z}$ such that $g_1^n(x) \in U_0[g_1]$, and which is unique unless x is in one of the T_n, in which case $g_1^{-n} \in T_0$ and $g_1^{-n+1} \in T_1$.

Set $\varphi(x) = g_2^{-n}(\varphi(g_1^n(x)))$ in this case; equation (6.1) guarantees our mapping is well defined.

Step 5: Gluing the constructions together. Finally, we must extend our mapping to $\Sigma^\pm$. Note first that there is a unique mapping $\varphi : \Sigma^\pm[g_1] \to \Sigma^\pm[g_1]$ which conjugates the dynamics and is close to the identity. So far our construction could, with a bit of extra work, have been made C^1; at this point the construction becomes inherently topological, and could not be smoothed.

The problem is to see that φ as defined by different rules on $\Sigma^\pm$ and on $S^3 - (\Sigma^+ \cup \Sigma^-)$ is continuous.

Remark. As far as Step 4 is concerned, we could have used any homeomorphism $U_0[g_1] \to U_0[g_2]$; the fact that it is a fiber homeomorphism was irrelevant. Now we will use this property of φ in a crucial way.

This follows from the following argument: if $x \in S^3 - (\Sigma^+ \cup \Sigma^-)$ is close to $y \in \Sigma^+[g_1]$, then the sequence

$$\Phi_0^+[g_1](x),\ \Phi_0^+[g_1](g_1(x)),\ \Phi_0^+[g_1](g_1^2(x)),\ \ldots$$

is close to the sequence

$$\Phi_0^+[g_1](y),\ \Phi_0^+[g_1](g_1(y)),\ \Phi_0^+[g_1](g_1^2(y)),\ \ldots$$

in the product topology on $(\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$. Our construction of φ now guarantees that

$$\Phi_0^+[g_2](\varphi(x)),\ \Phi_0^+[g_2](\varphi(g_1(x))),\ \Phi_0^+[g_2](\varphi(g_1^2(x))),\ \ldots$$

is close to the sequence

$$\Phi_0^+[g_2](\varphi(y)),\ \Phi_0^+[g_2](\varphi(g_1(y))),\ \Phi_0^+[g_2](\varphi(g_1^2(y))),\ \ldots$$

But this sequence determines $\varphi(y)$ up to $d-1$ choices, and our requirement that φ be close to the identity now guarantees that $\varphi(x)$ is close to $\varphi(y)$. Thus φ is continuous. □

7. OPEN PROBLEMS

There is much we don't know about the invariant ntl. For one thing, we don't know how complicated the non-transversality locus can be, though it is clear from the construction in Section 5 that it can be very complicated. There we took $h : U_0 \to U_0$ to be quite simple, but if it were taken more general it could surely generate very complicated non-transversality loci. More specifically:

- Is $\mathrm{ntl}(f)$ naturally a cycle, and, if so, is its homology class map $m-1$ times the class of the diagonal in $(\mathbb{R}/\mathbb{Z})^2$?
- What can happen to small perturbations of standard solenoidal mappings in degrees $d > 2$?
- Can the components of $X(f)$ knot? In our case, they are all linked like the core curves of the tori $T_i^{\pm}$. Can they link in some different way?
- With our definition of a linked solenoidal map f, the tori T_i are disjoint from $X(f)$. But maybe this definition is too restrictive. Can we deform such a map, away from the solenoids $\Sigma^{\pm}$, so that the stable and unstable manifolds of Σ^+ and Σ^- respectively are non-transversal on a continuum connecting the solenoids?

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JOHN H. HUBBARD:
Department of Mathematics
Cornell University
Ithaca, New York 14850, U. S. A.
EMAIL: `hubbard@math.cornell.edu`

RALPH W. OBERSTE-VORTH:
Department of Mathematics
University of South Florida
Tampa, Florida 33620-5700, U. S. A.
EMAIL: `ralph@math.usf.edu`

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