Symmetry Properties of Confined Convective States

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1 Introduction

This paper is a commentary on the experimental observation observations of Bensimon et al. [1] of convection of a binary fluid mixture in a fluid annulus. They observed a convective pattern that is time periodic, but with non-zero amplitude confined to a stationary sector of the annulus. The symmetry properties of this state are paradoxical. There is no subgroup of the spatio-temporal symmetry group of the experiment that preserves the confined state, but different aspects of the confined states can be associated with different symmetry subgroups. Solutions to a model partial differential equation are displayed here which are reminiscent of the confined states observed in the experiments of Bensimon et al.

The convection experiments that motivate this work were conducted with ethyl alcohol water mixtures in a "thin" annulus. The height and cross-sectional width of the annulus were comparable and much smaller than the circumference of the annulus. The states observed at the onset of convection are patterns of approximately fifty rolls that propagate at uniform speed as traveling waves. By first decreasing the Rayleigh number when the fluid is in a state of uniform traveling waves and increasing it again, Bensimon et al. were able to stabilize a pattern of traveling waves that was confined to a sector of the annulus. In one sector of the annulus, the fluid was conducting while, in the complementary sector, the fluid propagated as a pattern of rolls moving with constant speed from one boundary point of the sector to the other.

The symmetry properties of the confined states are puzzling. In particular, the combination of stationary boundaries to the convective sectors with the propagation of rolls within sectors calls for explanation. Solutions to a model system of equations with these properties are obtained by an amalgamation of two distinct strategies for the analysis of the dynamics of spatially extended systems. The first of these strategies has been employed by Kirchgassner, Mielke and Iooss [3] to describe periodic solutions of the Navier-Stokes equations with complicated spatial behavior. They represent periodic solutions as a complex exponential of time multiplying a fixed spatial function in appropriate coordinates. When this assumption is used to examine solutions of a partial differential equation with sinusoidal time dependence, the result is a system of differential equations for the spatial structure of the time periodic solution. If the spatial dependence is one dimensional, then this is a system of ordinary differential equations. Solutions of these ordinary differential equations matching the spatial boundary conditions give the spatial structure for time periodic solutions of the partial differential equation.

The procedure of Kirchgassner et al. is applied here to a one space dimension system of partial differential equations that describes spatial modulations of a system whose "local dynamics" are given by the normal form of a codimension two bifurcation with a double zero eigenvalue. This system was derived by an amplitude expansion procedure by Zimmerman et al. [7] as

$$\partial_T^2 A - \epsilon(R + (\partial_X - iP)^2)\partial_T A + \epsilon(f_2 + f_3)|A|^2\partial_T A +$$
$$\epsilon f_3 A^2 \partial_T A^* - (R + S + \alpha\partial_X^2 - f_1|A|^2)A = 0$$

Here $A(X,T)$ is a complex valued function of space and time. A systematic derivation of this amplitude equation from the fluid equations is not given here and no attempt has been made to draw a specific correspondence between the two. The goal here is to merely illustrate that the symmetry properties of the confined convective states are natural within the context of systems with the symmetry group $O(2)$. Periodic boundary conditions of period L for the amplitude equation represent the spatial periodicity in the annulus of the experiments. The amplitude equation can be regarded as the simplest nonlinear equation embodying invariance under space-time translation, space reflection, the symmetry $A \leftrightarrow -A$, and the degeneracy imposed on the zero eigenvalues. Time periodic solutions of the amplitude equation with a sinusoidal time dependence that separates from the spatial dependence of the solutions can be written in the form

$$A(X,T) = e^{i\omega T} C(X) \qquad A^*(X,T) = e^{-i\omega T} C^*(X)$$

Substituting these expressions into the partial differential equation yields a second order ordinary differential equation for C that can be expressed as a real four dimensional vector field.

In terms of the functions A and C, a quiescent, conducting sector for the convecting annulus corresponds to an interval over which $C(X) \approx 0$ in the representation of A. The amplitude equation has constant coefficients and is therefore invariant with respect to translations in both X and T. The zero order part (in ϵ) of the vector field is also invariant under the symmetry $X \leftrightarrow -X$ and $A \leftrightarrow A^*$. These symmetries lead to $SO(2)$ symmetries of the full vector field and a time reversal symmetry of the zero order vector field. There are explicit integrals for the zero order vector field and a complete analysis of its solutions are possible. The theory of averaging can then be used to give a discussion of the dynamics of the full vector field for small values of ϵ. The spatial structure of time periodic solutions of the partial differential equation are represented by periodic solutions of the vector field of period L. Thus, the primary objective in studying the vector field is to find periodic orbits of the vector field with quite long period that approach the vicinity of the origin. This is accomplished by first finding homoclinic orbits to the origin for the zero order vector field and determining which perturbations preserve the presence of these homoclinic orbits. By varying parameters of the vector field, the homoclinic orbits are deformed to long periodic (or quasiperiodic) solutions that give the solutions which are sought.

2 Analysis of the vector field

The equation satisfied by C has the form

$$C'' = (a - c|C|^2)C + ibC' + \epsilon i((d + f|C|^2)C + ieC')$$

in which the coefficients are readily computed in terms of the coefficients of the amplitude equation. The ordinary differential equation becomes a real vector field in the variables $C = x + iy$ and $C' = z + iw$:

$$x' = z \tag{1}$$
$$y' = w \tag{2}$$
$$z' = (a - c(x^2 + y^2))x - bw + \epsilon(-(d + f(x^2 + y^2))y + ez) \tag{3}$$
$$w' = (a - c(x^2 + y^2))y + bz + \epsilon((d + f(x^2 + y^2))x + ew) \tag{4}$$

This is the vector field whose solutions we proceed to analyze, first in the case in which $\epsilon = 0$. Though the independent variable in this vector field is X, it will also be labeled t and called time. The equations have an $SO(2)$ corresponding to simultaneous rotations in the (x, y) and (z, w) planes.

Consider the properties of the vector field when $\epsilon = 0$. There is a time reversal symmetry in this case given by $t \leftrightarrow -t, y \leftrightarrow -y, z \leftrightarrow -z$. The system has a pair of first integrals $f_1(x, y, z, w) = b(x^2 + y^2) + 2(yz - xw)$ and $f_2(x, y, z, w) = (z^2 + w^2) - a(x^2 + y^2) + c(x^2 + y^2)^2/2$. There are two types of equilibrium solutions: the origin and the circle defined by $z = w = 0$ and $a(x^2 + y^2) = c$. Note that the presence of the nontrivial equilibrium requires that a and c have the same sign. The characteristic polynomial for the linearization at the origin is $l^4 + (b^2 - 2a)l^2 + a^2$. The roots of this quartic are all imaginary if $b^2 > 4a$ or of the form $\pm p \pm qi$ with p and q nonzero if $b \neq 0$ and $b^2 < 4a$. We are primarily interested in homoclinic orbits to the origin, so we restrict attention to the second case in which the origin is a saddle. The stable and unstable manifolds of the saddle at the origin coincide and fill the surface defined by $f_1 = f_2 = 0$. Along the homoclinic orbits, solutions spiral away from the origin, reaching a maximum value of $x^2 + y^2$ when $xz + yw = 0$ and $(yz - xw)^2 = (x^2 + y^2)(z^2 + w^2)$. The relations $f_1 = f_2 = 0$ imply that the maximum value attained by $x^2 + y^2$ is $(4a - b^2)/2c$.

When $\epsilon > 0$ but small, perturbation calculations estimate which perturbations preserve homoclinic solutions at the origin. The condition for this to occur can be deduced from "Melnikov" criteria or from applying the averaging theorem to the homoclinic trajectories [2]. The derivatives of the functions f_1 and f_2 with respect to time are given by

$$f_1' = 2\epsilon(-(d + f(x^2 + y^2))(x^2 + y^2) + e(yz - xw))$$

$$f_2' = 2\epsilon((d + f(x^2 + y^2))(xw - yz) + e(z^2 + w^2))$$

The conditions to be satisfied for the existence of homoclinic orbits in the full equation are that

$$\int f_1' dt = 0$$

$$\int f_2' dt = 0$$

in the limit of vanishing ϵ. The integrals are taken over the homoclinic orbits of the vector field with $\epsilon = 0$. Since $f_1 = f_2 = 0$ along the homoclinic orbits, the integrals can be reduced to functions of $r = (x^2 + y^2)$ alone. Straightforward algebraic substitutions from $f_1 = f_2 = 0$ yield

$$(bd + 2ae) \int r dt + (bf - ce) \int r^2 dt = 0$$

$$-(be + 2d) \int r dt - 2f \int r^2 dt = 0$$

To proceed further, note that $r' = r\sqrt{(4a - b^2) - 2cr}$ which implies that

$$\int r dt = 2 \frac{\sqrt{(4a - b^2)}}{c}$$

$$\int r^2 dt = 2 \frac{(4a - b^2)\sqrt{(4a - b^2)}}{3c^2}$$

Thus (d, e, f) satisfies the pair of linear equations

$$6cd + 3bce + 2(4a - b^2)f = 0$$

$$3bcd + c(2a + b^2)e + b(4a - b^2)f = 0$$

Solutions of this pair of equations have (d, e, f) proportional to the vector $(4a - b^2, 0, -3c)$. Thus for perturbations to preserve the homoclinic orbit to first order in ϵ, $e = 0$ and $d/f = -(4a - b^2)/3c$. Figures 1 and 2 show trajectories that are near the homoclinic orbits. As noted by a referee, our calculations of parameter values yielding homoclinic orbits can be readily extended to compute parameter values yielding periodic or quasiperiodic solutions to the system in terms of complete elliptic integrals.

3 Discussion

Binary convection in a fluid annulus produces a rich set of observed dynamical states and spatial patterns. This is an excellent system for trying to bring computations, experiments and theory together to understand how spatial patterns displayed by the system arise from underlying symmetries. A full analysis of the experiments of Bensimon et al. and subsequent experiments of Kolodner [4, 5, 6] would start with the equations for fluid convection in a binary fluid mixture with physically realistic boundary conditions, produce a systematic reduction of this problem to a finite dimensional subspace in which the confined states observed in the experiments lie, and perform an analysis of this reduced dynamical system. As referenced in [4], Barten, Lucke and Kamps have performed some calculations beginning with the full set of fluid equations. One can hope that a more complete understanding of the dynamics of this system will emerge over the next few years. However, we have given an explanation for some of the puzzling symmetry properties observed in time periodic states of binary convection in an annulus without resorting to integrations of the full fluid equations. While not addressing the issue of asymptotic stablility, we have exhibited solutions for a model partial differential equation with the same symmetry properties as those observed in the physical system.

The confined convective states that we have found for an amplitude equation model occupy a sector of fixed length in the annulus which is determined by the "time" it takes a homoclinic loop of a system of ordinary differential equations to return to a small neighborhood of the origin. There are a variety of other spatial patterns that are observed. For example, there are observations of confined states that occupy sectors of fixed length in an annulus, whereas others appear to have variable length. In other cases, further experimentation revealed that some properties of observed states were apparently due to imperfections of the apparatus. We hope that analysis of the type presented in this paper can be extended to provide a mathematical basis for predicting which symmetry types of solutions are expected in physical systems with a given symmetry. Our main point is that these considerations go beyond those that come from an analysis of the fluid equations as a dynamical system with symmetry in an infinite dimensional phase space. The reduction of a system of partial differential equation to a system of ordinary differential equations by the use of center manifolds, Lyapunov-Schmidt techniques, etc.

discards important aspects of the way spatial symmetry manifests itself in observed patterns. We have not yet tried to exhibit solutions of the type of system studied in this paper that correspond to all of the observed states in binary convection experiments, though this would be a useful undertaking.

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Figure Captions

Figure 1: This figure shows the projection onto the (x,y) coordinate plane of a trajectory that is approximately homoclinic for the reversible system with $\epsilon = 0$. The parameters are $(a, b, c) = (1, 1, 1)$ and the initial conditions are (x,y,z,w) = (0,0.0001,0,-0.0001). The square shown in the (x,y) plane has vertices at $(-1.5, -1.5)$ and $(1.5, 1.5)$. Numerical integration was performed with a variable step fifth order Runge-Kutta algorithm.

Figure 2a: This figure shows the same data as in Figure 1, except that $\epsilon = 0.01$ and $(d, e, f) = (1, 0, -1)$. A long initial segment of the trajectory has been computed prior to the portion plotted.

Figure 2b: This shows an expanded view of the trajectory plotted in Figure 2a in the square with vertices at $(-0.01, -0.01)$ and $(0.01, 0.01)$.

References

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