

## A Robust Hybrid Stabilization Strategy for Equilibria

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**Abstract**—For an equilibrium of a general dynamical system, the domain of stability of a linear feedback controller is enlarged by the use of a general “hybrid” or “switching” strategy. The strategy is illustrated for numerical simulations of an inverted double pendulum on a cart.

### I. INTRODUCTION

Linear control strategies provide a means for the stabilization of equilibria under general hypotheses. When applied to nonlinear systems, the effectiveness of these strategies depends upon the size of the domain of stability that is produced for the stabilized equilibrium. If this domain is small compared to the accuracy of the measurements or the size of disturbances within the system, then the linear controller is likely to fail within a short period. Failure of the system can be catastrophic, with the system wandering far from the desired equilibrium. We present here a general procedure to recapture stability of a linear controller when a trajectory leaves its region of stability. By using a hybrid strategy based upon discrete switching events within the state space of the plant, the system returns to the region of stability for the linear controller from a much larger domain. The control procedure is robust and remains effective under large classes of perturbations of the underlying system. We illustrate the effectiveness of our technique by applying it to the control of an inverted double pendulum.

The stabilization of unstable equilibria is a fundamental problem for the control of engineering systems. A sufficient condition for stability of an equilibrium point of a smooth dynamical system is that the eigenvalues of its linearization lie in the left-half plane. This is easily proved by several means, for example, by defining a quadratic Lyapunov function in a neighborhood of the equilibrium. Control theory addresses the questions of when stabilization is possible with the modifications (controller) that can be built into the underlying system (plant). Over the past 50 years, an extensive theory of “linear control” has developed comprehensive procedures for determining when the stabilization problem is solvable and for the design of controllers that implement stabilization. This theory is widely employed in engineering for the design of controllers in communication systems, chemical process control, avionics, etc. However, linear feedback control is not a complete panacea for all control problems, even ones of stabilizing equilibria when complete state-space information is available at all times. One difficulty that is encountered in some applications is that the domain of attraction of a controlled equilibrium may be small. This leads to unacceptable constraints on system performance. Small random disturbances in the environment or the inability of the actual physical system to implement its model idealization lead to failures of the controller. The results of the failure can be catastrophic in terms of the design objectives. For example, in the double pendulum example we describe below, the failure of a linear controller leads to large motions of the

pendulum, and in the presence of damping, the pendulum eventually falls to rest at its naturally stable hanging position.

The goal of this work is to provide a simple, effective means of recovery from the failure of a linear controller. We want to design a “safety net” around the (small) domain of attraction of a linear controller, so that if a disturbance moves a system outside this domain of attraction, it will be guided back into the domain by the application of a different control strategy. The strategy that we describe is very general. It can be applied to any system that meets conditions of controllability and accessibility. Moreover, the computations that are required for the design of a controller are based on the linearization of the system at its equilibrium, just as with linear controllers. Verification of the effectiveness of a particular design requires more extensive simulation, but the design guidelines are based upon accessible information.

The framework in which our control strategy is implemented has precursors in the literature [2], [3], [11], [12], [15]. The terms “switching” system, “variable structure” system, and “hybrid” system have all been used to describe piecewise smooth vector fields in the context of control, but there does not yet seem to be an effective, systematic theory of such systems. We shall use the terms switching system and hybrid system interchangeably. One of the essential aspects of our work is the presence of “hysteresis” in a piecewise smooth system: there is a discrete component of the state of the system used by the controller in addition to its location in the underlying state space of the physical system. We recall the description of hybrid systems that we have used previously [1] and adopt here.

The problem domain is a disjoint union of open, connected subsets of  $R^n$ , called charts. Each chart has associated with it a vector field. Inside each chart is a patch—an open subset with closure contained inside the patch. The patch boundaries are assumed piecewise smooth. The evolution of the system is implemented as a sequence of trajectory segments where the endpoint of one segment is connected to the initial point of the next by a transformation, although the transformations are trivial in the examples studied in this note. However, states of a system have both a continuous and discrete part, and switches that change the discrete part of the system state do occur. Time is divided into contiguous periods separated by instances where a state transformation is applied at times referred to as events.

We end this section with a few “philosophical” comments underlying our approach to nonlinear control. Structural stability is a useful concept for dynamical systems, stating that perturbations of a system remain equivalent to the reference system by continuous changes of coordinates. In implementing hybrid control for stabilizing equilibria, we have sought to maintain this type of robustness to perturbations of the system itself. Nothing in the controller should be subject to the choice of exact values of any parameters. In particular, we have avoided the use of sliding modes or switches that must be implemented exactly to be effective. To a large extent, this strategy involves trade-offs to attempts to achieve optimization of some cost function in the control because that is likely to push one to select parameter values in a system that are borderline for a property. If the property is one that involves the stability of the system, then small perturbations cannot be relied upon to maintain the efficacy of the controller. Here the emphasis is squarely upon reliability rather than efficiency. It is quite possible that modifications of the strategy described here will lead to improved performance by returning a system to the stability region of a linear controller more quickly.

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## II. BARRIERS TO UNBOUNDED MOTION

This section discusses a strategy for maintaining the motion of a trajectory in a bounded region of an unstable equilibrium point with a piecewise constant control. Consider the following linear system as a model example:

$$\begin{aligned} \dot{x}_1 &= \lambda_1(x_1 - c) \\ \dot{x}_2 &= \lambda_2(x_2 - c). \end{aligned} \quad (1)$$

In (1),  $c$  represents a "control" that moves the equilibrium point of the systems along a line. While we have chosen a particular form for this system, most planar vector fields with real eigenvalues can be transformed to this representation by a linear change of coordinates. Such a transformation to a "normal form" exists if the system is controllable. Necessary and sufficient conditions for this are that 1) the eigenvalues are distinct and 2) the control moves the equilibrium along a line that is not an eigendirection. We assume that  $\lambda_1 > \lambda_2 > 0$ , so that equilibrium point is a source.

The goal is to define a feedback control  $c(x)$  so that the motion of the system remains within a moderate-sized bounded neighborhood of the origin. We approach this by defining a hybrid system with two patches that are half-planes  $H_+$  and  $H_-$  defined by  $y > x - e$ ,  $y < x + e$ , respectively. The boundaries of  $H_+$  and  $H_-$  are the lines  $L_{\pm}$  defined by  $y = x \mp e$ . These lines are parallel to the control line. In each of the two patches, the control  $c(x)$  takes a constant value  $c_{\pm}$ . The values of  $c$  are chosen with the object of making trajectories in the overlap strip  $H_+ \cap H_-$  stay in a bounded region of the origin. This defines a hybrid system with parameters  $c_{\pm}$  and  $e$ . The goal is now clear: to choose values of these parameters to create a trapping region surrounding the origin.

We can compute readily that the trajectories of the system (1) are defined by

$$\begin{aligned} x_1(t) &= c + \exp(t\lambda_1)(x_1(0) - c) \\ x_2(t) &= c + \exp(t\lambda_2)(x_2(0) - c). \end{aligned}$$

Given  $e$ , we would like to find  $c_{\pm}$  that creates a trapping region around the origin in the strip  $-e \leq x_2 - x_1 < e$ . Along a segment of the right boundary  $L_+$  of the strip, we would like the vector field associated with  $H_+$  to point to the left, toward the interior of the strip. Similarly, we would like the vector field associated with  $H_-$  to point to the right on a segment of the left boundary  $L_-$  of  $H_-$ . These segments are to be chosen so that the flow carries one into the other. See Fig. 1. To choose  $c_{\pm}$  with the desired properties, we argue as follows. For simplicity, we shall assume that  $c_+ = -c_-$  so that the system has a symmetry. The symmetry streamlines the analysis, but is not essential to the argument that we give.

Regard the value of  $c$  as fixed for the moment. We shall determine the conditions we would like it to satisfy. Along  $L_+$ , define the point  $p_+$  to be the point where the vector field has slope 1. The point  $p_+$  is obtained by solving the equations

$$\begin{aligned} x_2 &= x_1 - e \\ \lambda_1(x_1 - c) &= \lambda_2(x_2 - c) \end{aligned}$$

whose solution is

$$(x_1, x_2) = \left( c - \frac{\lambda_2}{\lambda_1 - \lambda_2}e, c - \frac{\lambda_1}{\lambda_1 - \lambda_2}e \right).$$

Above  $p_+$  on  $L_+$ , the vector field points to the right of  $L_+$ . Below  $p_+$  on  $L_+$ , the vector field points to the left of  $L_+$ . The trajectory starting at  $p_+$  should lie above the trapping region in the strip. For example, we might want the intersection of the  $x_2 - x_1$  axis with the strip to lie in the trapping region. For this to occur, it suffices that the trajectory with initial condition  $p_+$  intersect the strip at a point

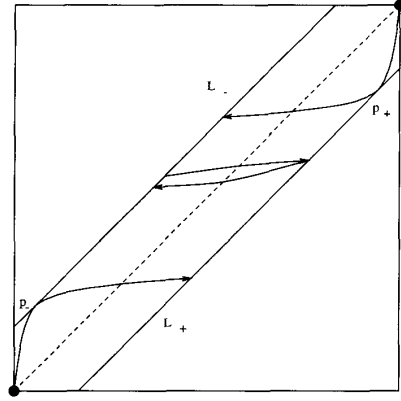


Fig. 1. The barriers  $L_{\pm}$  and region in which trajectories switch back and forth between the boundaries. The curves segments show trajectories, with arrows located where they encounter a patch boundary.

TABLE I

$\lambda$	$c/e$
1.5	9.444
2.0	5.828
2.5	4.614
3.0	4.0
3.5	3.629
5.0	3.063
10	2.518
100	2.065

with a nonnegative value of  $x_1$ . A lower bound for  $c_+$  satisfying this criterion is given by the value of  $c$  for which the trajectory with initial condition  $p_+$  passes through the point  $(0, e)$ . This yields an implicit equation for  $c/e$  in terms of the ratio  $\lambda = \lambda_1/\lambda_2$

$$\frac{\left(\frac{c}{e}\right)^{\lambda_1}}{\left(\frac{c-e}{e}\right)^{\lambda_2}} = \frac{(\lambda_1 - \lambda_2)^{(\lambda_1 - \lambda_2)} (\lambda_2)^{\lambda_2}}{\lambda_1^{\lambda_1}}.$$

Representative values of the solution of this equation are given in Table I to three-digit accuracy. This discussion leads to the following theorem.

*Theorem 2.1:* Let  $c_+/e$  be larger than the solution of the equation

$$\frac{\left(\frac{c}{e}\right)^{\lambda_1}}{\left(\frac{c-e}{e}\right)^{\lambda_2}} = \frac{(\lambda_1 - \lambda_2)^{(\lambda_1 - \lambda_2)} (\lambda_2)^{\lambda_2}}{\lambda_1^{\lambda_1}}$$

and set  $c_- = -c_+$ . Denote by  $W_{\pm}$  the trajectories of the vector fields  $X_{\pm}$  defined by

$$\begin{aligned} \dot{x}_1 &= \lambda_1(x_1 - c_{\pm}) \\ \dot{x}_2 &= \lambda_2(x_2 - c_{\pm}) \end{aligned}$$

passing through the points  $(0, \pm e)$ . Assume  $\lambda_2 < \lambda_1$ . On the lines  $l_{\pm}$  defined by  $x_2 = x_1 \mp e$ , consider the segment  $s_{\pm}$  with endpoints at  $(0, \mp e)$  and the intersections of  $W_{\pm}$  with  $l_{\pm}$ . Then trajectories of  $X_+$  with initial conditions on  $l_+$  intersect  $l_-$ , and trajectories of  $X_-$  with initial conditions on  $l_-$  intersect  $l_+$ .

To prove this theorem, we still need to verify that the trajectory of  $X_-$  with initial condition at  $(0, e)$  intersects the segment  $s_+$ . From  $X_-$ , we obtain the equation

$$\frac{dx_2}{dx_1} = \frac{\lambda_2(x_2 + c)}{\lambda_1(x_1 + c)}. \quad (2)$$

Equation (2) gives the trajectory of  $X_-$  with initial condition  $(0, e)$  in a form parameterized by  $x_1$ :

$$\left( x_1, -c + (c + e) \left( 1 + \frac{x_1}{c} \right)^\lambda \right)$$

with  $\lambda = \lambda_2/\lambda_1$ . Similarly, the trajectory of  $X_+$  with initial condition  $(0, e)$  is given by

$$\left( x_1, c + (c - e) \left( 1 - \frac{x_1}{c} \right)^\lambda \right).$$

These two trajectories intersect at one point in the right-half plane if the trajectory of  $X_-$  passes below the point  $(c, c)$ . Setting  $u = x_1/c$  and  $b = c/e$ , we want to determine when

$$-1 + (1 + b)(1 + u)^\lambda = 1 - (1 - b)(1 - u)^\lambda.$$

This equation is readily solved for  $b$  in terms of  $u$

$$b = \frac{2 - (1 - u)^\lambda - (1 + u)^\lambda}{(1 + u)^\lambda - (1 - u)^\lambda}.$$

At the solution to this equation, we want to have  $-1 + (1 + b)(1 + u)^\lambda - u + b > 0$ , so that the intersection point of the two trajectories lies above the line  $l_+$ . This yields the requirement that

$$b > \frac{(1 + u) - (1 + u)^\lambda}{(1 + u) - (1 + u)^\lambda}.$$

Substituting the value for  $b$  at the intersection point into this inequality and simplifying leads to the requirement that

$$2 - u(1 + u)^\lambda + u(1 - u)^\lambda - 2(1 - u^2) > 0.$$

This inequality holds throughout the unit square in the  $(u, l)$  plane, which is the domain of interest. We conclude that the intersection of the trajectories of  $X_+$  and  $X_-$  with common initial point  $(0, e)$  lies above the line  $l_+$  since we assumed  $\lambda < 1$ .

Recall that our hybrid system  $X$  applies a mode switch from  $X_-$  to  $X_+$  when a trajectory hits  $l_+$  from the left and, similarly, a mode switch from  $X_+$  to  $X_-$  when a trajectory hits  $l_-$  from the right. The theorem implies the following corollary.

*Corollary 2.1:* With the notations introduced above, define  $R$  to be the region bounded by  $W_\pm$ , the trajectories of  $X_\pm$  with initial conditions at  $(0, \mp e)$  and  $s_\pm$ . Then trajectories with initial conditions in  $R$  remain in  $R$  for all forward time.

Less formally, we say that  $R$  is a trapping region for the hybrid system  $X$ . We can say more still about the dynamics of  $X$  in  $R$ . There are passage maps  $\theta_\pm$  that map trajectories with initial conditions on  $s_\pm$  to their intersections with  $s_\mp$ . The maps  $\theta_\pm$  are monotone, so the composition  $\theta_- \circ \theta_+$  is a monotone map of the interval  $s_+$  into itself. It follows that this return map has a stable fixed point, representing a stable periodic orbit for the hybrid system. Additional computations lead to the conclusion that the return map is a **contradiction** and has only a single fixed point. To carry forward these computations, we use the coordinates which scale  $c$  to 1, writing  $u = x_1/c$  and  $b = e/c$  as in the proof of the theorem. The trajectory starting on the line  $l_-$  with initial value  $u = u_0$  is given by

$$\left( u_0, -1 + (1 + b + u_0) \left( \frac{1 + u}{1 + u_0} \right)^\lambda \right).$$

The intersection of the trajectory occurs at a value of  $u = u_1$  satisfying

$$-1 + (1 + b + u_0) \left( \frac{1 + u}{1 + u_0} \right)^\lambda - u_1 + b = 0.$$

We want to estimate  $du_1/du_0$  from this equation. From the equivalent equation

$$\frac{(1 + u_0)^\lambda}{1 + u_0 + b} = \frac{(1 + u_1)^\lambda}{1 + u_1 - b}$$

we deduce that  $du_1/du_0 < 1$ . The right-hand side of the last equation defines a function of  $u_1$  which intersects the function of  $u_0$  on the left-hand side crossing from above to below while decreasing. Therefore, implicitly differentiating the last equation gives  $du_1/du_0 < 1$ . From this, we conclude that the return map of our hybrid system has derivative smaller than 1 and is a contraction.

*Theorem 2.2:* With the same hypotheses of the previous theorem and corollary, there is a stable limit cycle for the hybrid system that is globally attracting for all initial conditions in the trapping region  $R$ .

*Remarks:*

1) Due to the symmetry of the hybrid system, the stable limit cycle is symmetric with respect to the origin.

2) In systems with equilibria that are saddle points with two-dimensional unstable manifolds, the procedure described above can be applied with switching surfaces that are hyperplanes tangent to the directions spanned by the stable manifold of the equilibrium and the tangent to the control curve. We have not investigated systems with unstable manifolds of dimension larger than two, but the strategy might work there as well. In that case, one can try to use switching surfaces that are tangent to the directions spanned by the tangent to the control curve and all but the largest two eigendirections.

### III. MULTIPLE BARRIERS

The results of the previous section can be extended and improved in a number of ways. We describe two.

The barriers described in the previous section can be combined with linear controllers. If one knows a region around the equilibrium that lies in the domain for a linear controller, then one can define a hybrid system with three patches: the system described in the previous section, and a domain cut from these two patches in which the linear controller will be applied. If the stable limit cycle of the switching system intersects the domain in which the linear controller is applied, then the barriers and switching system serve to guide the system to the domain of the linear controller from initial conditions between the two barriers. To prevent the system from exiting the domain of the linear controller, distinct boundaries can be defined to switch the linear controller on and off.

The second extension of the controller described in the previous section is to place multiple barriers in the system parallel to one another. Consider, for example, a planar system with six barriers that are parallel lines  $l_i$ ,  $i = 1, \dots, 6$ . The lines  $l_i$  divide the plane into seven closed "strips"  $S_i$ ,  $i = 1, \dots, 7$ .  $S_1$  and  $S_7$  are half-planes. From the  $S_i$ , we form six overlapping patches  $D_i = S_i \cup S_{i+1}$ . In each of these patches, define a constant control that increases in magnitude as one moves away from the control line. The transition conditions are defined so that if one crosses a patch boundary moving away from the control line, then the control value changes to one of smaller magnitude. The effect of these barriers is to guide a trajectory back toward the origin from farther away from the origin, while at the same time decreasing the amplitude of the control when feasible. Combining these multiple barriers with a linear controller in a neighborhood of the origin allows one to recover from disturbances of large size that move the system outside the region of stability for the linear controller. See Fig. 2 for an illustration of the patches

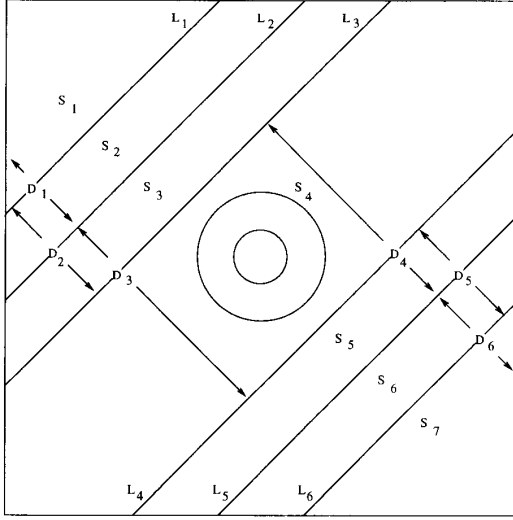


Fig. 2. The geometry of the state space for the system with three sets of barriers and a region in which linear feedback control is used.

associated with a system that has three pairs of barriers and a domain where the linear controller will be applied.

#### IV. AN EXAMPLE: THE DOUBLE PENDULUM ON A CART

We describe an example—a frictionless double pendulum on a zero mass cart whose acceleration along a track can be controlled. The object is to keep the pendulum in the fully upright position. The control of a pendulum on a cart has been a frequently studied problem [4]–[10], [13], [14]. This example provides a good illustration of the effectiveness of our stabilization strategy on a nonlinear system.

The double pendulum consists of two point masses  $m_1$  and  $m_2$  with body 1 attached to a fulcrum and body 2 attached to body 1 by massless rigid rods of lengths  $l_1$  and  $l_2$ . We want to include within the system of equations the additional effect of applying a horizontal acceleration. See Fig. 3. Choose units for which the acceleration of gravity is 1, let the magnitude of the acceleration be  $\alpha$ , and set  $\mu = 1 + m_1/m_2$ . Then the equations of motion are given by the following vector field  $X$  as shown in the equation at the bottom of the page. Here,  $q_1, q_2$  are angular coordinates and  $p_1, p_2$  are the conjugate momenta. The angles  $q_1, q_2$  are measured with respect to vertical rays pointing down, so the stable equilibrium with the pendulum hanging down is given by  $q_1 = q_2 = p_1 = p_2 = 0$ . The vertically upright position that we want to stabilize is given by  $q_1 = q_2 = \pi$  and  $p_1 = p_2 = 0$ .

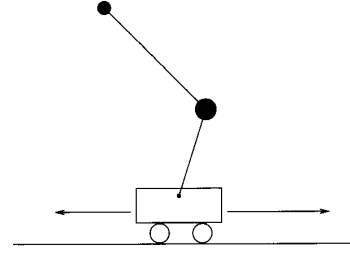


Fig. 3. A double pendulum on a cart.

The vertically upright position is an equilibrium of the pendulum equations (without horizontal acceleration) that has a two-dimensional stable manifold and a two-dimensional unstable manifold. Therefore, we are in a situation for which the theory described earlier can be applied. To do so, we need to construct a linear controller, a region in which the linear controller will be applied, barriers parallel to the hyperplanes spanned by the control line and the stable manifold at the vertical equilibrium, and control values for each of the patches to be used by the controller outside the patch of the linear controller.

The linear controller is defined by making the acceleration of the pendulum fulcrum a linear function of the location of the pendulum in phase space. For convenience, we shall use coordinates  $(-\sin(q_1), -\sin(q_2), p_1, p_2)$  near the upright equilibrium. We seek a vector  $\gamma = (g_1, g_2, g_3, g_4)$  so that setting  $\alpha = g_1 q_1 + g_2 q_2 + g_3 p_1 + g_4 p_2$  makes the upright equilibrium stable. Controllability of the system linearized at the upright equilibrium implies that we can find  $\gamma$  to place the eigenvalues of the linearly controlled system anywhere in the complex plane. We describe one approach to solving this problem. Treating the eigenvalues of the linearization as functions of the control coefficients  $g_i$  gives a system of equations that can be solved for the  $g_i$ . Let  $\lambda_i, i = 1 \dots 4$  be the desired eigenvalues for the controlled system. Denoting the Jacobian matrix of the vector field by  $A$  and  $v = \partial A / \partial \alpha$ , we seek vectors  $w_i \neq 0$  and  $\gamma$  so that

$$(A + \gamma \cdot v^\dagger) w_i = \lambda_i w_i.$$

If  $A + \lambda_i I$  is invertible, rewrite this equation as

$$-\gamma(w_i)(A - \lambda_i I)^{-1} v = w_i.$$

It follows that  $w_i$  is a multiple of  $(A - \lambda_i I)^{-1} v$  and

$$\gamma((A - \lambda_i I)^{-1} v) = -1.$$

As  $i$  varies in  $\{1, 2, 3, 4\}$ , this yields a system of linear equations for  $\gamma$ . If the system is nonsingular, then it uniquely determines  $\gamma$  in terms of the eigenvalues  $\lambda_i$ .

We have investigated a numerical example with parameters  $l_1 = 1/2, l_2 = 3/4, m_1 = 2, m_2 = 1$ , and

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{q}_2 &= p_2 \\ \dot{p}_1 &= \frac{\sin(q_2) \cos(q_1 - q_2) - \mu \sin(q_1) - (l_2 p_2^2 + l_1 p_1^2 \cos(q_1 - q_2)) \sin(q_1 - q_2)}{l_1(\mu - \cos^2(q_1 - q_2))} \\ &\quad + \frac{\alpha(-\mu \cos(q_1) + \cos(q_2) \cos(q_1 - q_2))}{l_1(\mu - \cos^2(q_1 - q_2))} \\ \dot{p}_2 &= \frac{\mu(\sin(q_1) \cos(q_1 - q_2) - \sin(q_2)) + (\mu l_1 p_1^2 + l_2 p_2^2 \cos(q_1 - q_2)) \sin(q_1 - q_2)}{l_2(\mu - \cos^2(q_1 - q_2))} \\ &\quad + \frac{\alpha(-\mu \cos(q_2) + \mu \cos(q_1) \cos(q_1 - q_2))}{l_2(\mu - \cos^2(q_1 - q_2))}. \end{aligned}$$

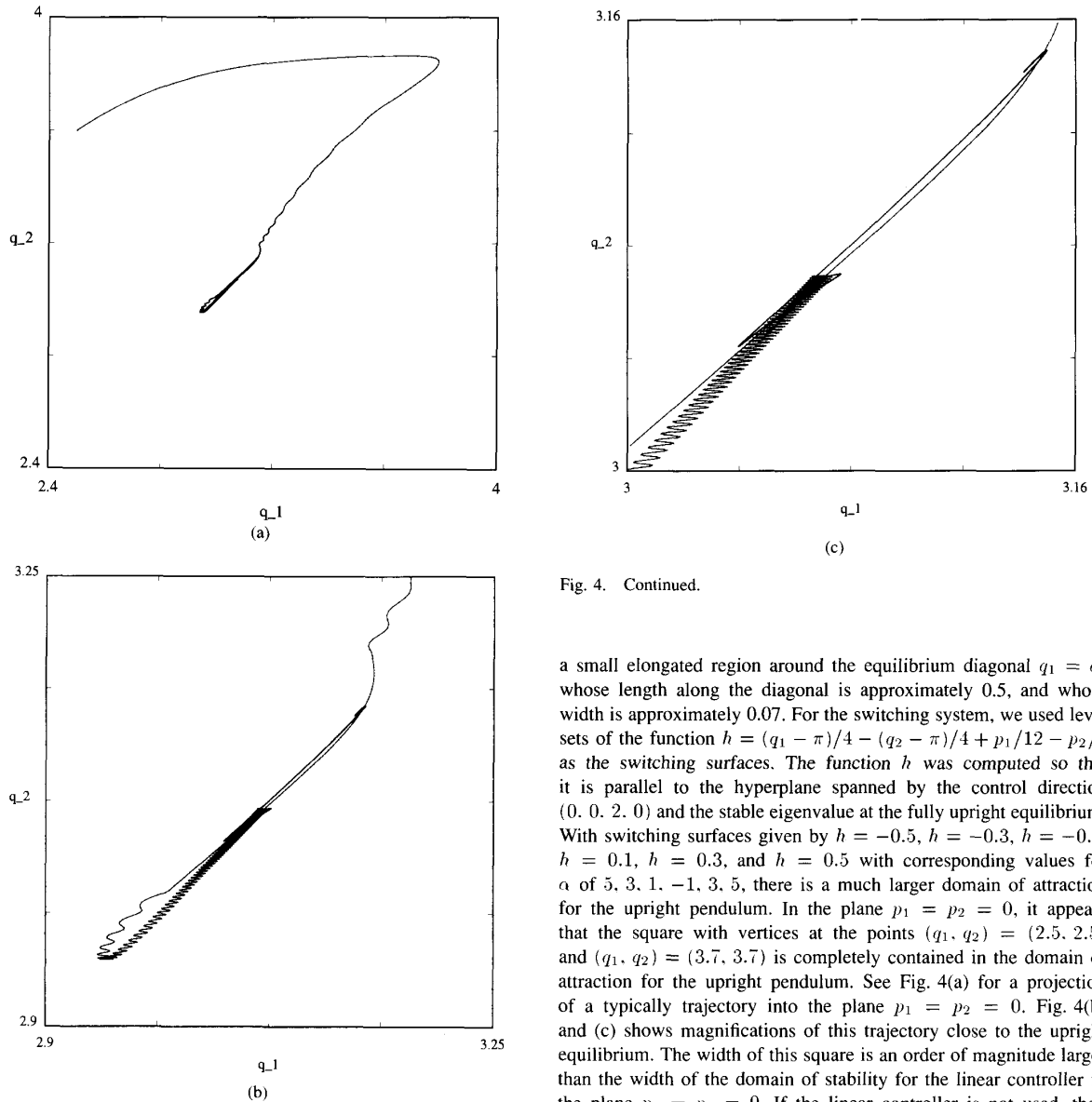


Fig. 4. Continued.

a small elongated region around the equilibrium diagonal  $q_1 = q_2$  whose length along the diagonal is approximately 0.5, and whose width is approximately 0.07. For the switching system, we used level sets of the function  $h = (q_1 - \pi)/4 - (q_2 - \pi)/4 + p_1/12 - p_2/6$  as the switching surfaces. The function  $h$  was computed so that it is parallel to the hyperplane spanned by the control direction  $(0, 0, 2, 0)$  and the stable eigenvalue at the fully upright equilibrium. With switching surfaces given by  $h = -0.5, h = -0.3, h = -0.1, h = 0.1, h = 0.3,$  and  $h = 0.5$  with corresponding values for  $\alpha$  of 5, 3, 1, -1, 3, 5, there is a much larger domain of attraction for the upright pendulum. In the plane  $p_1 = p_2 = 0$ , it appears that the square with vertices at the points  $(q_1, q_2) = (2.5, 2.5)$  and  $(q_1, q_2) = (3.7, 3.7)$  is completely contained in the domain of attraction for the upright pendulum. See Fig. 4(a) for a projection of a typically trajectory into the plane  $p_1 = p_2 = 0$ . Fig. 4(b) and (c) shows magnifications of this trajectory close to the upright equilibrium. The width of this square is an order of magnitude larger than the width of the domain of stability for the linear controller in the plane  $p_1 = p_2 = 0$ . If the linear controller is not used, then the asymptotic state is the limit cycle described in the first section. Switching to and from the linear controller when the square of the distance to the upright equilibrium is 0.002 and 0.005, respectively, appears to robustly stabilize the pendulum at the precise upright state. Note that  $0.005 \approx (0.07)^2$  so that the disk for switching the linear controller off could not be chosen smaller and still remain in the basin of attraction of the vertical equilibrium for the linear controller. Addition of stochastic perturbations to the vector field does not appear to significantly diminish the size of the domain of attraction for the upright equilibrium.

As illustrated by this example, the hybrid or switching strategy that we have presented for the stabilization of equilibria appears to be robust. All aspects of the strategy seem to be "structurally stable" and persistent with respect to very general types of perturbations. It augments linear control for stabilizing equilibria by guiding a trajectory back into the domain of attraction for a linear controller from a much larger region.

Fig. 4. A typical hybrid trajectory converging to the upright position, with magnifications showing increasing detail near the equilibrium position. The initial point of the trajectory is (2.5, 3.6), near the left side of (a).

$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-0.4, -0.5, -0.6, -0.7\}$ . This gives (exactly)  $\gamma = (-3.395, 2.416, -1.1, 1.2595)$ . Using these parameter values, we investigated numerically a model of the double pendulum with the hybrid, switching control strategy described in the last section. The only modification of the strategy was to define two neighborhoods of the vertical pendulum on which switches to and from the linear controller were applied. This was done because many of the linearly controlled trajectories do not approach the equilibrium with monotonically decreasing distance.

To test the effectiveness of this controller, trajectories were computed on a grid of initial conditions in the plane  $p_1 = p_2 = 0$ . The domain of stability of the linear controller in this plane contains

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## Adaptive Control of Systems with Unknown Output Backlash

Gang Tao and Petar V. Kokotović

**Abstract**—Adaptive control schemes for systems with unknown backlash at the plant output are developed. In the case of known backlash, a backlash inverse controller guarantees exact output tracking. When the backlash characteristics are unknown, adaptive laws are designed to update the controller parameters and to guarantee bounded input-output stability. Simulations show significant improvements of the system performance achieved by such adaptive backlash inverse controllers.

## I. INTRODUCTION

Backlash is common in many components of control systems, such as actuators, sensors, and mechanical connections. A typical backlash example is the mechanical motion due to the imperfect contact of two gears. From the early days of classical control theory, the backlash nonlinearity has been recognized as one of the factors severely limiting the performance of feedback systems by causing delays, oscillations, and inaccuracy.

The backlash characteristic is a nondifferentiable nonlinearity which is often poorly known. Therefore, the control of systems with unknown backlash is an open theoretical problem of major relevance to applications. In [1] and [2], we proposed an adaptive control scheme for systems with unknown backlash at input of the plant, that is, in the actuator. In this note, we address the problem with unknown backlash at the plant output, that is, in the sensor, as shown in Fig. 1. Perhaps the most common example is a position servo: the block  $G(D)$  represents the power amplifier motor, and the backlash is in the position sensor such as a potentiometer connected to the motor shaft through a gear box.

A contribution of this note is the construction of new adaptive controller structures for the output backlash problem which is essentially different from the actuator backlash problem. Our new controller structures can be initialized to achieve exact output matching when the plant is known. They lead to a linear parameterization of the closed-loop plant when the backlash is unknown. Such a linear parameterization is crucial for the development of an adaptive scheme to deal with the unknown backlash. Our approach is to develop an adaptive backlash inverse to cancel the unknown backlash effect. A feedback-feedforward controller structure is then combined with such an adaptive backlash inverse to achieve the desired tracking performance.

The note is organized as follows. In Section II, we present the model of the backlash at the output of a linear part and formulate the control problem. In Section III, assuming that the backlash is known, we present a backlash inverse and introduce the idea of backlash inverse control. In Section IV, we develop two adaptive backlash inverse controller structures when the backlash is unknown: one for the linear part known, and the other for the linear part unknown.

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