Applications of Klee's Dehn-Sommerville relations.

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Abstract

We use Klee's Dehn-Sommerville relations and other results on face numbers of homology manifolds without boundary to (i) prove Kalai's conjecture providing lower bounds on the f-vectors of an even-dimensional manifold with all but the middle Betti number vanishing, (ii) verify Kühnel's conjecture that gives an upper bound on the middle Betti number of a $2k$-dimensional manifold in terms of k and the number of vertices, and (iii) partially prove Kühnel's conjecture providing upper bounds on other Betti numbers of odd- and even-dimensional manifolds. For manifolds with boundary, we derive an extension of Klee's Dehn-Sommerville relations and strengthen Kalai's result on the number of their edges.

1 Introduction

In this paper we study face numbers of triangulated manifolds (and, more generally, homology manifolds) with and without boundary. Here we discuss our results deferring most of definitions to subsequent sections.

Our starting point is a beautiful theorem known as the Dehn-Sommerville relations. It asserts that the upper half of the face vector of a triangulated manifold without boundary is determined by its Euler characteristic together with the lower half of the face vector. In this generality the theorem is due to Vic Klee [8].

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Perhaps the most elegant way to present the Dehn-Sommerville relations is via the h-vector of a manifold. The entries of this vector are certain (alternating) linear combinations of the face numbers. On the level of h-vectors, the Dehn-Sommerville relations for triangulated spheres and odd-dimensional manifolds merely state that the h-vector of these complexes is symmetric. In the case of spheres, the components of the h-vector are also known to be positive as they equal dimensions of algebraically determined nonzero vector spaces [20, Chapter 2].

Motivated by Dehn-Sommerville relations together with several commutative algebra results on Stanley-Reisner rings of triangulated manifolds, Kalai suggested [15, Section 7] a modification of the h-vector, the h''-vector, as the "correct" h-vector for (orientable) manifolds without boundary. The h''-vector of orientable manifolds (both odd-dimensional and even-dimensional) has since been shown to be symmetric [15] and nonnegative [16].

Our first result is an extension of Klee's Dehn-Sommerville relations to manifolds with boundary. Specifically, we show that for a triangulated manifold with a fixed boundary Γ, the upper half of the h-vector is determined by the Euler characteristic, its lower half, and the h-vector of Γ. This result is not entirely new. In the language of f-vectors it was first worked out by Macdonald [12], and then rediscovered by Klain [7], and Chen and Yan [2]. However its h-vector form appears to be absent from the literature. We then use this result to define a suitable version of the h''-vector for manifolds with boundary as well as show that it is symmetric and nonnegative.

Our next result concerns new inequalities on the face numbers and Betti numbers of manifolds without boundary. Kalai conjectured (private communication) that the face numbers of a $2k$-dimensional manifold with all but the middle Betti number vanishing are simultaneously minimized by the face numbers of a certain neighborly $2k$–dimensional manifold. We verify this conjecture. We also prove a part of a conjecture by Kühnel [11, Conjecture 18] that provides an upper bound on the middle Betti number of a $2k$-dimensional manifold in terms of k and the number of vertices. Both results turn out to be a simple consequence of the Dehn-Sommerville relations and results from [16].

Kühnel further conjectured [11, Conjecture 18] an upper bound on the i-th Betti number (for all i) of a $(d-1)$-dimensional manifold with n vertices in terms of i, d, and n. We prove that this conjecture is implied by the g-conjecture for spheres. In particular, Kühnel's conjecture holds for manifolds all of whose vertex links are polytopal.

In the last section we return to discussing manifolds with boundary. Here we derive a strengthening of Kalai's theorem [6, Theorem 1.3] that provides a lower bound on the number of edges of a manifold in terms of its dimension, total number of vertices, and the number of interior vertices. Our new bound also depends on the Betti numbers of the boundary.

The structure of the paper is as follows. In Section 2 we review necessary background material. In Section 3 we derive the Dehn-Sommerville relations and define the h''-vector for manifolds with boundary. In Section 4 we deal with Kalai's and Kühnel's conjectures. Finally, in Section 5 we prove a new lower bound on the number of edges of manifolds with boundary.

2 Simplicial complexes and face numbers

In this section we review necessary background material on simplicial complexes, Dehn-Sommerville relations, and Stanley-Reisner rings of homology manifolds. We refer our readers to [20, Chapter 2] and the recent paper [16] for more details on the subject.

Recall that a *simplicial complex* Δ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ that is closed under inclusion and contains all singletons $\{i\}$ for $i \in [n]$. The elements of Δ are called *faces*. The maximal faces (with respect to inclusion) are called *facets*. The *dimension of a face* $F \in \Delta$ is $\dim F := |F| - 1$ and the *dimension of* Δ is the maximal dimension of its faces. For a simplicial complex Δ and its face F, the *link* of F in Δ, $\mathrm{lk}\,(F)$, is the subcomplex of Δ defined by

$$\mathrm{lk}\,(F) = \mathrm{lk}_{\Delta}(F) := \{G \in \Delta \,|\, G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}.$$

In particular, the link of the empty face is the complex itself.

A basic combinatorial invariant of a simplicial complex Δ on the vertex set $[n]$ is its *f-vector*, $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$. Here, $d - 1 = \dim \Delta$ and f_i denotes the number of i-dimensional faces of Δ. Thus $f_{-1} = 1$ (there is only one empty face) and $f_0 = n$. An invariant that contains the same information as the f-vector, but sometimes is more convenient to work with, is the *h-vector* of Δ, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ whose entries are defined by the following relation:

$$\sum_{i=0}^{d} h_i \lambda^i = \sum_{i=0}^{d} f_{i-1} \lambda^i (1 - \lambda)^{d-i}. \tag{1}$$

A central object of this paper is a *homology manifold* (over a field $\mathbf{k}$), that is, a $(d-1)$-dimensional pure simplicial complex Δ such that for all $\emptyset \neq F \in \Delta$, the reduced simplicial homology $\tilde{H}_i(\mathrm{lk}\, F; \mathbf{k})$ vanishes if $i < d - |F| - 1$ and is isomorphic to $\mathbf{k}$ or 0 if $i = d - |F| - 1$. A complex is *pure* if all of its facets have the same dimension. The *boundary faces* of Δ are those faces $F \neq \emptyset$ such that $\tilde{H}_{d-|F|-1}(\mathrm{lk}\, F; \mathbf{k}) = 0$. When Δ has no boundary faces, we write $\partial\Delta = \emptyset$ and Δ is called a homology manifold without boundary. Otherwise, $\partial\Delta$ is the set of boundary faces together with the empty set. We will assume that $\partial\Delta$ is a $(d-2)$-dimensional homology manifold without boundary. Under certain conditions this assumption is superfluous, see, for instance [13]. As demonstrated by the suspension of the real projective plane whose 'boundary' would be the two suspension points for any field whose characteristic is not two, some additional assumption is required. We say that Δ is *orientable* if the pair $(\Delta, \partial\Delta)$ satisfies the usual Poincaré-Lefschetz duality associated with orientable compact manifolds with boundary. The prototypical example of a homology manifold (with or without boundary) is a triangulation of a topological manifold (with or without boundary).

A beautiful theorem due to Klee [8] asserts that if Δ is a homology manifold without boundary, then the f-numbers of Δ satisfy linear relations known as the *Dehn-Sommerville relations*:

$$h_{d-i} - h_i = (-1)^i \binom{d}{i} \left((-1)^{d-1} \tilde{\chi}(\Delta) - 1\right) \quad \text{for all } 0 \leq i \leq d. \tag{2}$$

Here $\tilde{\chi}(\Delta) := \sum_{i=-1}^{d-1}(-1)^i f_i$ is the *reduced Euler characteristic* of Δ. Proofs of several results in this paper rely heavily on Klee's formula (2) and its variations, while other results are concerned with deriving analogs of this formula for manifolds with boundary.

In addition to the Dehn-Sommerville relations we exploit several results on the Stanley-Reisner rings of homology manifolds. If Δ is a simplicial complex on $[n]$, then its *Stanley-Reisner ring* (also called the *face ring*) is

$$\mathbf{k}[\Delta] := \mathbf{k}[x_1, \ldots, x_n]/I_\Delta, \quad \text{where } I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_k} : \{i_1 < i_2 < \cdots < i_k\} \notin \Delta).$$

(Here and throughout the paper $\mathbf{k}$ is an infinite field of an arbitrary characteristic.) Since I_Δ is a monomial ideal, the ring $\mathbf{k}[\Delta]$ is graded, and we denote by $\mathbf{k}[\Delta]_i$ its ith homogeneous component. The Hilbert series of $\mathbf{k}[\Delta]$, $F(\mathbf{k}[\Delta], \lambda) := \sum_{i=0}^{\infty} \dim_{\mathbf{k}} \mathbf{k}[\Delta]_i \cdot \lambda^i$, has the following properties.

Theorem 2.1 **(Stanley)** *Let Δ be a $(d-1)$-dimensional simplicial complex. Then*

$$F(\mathbf{k}[\Delta], \lambda) = \frac{\sum_{i=0}^{d} h_i \lambda^i}{(1-\lambda)^d}.$$

Theorem 2.2 **(Schenzel)** *Let Δ be a $(d-1)$-dimensional homology manifold, and let $\theta_1, \ldots, \theta_d \in \mathbf{k}[\Delta]_1$ be such that $\mathbf{k}[\Delta]/\Theta := \mathbf{k}[\Delta]/(\theta_1, \cdots, \theta_d)$ is a finite-dimensional vector space over $\mathbf{k}$. Then*

$$F(\mathbf{k}[\Delta]/\Theta, \lambda) = \sum_{i=0}^{d} \left(h_i(\Delta) + \binom{d}{i} \sum_{j=1}^{i-1} (-1)^{i-j-1} \beta_{j-1}(\Delta) \right) \cdot \lambda^i,$$

where $\beta_{j-1} := \dim_{\mathbf{k}} \tilde{H}_{j-1}(\Delta; \mathbf{k})$.

Theorem 2.1 can be found in [20, Theorem II.1.4], while Theorem 2.2 is from [18]. In view of Theorem 2.2, for a $(d-1)$-dimensional homology manifold Δ, define

$$h'_i(\Delta) := h_i(\Delta) + \binom{d}{i} \sum_{j=1}^{i-1} (-1)^{i-j-1} \beta_{j-1}(\Delta). \tag{3}$$

We remark that if $|\mathbf{k}| = \infty$ then a set of linear forms $\{\theta_1, \ldots, \theta_d\}$ satisfying the assumptions of Theorem 2.2 always exists, e.g., choosing "generic" $\theta_1, \ldots, \theta_d$ does the job .

The following theorem summarizes several results on the h'-numbers of homology manifolds that will be needed later on. For $0 < m = \binom{x}{i} := x(x-1)\cdots(x-i+1)/i!$ where $0 < x \in \mathbb{R}$, define $m^{<i>} := \binom{x+1}{i+1}$. Also set $0^{<i>} := 0$.

Theorem 2.3 *Let Δ be a $(d-1)$-dimensional homology manifold. Then*

1. *$h'_0 = 1$, $h'_1 = f_0 - d$, and for all $1 \leq i \leq d$*

$$h'_i \geq \binom{d}{i}\beta_{i-1} \qquad \text{and} \qquad h'_{i+1} \leq \left(h'_i - \binom{d}{i}\beta_{i-1}\right)^{<i>}.$$

2. *Moreover, if Δ is a homology manifold without boundary that is orientable over* $\mathbf{k}$*, i.e.,* $\beta_{d-1}(\Delta) = \beta_0(\Delta) + 1$*, then*

$$h'_{d-i} - h'_i = \binom{d}{i}(\beta_i - \beta_{i-1}) \qquad \text{for all } 0 \le i \le d. \tag{4}$$

Part 1 of this theorem was recently proved in [16] (see Theorems 3.5 and 4.3 there). Part 2 is a simple variation of Klee's Dehn-Sommerville relations, see [15, Lemma 5.1]. It is obtained by combining equations (2) and (3) with Poincaré duality for homology manifolds.

Eq. (2) implies that all homology spheres and odd-dimensional manifolds without boundary satisfy $h_i = h_{d-i}$ for all i. While this symmetry fails for even-dimensional manifolds with $\tilde{\chi} \neq 1$, Theorem 2.3 together with Poincaré duality suggests we consider the following modification of the h-vector and yields the following algebraic version of (2).

Proposition 2.4 *Let Δ be a $(d-1)$-dimensional homology manifold without boundary. Assume further that Δ is connected and orientable over* $\mathbf{k}$*. Let*

$$h''_d := h'_d \text{ and } h''_i(\Delta) := h'_i(\Delta) - \binom{d}{i}\beta_{i-1}(\Delta) = h_i - \binom{d}{i}\sum_{j=1}^{i}(-1)^{i-j}\beta_{j-1}, \text{ for } 0 \le i \le d-1.$$

Then $h''_i \ge 0$ and $h''_i(\Delta) = h''_{d-i}(\Delta)$ for all $0 \le i \le d$.

In view of Proposition 2.4 and results of [17] that interpret h''-numbers as dimensions of homogeneous components of a Gorenstein ring, h'' can be regarded as the "correct" h-vector for orientable homology manifolds without boundary. What is the analog of h'' for manifolds with boundary? We deal with this question in the following section.

3 Dehn-Sommerville for manifolds with boundary

Klee's equations (2) generate a complete set of linear relations satisfied by the h-vectors of homology manifolds with empty boundary. More generally, one can fix a (non-empty) homology manifold Γ and ask for the set of all linear relations satisfied by the h-vectors of homology manifolds whose boundary is Γ. Deriving such relations and defining what seems to be the "correct" version of the h''-vector is the goal of this section.

We have the following version of Dehn-Sommerville relations:

Theorem 3.1 *Let Δ be a $(d-1)$-dimensional homology manifold with boundary. Then*

$$h_{d-i}(\Delta) - h_i(\Delta) = \binom{d}{i}(-1)^{d-1-i}\tilde{\chi}(\Delta) - g_i(\partial\Delta) \quad \text{for all } 0 \le i \le d,$$

where $g_i(\partial\Delta) := h_i(\partial\Delta) - h_{i-1}(\partial\Delta)$.

Proof: Write $f_i := f_i(\Delta)$ and $h_i := h_i(\Delta)$. Let $f_i^b := f_i(\partial\Delta)$, and define h_i^b and g_i^b in a similar way. Also let $f_i^\circ := f_i(\Delta) - f_i^b$ be the "interior" f-vector, and let h_i° be defined from f° according to Eq. (1). With this notation, we obtain from [20, Corollary II.7.2] that

$$(-1)^d F(\mathbf{k}[\Delta], 1/\lambda) = (-1)^{d-1}\tilde{\chi}(\Delta) + \sum_{i=1}^{d} \frac{f_{i-1}^\circ \lambda^i}{(1-\lambda)^i}.$$

Substituting Theorem 2.1 in the above formula yields

$$(-1)^d \frac{\sum_{i=0}^d h_{d-i}\lambda^i}{(\lambda-1)^d} = (-1)^{d-1}\tilde{\chi}(\Delta) + \sum_{i=1}^{d} \frac{f_{i-1}^\circ \lambda^i (1-\lambda)^{d-i}}{(1-\lambda)^d} = (-1)^{d-1}\tilde{\chi}(\Delta) + \frac{\sum_{i=0}^d h_i^\circ \lambda^i}{(1-\lambda)^d},$$

which is equivalent to

$$\sum_{i=0}^{d} (h_{d-i} - h_i^\circ)\lambda^i = (-1)^{d-1}\tilde{\chi}(\Delta)(1-\lambda)^d.$$

Subtracting $\sum_{i=0}^d g_i^b \lambda^i$ from both sides and noting that $h_i^\circ + g_i^b = h_i$, implies the result. □

While Theorem 3.1 appears to be new, f-vector forms of the same equality have appeared before. Chen and Yan gave a generalization which applies to more general stratified spaces [2]. However, we believe that the first place where an equivalent formula appears is due to Macdonald [12].

We now turn to finding the right definition of h'' for orientable homology manifolds with boundary. Recall that a connected $(d-1)$-dimensional homology manifold Δ is orientable over $\mathbf{k}$ if $H_{d-1}(\Delta, \partial\Delta; \mathbf{k}) \cong \mathbf{k}$. By Poincaré-Lefschetz duality, if Δ is such a manifold, then $H_{i-1}(\Delta, \partial\Delta) \cong H_{d-i}(\Delta)$. Write $\beta_{i-1}(\Delta, \partial\Delta)$ to denote $\dim H_{i-1}(\Delta, \partial\Delta)$.

We start by expressing $g_i(\partial\Delta)$ in terms of its Betti and h'-numbers. Substituting Eq. (3) in $g_i(\partial\Delta) = h_i(\partial\Delta) - h_{i-1}(\partial\Delta)$ and recalling that $\dim(\partial\Delta) = d-2$, we obtain

$$g_i(\partial\Delta) = \left[h_i'(\partial\Delta) - h_{i-1}'(\partial\Delta) + \binom{d-1}{i-1}\beta_{i-2}(\partial\Delta) \right] + \binom{d}{i} \sum_{j=1}^{i-1} (-1)^{i-j} \beta_{j-1}(\partial\Delta). \quad (5)$$

Theorem 3.2 *Let Δ be a $(d-1)$-dimensional homology manifold with nonempty boundary. If Δ is orientable, then for all $0 \le i < d$,*

$$h_{d-i}'(\Delta) - \binom{d}{d-i}\beta_{d-i-1}(\Delta) = h_i'(\Delta) - \overline{g}_i(\partial\Delta) - \binom{d}{i} \dim \operatorname{Im}\left(H_{i-1}(\Delta) \xrightarrow{\psi} H_{i-1}(\Delta, \partial\Delta)\right),$$

where $\overline{g}_i(\partial\Delta) := h_i'(\partial\Delta) - h_{i-1}'(\partial\Delta) + \binom{d-1}{i-1}\beta_{i-2}(\partial\Delta)$ and ψ is the map in the long exact sequence of the pair $(\Delta, \partial\Delta)$.

Proof: If $i = 0$, then both sides are equal to 0. For $0 < i < d$, using Eq. (3) and Theorem 3.1, we obtain

$$\begin{aligned} h'_{d-i}(\Delta) & - \binom{d}{d-i}\beta_{d-i-1}(\Delta) \\ & = h_i(\Delta) - g_i(\partial\Delta) + (-1)^{d-1-i}\binom{d}{i}\left[\tilde{\chi}(\Delta) + \sum_{j=1}^{d-i}(-1)^j\beta_{j-1}(\Delta)\right] \\ & = h_i(\Delta) - g_i(\partial\Delta) + (-1)^{d-1-i}\binom{d}{i}\left[\sum_{j=d-i+1}^{d}(-1)^{j-1}\beta_{j-1}(\Delta)\right] \\ & = h_i(\Delta) - g_i(\partial\Delta) + (-1)^{i}\binom{d}{i}\sum_{j=0}^{i-1}(-1)^j\beta_j(\Delta, \partial\Delta), \end{aligned}$$

where the last step is by Poincaré-Lefschetz duality. Substituting equations (3) and (5) in the last expression then yields,

$$\begin{aligned} h'_{d-i}(\Delta) & - \binom{d}{d-i}\beta_{d-i-1}(\Delta) \\ & = h'_i(\Delta) - \overline{g}_i(\partial\Delta) - \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{j-i-1}\left[\beta_j(\Delta, \partial\Delta) - \beta_{j-1}(\partial\Delta) + \beta_{j-1}(\Delta)\right]. \end{aligned}$$

The result follows, since by long exact homology sequence of the pair $(\Delta, \partial\Delta)$, the last summand equals $-\binom{d}{i}\dim \mathrm{Im}\,(H_{i-1}(\Delta) \to H_{i-1}(\Delta, \partial\Delta))$. □

Theorem 3.2 suggests the following definition of the h''-vector and shows (together with theorem 2.3) that it is symmetric and non-negative.

Definition 3.3 *For Δ — a $(d-1)$-dimensional orientable homology manifold with a nonempty boundary, define*

$$h''_i(\Delta) = \begin{cases} h'_i(\Delta) - \overline{g}_i(\partial\Delta) - \binom{d}{i}\dim \mathrm{Im}\,(H_{i-1}(\Delta) \to H_{i-1}(\Delta, \partial\Delta)) & \text{for } i \le d/2 \\ h'_i(\Delta) - \binom{d}{i}\beta_{i-1}(\Delta) & \text{for } i > d/2. \end{cases}$$

Note that in the case of the empty boundary and $i < d$, this definition agrees with the one given in Proposition 2.4.

4 Manifolds without boundary: Kalai's and Kühnel's conjectures

In this section we settle a conjecture of Kalai that provides lower bounds for the face numbers of even-dimensional homology manifolds with all Betti numbers but the middle one vanishing. We also partially settle a conjecture by Kühnel on the Betti numbers of homology manifolds. Throughout this section, Δ denotes a $(d-1)$-dimensional orientable homology manifold without boundary. Note that if $\mathbf{k}$ is a field of characteristic two, then this class includes all triangulated topological manifolds without boundary.

We start by discussing even-dimensional manifolds. The following result was conjectured by Kühnel [11, Conjecture 18].

Theorem 4.1 *Let Δ be a $2k$-dimensional orientable homology manifold with n vertices. Then*

$$\binom{2k+1}{k}\beta_k(\Delta) \le \binom{n-k-2}{k+1}.$$

Moreover, if equality is attained then $\beta_i = 0$ for all $i < k$.

Proof: Choose a nonnegative real number x such that

$$h'_k - \binom{2k+1}{k}\beta_{k-1} = \binom{x}{k}.$$

It exists since according to Theorem 2.3, $h'_k - \binom{2k+1}{k}\beta_{k-1} \ge 0$. Moreover, the same theorem implies that $h'_{k+1} \le \binom{x+1}{k+1}$. Thus

$$\binom{2k+1}{k}\beta_k \overset{\text{by (4)}}{=} h'_{k+1} - h'_k + \binom{2k+1}{k}\beta_{k-1} \le \binom{x+1}{k+1} - \binom{x}{k} = \binom{x}{k+1}.$$

Finally, since $h'_1 = n-2k-1$, another application of Theorem 2.3 shows that $h'_k \le \binom{n-k-2}{k}$, hence $x \le n-k-2$, and $\binom{2k+1}{k}\beta_k \le \binom{n-k-2}{k+1}$, as required. Furthermore, equality implies that $h'_i = \binom{n-k-2}{i} = (h'_{i-1})^{<i>}$ for all $2 \le i \le k+1$, which by Theorem 2.3 is possible only if $\beta_i = 0$ for all $i < k$. $\square$

Theorem 4.1 implies that if $\beta_k \ge 1$, then $n-k-2 \ge 2k+1$, or equivalently, $n \ge 3k+3$. In other words, having a non-vanishing middle Betti number requires at least $3k+3$ vertices. (This result was originally proved by Brehm and Kühnel for PL-triangulations [1].) Moreover, if such a homology manifold, $\mathcal{M}_k$, has exactly $3k+3$ vertices, then

$$\beta_k(\mathcal{M}_k) = 1,\ \beta_i(\mathcal{M}_k) = 0 \text{ for } i < k, \text{ and } h_i(\mathcal{M}_k) = h'_i(\mathcal{M}_k) = \binom{k+1+i}{i} \text{ for } i \le k+1.$$

In particular, the face numbers of $\mathcal{M}_k$ (whether it exists or not) are uniquely determined by Eqs. (1) and (2). These face numbers turn out to be minimal in the following sense (as was conjectured by Gil Kalai, personal communication):

Theorem 4.2 *Let Δ be a $2k$-dimensional orientable homology manifold with $\beta_k \neq 0$ being the only non vanishing Betti number out of all β_l, $l \le k$. Then*

$$f_{i-1}(\Delta) \ge f_{i-1}(\mathcal{M}_k) \quad \textit{for all} \quad 1 \le i \le 2k+1.$$

Proof: Substituting $\beta_l = 0$, $l < k$, in Theorem 2.2 and Eq. (2), we obtain that

$$h_j(\Delta) = h'_j(\Delta) \text{ and } h_{k+j+1}(\Delta) = h_{k-j}(\Delta) + (-1)^j \binom{2k+1}{k-j}\beta_k(\Delta) \quad \text{for } 0 \le j \le k.$$

Eq. (1) then implies

$$f_{i-1}(\Delta) = \sum_{j=0}^{i} \binom{2k+1-j}{2k+1-i} h'_j(\Delta), \quad \text{if } i \le k, \quad \text{and} \tag{6}$$

$$\begin{aligned} f_{i-1}(\Delta) &= \sum_{j=0}^{k} \left[\binom{2k+1-j}{2k+1-i} + \binom{j}{2k+1-i}\right] h'_j(\Delta) \\ &+ \beta_k(\Delta)\left[\sum_{j=0}^{i-k-1} (-1)^j \binom{k-j}{2k+1-i}\binom{2k+1}{k-j}\right] \quad \text{if } i \ge k+1. \end{aligned} \tag{7}$$

Since (i) the same formulas apply to the f-numbers of $\mathcal{M}_k$, (ii) the coefficients of the h'-numbers in Eqs. (6) and (7) are nonnegative, and (iii) $\beta_k(\Delta) \ge 1 = \beta_k(\mathcal{M}_k)$, to complete the proof it only remains to show that $h'_i(\Delta) \ge h'_i(\mathcal{M}_k)$ for all $i \le k$ and that the coefficient of β_k in Eq. (7) is nonnegative for all $i \ge k+1$.

The latter assertion follows by noting that the sequence

$$a_j = \binom{k-j}{2k+1-i}\binom{2k+1}{k-j}, \quad 0 \le j \le i-k-1$$

is decreasing (indeed, $a_j/a_{j+1} = (k+2+j)/(i-k-1-j) > 1$), and hence $a_0 - a_1 + \cdots + (-1)^{i-k-1}a_{i-k-1} \ge 0$.

To verify the former assertion, we use the same trick as in the proof of Theorem 4.1. Let $0 \le x \in \mathbb{R}$ be such that $h'_k(\Delta) = \binom{x}{k}$. Then according to Theorem 2.3, $h'_{k+1}(\Delta) \le \binom{x+1}{k+1}$ while $h'_{k+1}(\Delta) - h'_k(\Delta) = \binom{2k+1}{k+1}\beta_k \ge \binom{2k+1}{k+1}$. Thus we have

$$\binom{2k+1}{k+1} \le h'_{k+1}(\Delta) - h'_k(\Delta) \le \binom{x+1}{k+1} - \binom{x}{k} = \binom{x}{k+1}.$$

Hence $x \ge 2k+1$, and so $h'_k(\Delta) \ge \binom{2k+1}{k}$. Applying Theorem 2.3 once again, we infer that $h'_i(\Delta) \ge \binom{k+1+i}{i} = h'_i(\mathcal{M}_k)$ for all $i \le k$. □

In addition to Theorem 4.1, Kühnel conjectured (see [11, Conjecture 18]) that a $(d-1)$-dimensional manifold with n vertices satisfies $\binom{d+1}{j+1}\beta_j(\Delta) \le \binom{n-d+j-1}{j+1}$ for all $0 \le j \le \lfloor d/2 \rfloor - 1$. The case of $j = 0$ merely says that every connected component of Δ has at least $d+1$ vertices. The case of $j = 1$ is equivalent to Kalai's lower bound conjecture [6, Conjecture 14.1] that was recently settled in [16, Theorem 5.2]. For other values of j we have the following partial result. We recall that a $(d-1)$-dimensional homology sphere Γ is said to have the *hard Lefschetz property* if for a generic choice of $\theta_1, \ldots, \theta_d, \omega \in \mathbf{k}[\Gamma]_1$, the map

$$\mathbf{k}[\Gamma]/(\theta_1, \ldots, \theta_d)_i \xrightarrow{\cdot \omega^{d-2i}} \mathbf{k}[\Gamma]/(\theta_1, \ldots, \theta_d)_{d-i}$$

is an isomorphism of $\mathbf{k}$-spaces for all $i \leq d/2$. It is a result of Stanley [19] that in the case of $\operatorname{char}\mathbf{k} = 0$ all simplicial polytopes have this property, and it is the celebrated g-conjecture that all homology spheres do.

Theorem 4.3 *Let Δ be a $(d-1)$-dimensional orientable homology manifold with n vertices. If for every vertex v of Δ the link of v has the hard Lefschetz property (e.g., $\operatorname{char}\mathbf{k} = 0$ and all vertex links are polytopal spheres), then*

$$\binom{d+1}{j+1}\beta_j(\Delta) \leq \binom{n-d+j-1}{j+1} \quad \text{for all } 0 \leq j \leq \lfloor \frac{d}{2} \rfloor - 1.$$

If equality is attained for some $j = j_0$, then $\beta_i = 0$ for all $i \neq j_0$, $0 \leq i \leq \lfloor d/2 \rfloor - 1$.

Proof: Since all vertex links of Δ have the hard Lefschetz property, Theorem 4.26 of [21] implies that for a sufficiently generic choice of $\theta_1, \ldots, \theta_d, \omega \in \mathbf{k}[\Delta]_1$ and every $j \leq \lfloor d/2 \rfloor - 1$, the linear map

$$\mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_d)_{d-j-1} \xrightarrow{\cdot\omega} \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_d)_{d-j}$$

is surjective. The dimensions of the spaces involved are h'_{d-j-1} and h'_{d-j}, respectively (see Theorem 2.2). Also, by [16, Cor. 3.6], the dimension of the kernel of this map is at least $\binom{d}{d-j-1}\beta_{d-j-2}$. Therefore,

$$h'_{d-j} \leq h'_{d-j-1} - \binom{d}{d-j-1}\beta_{d-j-2} \quad \text{for all } j \leq \lfloor d/2 \rfloor - 1. \tag{8}$$

Apply Poincaré duality and Eq. (4) to rewrite this inequality in the form

$$h'_j + \binom{d}{j}(\beta_j - \beta_{j-1}) \leq h'_{j+1} - \binom{d}{j+1}\beta_j,$$

or, equivalently,

$$\binom{d+1}{j+1}\beta_j \leq h'_{j+1} - \left[h'_j - \binom{d}{j}\beta_{j-1}\right]. \tag{9}$$

Let $0 \leq x \in \mathbb{R}$ be such that $h'_{j+1} = \binom{x+1}{j+1}$. Then by Theorem 2.3, $h'_j - \binom{d}{j}\beta_{j-1} \geq \binom{x}{j}$, and so the right-hand-side of (9) is $\leq \binom{x}{j+1}$. Also, since $h'_1 = n-d$, $h'_{j+1} \leq \binom{n-d+j}{j+1}$, and hence $x \leq n-d+j-1$. Thus $\binom{d+1}{j+1}\beta_j \leq \binom{x}{j+1} \leq \binom{n-d+j-1}{j+1}$, as required.

If equality occurs for some $j = j_0$, then $x = n-d+j_0-1$, and we obtain that $h'_{i+1} = \binom{n-d+i}{i+1} = (h'_i)^{<i>}$ for all $i \leq j_0$. By Theorem 2.3 this can happen only if $\beta_{i-1} = 0$ for all $i \leq j_0$. Moreover in this case, $h_{i+1} = \binom{n-d+i}{i+1}$ for all $i \leq j_0$, hence Δ is (j_0+1)-neighborly (that is, every set of j_0+1 vertices of Δ is a face of Δ).

What about β_i for $i > j_0$? To prove that all these Betti numbers vanish as well, note that for equality $\binom{d+1}{j_0+1}\beta_{j_0} = \binom{n-d+j_0-1}{j_0+1}$ to happen, the inequality in (8) should hold as equality for $j = j_0$. The same argument as in the proof of [16, Theorem 5.2] then shows that $h_{j_0}(\operatorname{lk} v) = h_{j_0+1}(\operatorname{lk} v)$ for every vertex v of Δ. Since, by our assumptions, all vertex links of Δ satisfy the g-conjecture, and since Δ is (j_0+1)-neighborly, we conclude that for every vertex v, $h_i(\mathrm{lk}\, v) = h_{j_0}(\mathrm{lk}\, v)$ for all $j_0 \le i \le (d-1)/2$, and that $h(\mathrm{lk}\, v) = h(\mathrm{lk}\, w)$ for all vertices v and w of Δ. This information about links turns out to be enough to compute the entire h-vector of Δ. Indeed, it follows from [5, Remark 4.3] that

$$h_r(\Delta) = (-1)^r \binom{d}{r} + \sum_{i=0}^{r-1} (-1)^{r-i-1} \frac{(d-1-i)!i!}{(d-r)!r!} \cdot n \cdot h_i(\mathrm{lk}\, v).$$

Hence

$$g_{r+1}(\Delta) = h_{r+1} - h_r = \binom{d+1}{r+1} \left[(-1)^{r+1} + n \sum_{i=0}^{r-1} \frac{(-1)^{r-i} h_i(\mathrm{lk}\, v)}{(d-i)\binom{d}{i}} + \frac{n \cdot h_r(\mathrm{lk}\, v)}{(r+1)\binom{d+1}{r+1}} \right],$$

and since $h_{j_0}(\mathrm{lk}\, v) = h_{j_0+1}(\mathrm{lk}\, v) = \cdots$, we infer that for all $j_0 + 1 \le r \le (d-1)/2$,

$$\frac{g_{r+1}}{\binom{d+1}{r+1}} + \frac{g_r}{\binom{d+1}{r}} = n \cdot h_{j_0}(\mathrm{lk}\, v) \left[-\frac{1}{(d-r+1)\binom{d}{r-1}} + \frac{1}{(r+1)\binom{d+1}{r+1}} + \frac{1}{r\binom{d+1}{r}} \right] = 0.$$

Therefore,

$$(-1)^{r-j_0} \frac{g_{r+1}}{\binom{d+1}{r+1}} = \frac{g_{j_0+1}}{\binom{d+1}{j_0+1}} = \frac{h_{j_0+1} - h_{j_0}}{\binom{d+1}{j_0+1}} = \beta_{j_0}.$$

Substituting this result in Eq. (9) with $j = j_0 + 1$ and using that all β_i for $i < j_0$ vanish, yields $\binom{d+1}{j_0+1}\beta_{j_0+1} \le \left[h_{j_0+2} + \binom{d}{j_0+2}\beta_{j_0}\right] - \left[h_{j_0+1} - \binom{d}{j_0+1}\beta_{j_0}\right] = g_{j_0+2} + \binom{d+1}{j_0+2}\beta_{j_0} = 0$, and so $\beta_{j_0+1} = 0$. Assuming by induction that $\beta_{j_0+1} = \ldots = \beta_{r-1} = 0$, a similar computation using Eq. (9) with $j = r$ then implies that $\beta_r = 0$ for all $j_0 < r \le (d-1)/2$. □

Kalai conjectured [6, Conj. 14.2] that if Δ is a $(d-1)$-dimensional manifold without boundary, then $h''_{j+1}(\Delta) - h''_j(\Delta) \ge \binom{d}{j}\beta_j(\Delta)$. This is an immediate consequence of Eq. (9). Thus Kalai's conjecture holds for all manifolds whose vertex links have the hard Lefschetz property.

5 Rigidity inequality for manifolds with boundary

In this section we return to our discussion of the face numbers of homology manifolds with nonempty boundary. The goal here is to strengthen Kalai's result [6, Theorem 1.3] asserting that if Δ is a $(d-1)$-dimensional manifold with boundary and $d \ge 3$, then $h_2(\Delta) \ge f_0^\circ(\Delta)$, where as in Section 3, $f_0^\circ(\Delta)$ denotes the number of interior vertices of Δ. Our main result is

Theorem 5.1 *If Δ is a connected $(d-1)$-dimensional homology manifold with nonempty orientable boundary and $d \ge 5$, then*

$$h_2(\Delta) \ge f_0^\circ(\Delta) + \binom{d}{2}\beta_1(\partial\Delta) + d\ \beta_0(\partial\Delta). \tag{10}$$

If $d=4$ *and the characteristic of* $\mathbf{k}$ *is two, then*

$$h_2(\Delta) \geq f_0^\circ(\Delta) + 3\ \beta_1(\partial\Delta) + 4\ \beta_0(\partial\Delta).$$

Since the boundary of a 3-manifold with boundary is a collection of closed surfaces, using a field whose characteristic is two maximizes the relevant Betti numbers, so we have restricted ourselves to this case. Before beginning the proof of this theorem we establish some preliminary results pertaining to rigidity in characteristic $p > 0$. In characteristic zero the cone lemma, gluing lemma, and Proposition 5.5 follow easily from the work of Kalai [6] and Lee [10].

Definition 5.2 *A* $(d-1)$*-dimensional complex* Δ *is* **k-rigid** *if for generic* $\theta_1, \ldots, \theta_{d+1}$ *linear forms and* $1 \leq i \leq d+1$*, multiplication* $\cdot\theta_i : \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1})_1 \rightarrow \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1})_2$ *is injective.*

It follows from Proposition 5.5 below that if Δ is $\mathbf{k}$-rigid, then the $\mathbf{k}$-dimension of $\mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_d)$ is $h_2(\Delta)$ and of $\mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{d+1})$ is $g_2(\Delta)$. In fact, it is not hard to see (but we will not use it here) that the converse holds as well.

Lemma 5.3 *(Cone lemma) If* Δ *is* $\mathbf{k}$*-rigid, then the cone on* $\Delta, C(\Delta)$*, is* $\mathbf{k}$*-rigid.*

Proof: Observe that $\mathbf{k}[C(\Delta)] \cong \mathbf{k}[\Delta] \otimes_{\mathbf{k}} \mathbf{k}[x_0]$. Hence for any θ_0 of the form $x_0 + \sum_{i=1}^n \alpha_i x_i$, θ_0 is a non-zero-divisor on $\mathbf{k}[C(\Delta)]_1$ and the quotient ring $\mathbf{k}[C(\Delta)]/(\theta_0)$ is isomorphic to $\mathbf{k}[\Delta]$. The assertion follows. $\square$

Lemma 5.4 *(Gluing lemma) If* Δ_1 *and* Δ_2 *are* $(d-1)$*-dimensional* $\mathbf{k}$*-rigid complexes and there are at least* d *vertices in* $\Delta_1 \cap \Delta_2$*, then* $\Delta_1 \cup \Delta_2$ *is* $\mathbf{k}$*-rigid.*

Proof: Set $\Delta = \Delta_1 \cup \Delta_2$. Since Δ_l $(l = 1, 2)$ is a subcomplex of Δ, there is a natural surjection $\mathbf{k}[\Delta] \longrightarrow \mathbf{k}[\Delta]_l$. Consider the following commutative square.

$$\begin{array}{ccccc} \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1})_2 & \longrightarrow & \mathbf{k}[\Delta_1]/(\overline{\theta}_1, \ldots, \overline{\theta}_{i-1})_2 & \longrightarrow & 0 \\ \uparrow \cdot\theta_i & & \uparrow \cdot\theta_i & & \\ \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1})_1 & \longrightarrow & \mathbf{k}[\Delta_1]/(\overline{\theta}_1, \ldots, \overline{\theta}_{i-1})_1 & \longrightarrow & 0. \end{array} \tag{11}$$

Here, $\overline{\theta}$ is the image of θ in $\mathbf{k}[\Delta_1]$. Suppose ω is in the kernel of the left-hand vertical map. Then its image in $\mathbf{k}[\Delta_1]/(\theta_1, \ldots, \theta_{i-1})_1$ must be in the kernel of the right-hand vertical map, and hence zero when restricted to $\mathbf{k}[\Delta_1]/(\theta_1, \ldots, \theta_{i-1})_1$. Similarly, ω is zero when restricted to $\mathbf{k}[\Delta_2]/(\theta_1, \ldots, \theta_{i-1})_1$. But, if there are at least d vertices in $\Delta_1 \cap \Delta_2$, then $\omega = 0$ in $\mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1})_1$. $\square$

Proposition 5.5 *Let $\Delta_1, \ldots, \Delta_b$ be* **k**-*rigid $(d-1)$-dimensional complexes with disjoint sets of vertices. If $\Delta = \cup \Delta_i$, then for generic linear forms $\Theta = (\theta_1, \ldots, \theta_d)$ and ω,*

$$\dim_{\mathbf{k}} (\mathbf{k}[\Delta]/\Theta)_2 = h_2(\Delta) + \binom{d}{2}(b-1) \quad \text{and} \quad \dim_{\mathbf{k}} \ker \left[(\mathbf{k}[\Delta]/\Theta)_1 \xrightarrow{\cdot\omega} (\mathbf{k}[\Delta]/\Theta)_2 \right] = d(b-1).$$

Proof: Suppose that w is in the kernel of $\cdot\theta_1 : \mathbf{k}[\Delta]_1 \to \mathbf{k}[\Delta]_2$. Using a commutative square analogous to (11), we see that restricted to each vertex set w is zero. Hence $w = 0$. Therefore,

$$\dim_{\mathbf{k}} (\mathbf{k}[\Delta]/(\theta_1))_2 = \dim_{\mathbf{k}} \mathbf{k}[\Delta]_2 - \dim_{\mathbf{k}} \mathbf{k}[\Delta]_1 = (f_1 + f_0) - f_0 = f_1.$$

Now replace $\mathbf{k}[\Delta]$ with $\mathbf{k}[\Delta]/(\theta_1)$ in (11) and consider multiplication by θ_2. The same argument shows that any w in the kernel must restrict to a multiple of θ_1 on the vertex set of each Δ_j. The dimension of the space of such w in $(\mathbf{k}[\Delta]/(\theta_1))_1$ is $b-1$. Thus,

$$\dim_{\mathbf{k}}(\mathbf{k}[\Delta]/(\theta_1, \theta_2))_2 = f_1 - (f_0 - 1) + (b-1).$$

Continuing with this reasoning we see that for each i the dimension of the kernel of multiplication by θ_i on $(\mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1}))_1$ is $(i-1)(b-1)$. Hence, for $i \geq 2$,

$$\dim_{\mathbf{k}}(\mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_{i-1}))_2 = f_1 - (i-2)f_0 + \binom{i-1}{2} + \binom{i-1}{2}(b-1).$$

Setting $i = d+1$ finishes the proof. □

Proof of Theorem 5.1: First we consider the situation when $d \geq 5$. Let Γ be the simplicial complex obtained from Δ by coning off each component of the boundary of Δ. Specifically, let $c_1, \ldots, c_b$ be the components of the boundary of $\partial\Delta$. We introduce new vertices $\mathbf{n+1}, \ldots, \mathbf{n+b}$ and set

$$\Sigma = ((\mathbf{n+1}) * c_1) \cup \cdots \cup ((\mathbf{n+b}) * c_b) \quad \text{and} \quad \Gamma = \Delta \cup \Sigma.$$

Then Γ is a $(d-1)$-dimensional pseudomanifold that is **k**-rigid. The proof is by induction on d. Any Δ homeomorphic to S^2 is **k**-rigid. This follows from [14, Cor. 3.5]. So the cone lemma implies that the closed star of a vertex in a three-dimensional **k**-homology sphere is **k**-rigid. Now, using the gluing lemma we can take the union with closed stars of other vertices until we see that an arbitrary three-dimensional **k**-homology sphere is **k**-rigid. Since for every vertex $v \in \Delta$, the link of v in Γ is a **k**-homology sphere, induction on d implies that this link is **k**-rigid. Hence the closed star of v in Γ is **k**-rigid for all $v \in \Delta$. Taking the union of the closed stars of the noncone points using the gluing lemma shows that Γ is **k**-rigid.

Observe that $f_0(\Gamma) = f_0(\Delta) + b$ and $f_1(\Gamma) = f_1(\Delta) + f_0(\partial\Delta)$. Thus $h_2(\Gamma) = h_2(\Delta) + h_1(\partial\Delta) - (d-1)\beta_0(\partial\Delta)$. For Σ we have $f_0(\Sigma) = f_0(\partial\Delta) + b$ and $f_1(\Sigma) = f_1(\partial\Delta) + f_0(\partial\Delta)$. Hence, $h_2(\Sigma) = h_2(\partial\Delta) - (d-1)\beta_0(\partial\Delta)$.

Consider the face rings $\mathbf{k}[\Gamma]$ and $\mathbf{k}[\Sigma]$, and let $\theta_1, \ldots, \theta_d, \omega \in \mathbf{k}[\Gamma]_1$ be generic linear forms. Since Σ is a subcomplex of Γ, there is a natural surjection $\phi : \mathbf{k}[\Gamma] \longrightarrow \mathbf{k}[\Sigma]$. Let $\overline{\theta}_i$ denote the image of θ_i under ϕ, and consider $\mathbf{k}(\Gamma) := \mathbf{k}[\Gamma]/(\theta_1, \ldots, \theta_d)$ and $\mathbf{k}(\Sigma) := \mathbf{k}[\Sigma]/(\overline{\theta}_1, \ldots, \overline{\theta}_d)$. Then ϕ induces a surjection $\mathbf{k}(\Gamma) \longrightarrow \mathbf{k}(\Sigma)$. Denoting by $I \subset \mathbf{k}(\Gamma)$ its kernel, we obtain the following commutative diagram whose rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_2 & \longrightarrow & \mathbf{k}(\Gamma)_2 & \longrightarrow & \mathbf{k}(\Sigma)_2 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \cdot\omega} & & \uparrow{\scriptstyle \cdot\omega} & & \uparrow{\scriptstyle \cdot\omega} & & \\ 0 & \longrightarrow & I_1 & \longrightarrow & \mathbf{k}(\Gamma)_1 & \longrightarrow & \mathbf{k}(\Sigma)_1 & \longrightarrow & 0. \end{array} \tag{12}$$

Since Γ is $\mathbf{k}$-rigid, $\dim \mathbf{k}(\Gamma)_2 = h_2(\Gamma)$ and the middle vertical map is an injection. Hence the left vertical map is also an injection. By the cone lemma and the argument which proved that Γ is $\mathbf{k}$-rigid, each of the b components of Σ is $\mathbf{k}$-rigid. Proposition 5.5 says that $\dim_{\mathbf{k}} \mathbf{k}(\Sigma)_2 = h_2(\Sigma) + \binom{d}{2}\beta_0(\partial\Delta)$.

By Proposition 5.5, the dimension of the kernel of the right vertical map is $d\beta_0(\partial\Delta)$. Applying the snake lemma, we find that the dimension of the cokernel of $\cdot\omega : I_1 \to I_2$ is at least $d\beta_0(\partial\Delta)$ and thus $\dim I_1 + d\beta_0(\partial\Delta) \leq \dim I_2$.

What are the dimensions of I_1 and I_2? From exactness of rows, we infer that

$$\dim I_1 = \dim \mathbf{k}(\Gamma)_1 - \dim \mathbf{k}(\Sigma)_1 = (f_0(\Delta) + b - d) - (f_0(\partial\Delta) + b - d) = f_0^\circ(\Delta), \tag{13}$$

and

$$\begin{aligned} \dim I_2 &= \dim \mathbf{k}(\Gamma)_2 - \dim \mathbf{k}(\Sigma)_2 = \\ &= h_2(\Gamma) - h_2(\Sigma) - \binom{d}{2}\beta_0(\partial\Delta) \\ &= h_2(\Delta) - g_2(\partial\Delta) - \binom{d}{2}\beta_0(\partial\Delta) \\ &\leq h_2(\Delta) - \left[\binom{d}{2}\beta_1(\partial\Delta) - \binom{d}{2}\beta_0(\partial\Delta)\right] - \binom{d}{2}\beta_0(\partial\Delta) \\ &= h_2(\Delta) - \binom{d}{2}\beta_1(\partial\Delta), \end{aligned} \tag{14}$$

where the penultimate step follows from [16, Theorem 5.2] applied to connected components of $\partial\Delta$ and from the observation that for a $(d-2)$-dimensional complex $\partial\Delta$, its g_2-number equals the sum of the g_2-numbers of its connected components minus $\binom{d}{2}\beta_0(\partial\Delta)$. Comparing the right-hand-sides of (13) and (14) and using $\dim I_1 + d\beta_0 \leq \dim I_2$, implies the result.

Two modifications are necessary when $d = 4$. First, each component of the boundary of Δ is a closed surface, so the Dehn-Sommerville relations tell us that the g_2 of each component is $3\beta_1$. Second, to show that Σ is $\mathbf{k}$-rigid the induction must begin with any closed surface instead of just S^2. In his thesis [4], Fogelsanger proved that any triangulation of a closed surface is generically 3-rigid in the graph-theoretic sense. Fogelsanger used three properties of generic 3-rigidity: a cone lemma, a gluing lemma, and a result of Whiteley's concerning vertex splitting [22]. Our cone lemma and gluing lemma cover the first two. Whiteley's vertex splitting result, combined with [10, Theorem 10] due to Carl Lee, is characteristic independent. Hence, Fogelsanger's proof shows that a triangulation of a closed surface is **k**-rigid. □

Now we give a series of examples that show that for any $d \geq 5, \beta_1, \beta_0$ and f_0°, Theorem 5.1 is optimal. We recall a family of complexes introduced by Kühnel and Lassman.

Theorem 5.6 *[9] For every $d \geq 4$ and $n \geq 2d-1$ there exists a complex $M^d(n)$ with n vertices such that*

- *$M^d(n)$ is a B^{d-2}-bundle over the circle. In particular, $M^d(n)$ is a manifold with boundary.*
- *Depending on the parity of n and d the boundary of $M^d(n)$ is either $S^{d-2} \times S^1$ or the nonorientable S^{d-2}-bundle over the circle. Hence, for $d \geq 5$, the first Betti number of $\partial M^d(n)$ is one for any field. When $d = 4$ and the characteristic of $\mathbf{k}$ is 2, then $\beta_1(\partial M^4(n)) = 2$.*
- $h_2(M^d(n)) = \binom{d}{2}$.
- *All of the vertices are on the boundary of $M^d(n)$. The link of every vertex is combinatorially equivalent to a stacked polytope.*

Evidently, $M^d(n)$ for $d \geq 5$ is an example of equality in Theorem 5.1 with $\beta_1(\partial\Delta) = 1$ and $f_0^\circ = 0$. For spaces with $\beta_1(\partial\Delta) > 1$, begin with two disjoint copies of $M^d(n)$. Choose two $(d-2)$-faces on their respective boundaries and a bijection between their vertices. Now identify these vertices and associated faces according to the chosen bijection. The resulting space has no interior vertices and is a manifold with boundary whose boundary is topologically the connected sum of two copies of the boundary of $M^d(n)$. Thus the first Betti number is now two. Direct computation shows that h_2 of the new space is $2\binom{d}{2}$. Repeating this operation of connected sum along the boundary b times with $M^d(n)$ produces an example of equality in Theorem 5.1 with $\beta_1(\partial\Delta) = b$ and $f_0^\circ = 0$. To construct Δ with $f_0^\circ = m > 0$ simply take a complex with $f_0^\circ = 0$ and subdivide a facet m times. Each such subdivision increases h_2 and f_0° by one while leaving the topological type of the complex unchanged.

To produce spaces Δ with $\beta_0(\partial\Delta) > 0$, begin with any of the above examples. It is possible to subdivide a facet d times so that there is now a facet with interior vertices. See [3] for the algorithm. Removing the open facet leaves a manifold whose boundary has two components, the original and the boundary of the simplex. The new space will have the same number of interior vertices and its h_2 will have increased by d.

In dimension three, the same constructions lead to examples of equality in Theorem 5.1 with arbitrary f°, β_0, and even β_1. Since the boundary of a three-dimensional manifold Δ must have even Euler characteristic, this is the best we can hope for.

All of the complexes constructed using the procedures have the property that the link of every boundary vertex is combinatorially a stacked polytope, and the link of every interior vertex is a stacked sphere.

Conjecture 5.7 *If Δ is a connected $(d-1)$-dimensional homology manifold with nonempty orientable boundary and $d \geq 4$, then equality occurs in Theorem 5.1 if and only if all of the links of Δ are combinatorially equivalent to stacked polytopes or stacked spheres.*

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