ON ASYMPTOTIC VALUES OF CANONICAL QUADRATIC L-FUNCTIONS

NICOLAS TEMPLIER

Abstract. We derive an asymptotic for the first moment of Hecke L-series associated to canonical quadratic characters. This provides another proof and slightly generalizes recent results by Masri and Kim-Masri-Yang.

Résumé. On établit le comportement asymptotique du premier moment des séries L de Hecke associées aux caractères quadratiques canoniques. On donne ainsi une nouvelle démonstration ainsi qu'une légère amélioration de résultats récents de Masri et Kim-Masri-Yang.

Contents

1. Introduction

We shall study L-functions associated to certain *ramified* Hecke characters on a quadratic field $\mathbb{Q}(\sqrt{D})$ traditionally called "canonical characters". The special values of these L-functions have been studied in several aspects: Rohrlich [13–15] and Miller-Yang [9] (nonvanishing), Duke-Friedlander-Iwaniec [2] (implies subconvexity), Fouvry-Iwaniec [3] (low-lying zeros), Villegas-Zagier [11,12] and Villegas-Yang [10] (period formula), Liu-Xu [6], Masri [7,8], Kim-Masri-Yang [5] (first moment).

This article provides a "rudimentary" proof of the asymptotic expansion for the first moment:

$$\mathcal{L}(\eta) := \frac{1}{h(D)} \sum_{\chi \in \widehat{\mathrm{Cl}}(D)} L(1/2, \eta\chi), \quad \text{as } D \to -\infty. \tag{1.1}$$

(η is a canonical character and χ runs over class group characters ; our notations are defined below, and see (1.4) for the final result). By rudimentary we mean that we only make use of the functional equation of the L-functions $L(s, \eta\chi)$ and estimates for character sums (Burgess estimate). We shall also consider a variant where the average is over a subgroup of characters, see below. In the proof we avoid the use of the following deep results:

- Villegas-Zagier period formulas;
- Duke's equidistribution Theorem.

As a consequence our results hold in a better generality.

Although for those familiar with the theory of moments of special values of L-functions the existence of such a proof may not be very surprising, we believe it is important to have the details clearly written down for the sake of further research. For instance, from the work of Masri and Kim-Masri-Yang, one was inclined to think that the moment (1.1) is governed by a $GL(2)$-subconvexity estimate (Duke-Iwaniec

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bounds) while in the contrary we show in this paper that it is governed by a $GL(1)$-subconvexity estimate (Burgess bound).

Another feature of our work (Theorem 2) is the observation that the asymptotic of the moment over characters χ of the subgroup $\widehat{\mathrm{Cl}}(D)^2$ may be dealt with thanks to the well-known explicit description of the 2-torsion ideal classes [1]. This might turn to be useful in other contexts because characters in $\widehat{\mathrm{Cl}}(D)^2$ tend to have a more stable sign of functional equation. In the proof the off-diagonal terms appear complicated at first sight (which might explain why the observation has not been made earlier), but we show that a straightforward simplification occurs. Before stating the results, we now proceed to recall classical facts on Hecke characters. Since the tools involved in the proofs are by now very standard we have made effort to keep the paper as short and elementary as possible.

1.1. **Hecke characters.** An introduction to Hecke characters in the classical language may be found in section 3.8 of the book [4] and we shall mainly follow their notations and conventions. A character shall always mean *unitary character*. Let K be a number field, $\mathcal{O}$ its ring of integers and Cl_K the ideal class group. A Hecke character η comes with a conductor $\mathfrak{m}$ (integral ideal of $\mathcal{O}$) and an archimedean part η_∞. More precisely it is a character $\eta : I_\mathfrak{m} \to S^1$ from the group of fractional ideals of $\mathbb{Q}(\sqrt{D})$ prime to $\mathfrak{m}$ which satisfies:

$$\eta((\alpha)) = \eta_f(\alpha)\eta_\infty(\alpha), \quad \forall \alpha \in \mathcal{O} \text{ with } (\alpha, \mathfrak{m}) = 1. \tag{1.2}$$

Here $\eta_f : (\mathcal{O}/\mathfrak{m})^\times \to S^1$ is a Dirichlet character and $\eta_\infty : K_\infty^\times \to S^1$. One says that η is *unramified* when $\mathfrak{m} = \mathcal{O}$ and η_∞ is trivial. An unramified Hecke character is a character on Cl_K, i.e., a *class group character*.

Conversely if $\mathfrak{m}$, η_f and η_∞ are such that $\eta_f(\epsilon)\eta_\infty(\epsilon) = 1$ for all *units* $\epsilon \in \mathcal{O}^\times$, then there exist Hecke characters with finite part η_f and archimedean part η_∞ (this is a consequence of the strong approximation theorem). Those characters with the same archimedean part, conductor and underlying finite part differ by multiplication by a class group character.

The L-function associated to a Hecke character η is:

$$L(s, \eta) := \sum_{\mathfrak{a} \subset \mathcal{O}} \frac{\eta(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s} \tag{1.3}$$

where $\mathfrak{a}$ runs through all integral $\mathcal{O}$-ideals prime to $\mathfrak{m}$.

Now we specialize the discussion to $K = \mathbb{Q}(\sqrt{D})$ and describe some examples. If $D < 0$, then $K_\infty \simeq \mathbb{C}$ and $\eta_\infty(z) = (\frac{z}{|z|})^l$ for a *weight* $l \in \mathbb{Z}$ (the "frequency" in the terminology of [4, p. 59]).

1.1.1. *Class group characters.* We shall denote the ideal class group by $\mathrm{Cl}(D)$ and the dual group of *class group characters* by $\widehat{\mathrm{Cl}}(D)$. Let $h(D)$ be the class number. If $\mathfrak{a}$ is an ideal of $\mathcal{O}_D$, we denote by $[\mathfrak{a}] \in \mathrm{Cl}(D)$ its ideal class.

The subgroup $\widehat{\mathrm{Cl}}(D)^2 \subset \widehat{\mathrm{Cl}}(D)$, image of the map $\chi \mapsto \chi^2$, is of index $2^{\omega(D)-1}$ by Gauss genus theory. It is orthogonal to the 2-torsion $\mathrm{Cl}_2(D) \subset \mathrm{Cl}(D)$.

More classical are the characters belonging to the 2-torsion $\widehat{\mathrm{Cl}}_2(D) \subset \widehat{\mathrm{Cl}}(D)$ in other words the characters which take real values. They are called *genus characters*. This subgroup is orthogonal to $\mathrm{Cl}(D)^2$ thus we may view genus characters as characters on the quotient $\mathrm{Cl}(D)/\mathrm{Cl}(D)^2$ which consists of *genus classes*. If $\mathfrak{a}$ is an ideal, its genus class is traditionally denoted by $\{\mathfrak{a}\}$.

1.1.2. *Canonical characters.* The Kronecker character χ_D is the real primitive Dirichlet character of conductor $|D|$. We may view it as a character on $(\mathbb{Z}/D\mathbb{Z})^\times \simeq (\mathcal{O}_D/(\sqrt{D}))^\times$.

We assume from now on that $D \equiv 1(4)$ and $D < -3$ so that the group of units is trivial: $\mathcal{O}_D^\times = \{\pm 1\}$. Let ψ be an Hecke character of conductor $(\sqrt{D})$, finite part χ_D and weight 1 (the units compatibility condition is satisfied since $\chi_D(-1) = -1$). Such a ψ is an example of a canonical character in the the sense of Rohrlich [15].

[1]in spirit this is close to how the Villegas-Zagier formula is proved in [11,12]

1.1.3. *Base change characters.* Let ϵ be a primitive Dirichlet character of modulus q, with $(q, D) = 1$. Then $\rho(\mathfrak{a}) := \epsilon(\mathbf{N}\mathfrak{a})$ defines a primitive Hecke character $\rho : I_{(q)} \to S^1$ of conductor (q) and trivial infinite part. Its finite part $\rho_f : (\mathcal{O}_D/(q))^\times \to S^1$ is the composition $\epsilon \circ \mathbf{N}$ of ϵ and the norm map.

1.2. **Main result.**

Theorem 1. *Let $k \geqslant 1$ and ϵ be a primitive real Dirichlet character of modulus $q \geqslant 1$ coprime with D. For each D, let η be a Hecke character of conductor $(q\sqrt{D})$, weight $2k-1$, finite part $\epsilon \circ \mathbf{N} \cdot \chi_D$ and such that the sign of the functional equation of $L(s, \eta)$ is $+1$ (see § 2.2). As $D \to -\infty$ we have*

$$\mathcal{L}(\eta) = 2L^{(q)}(1, \chi_D) + O_{k,q}(|D|^{-1/16}). \tag{1.4}$$

Here the notation $L^{(q)}$ means that the prime factors above q in the Euler product have been removed.

The value $\mathcal{L}(\eta)$ does not depend on the choice of the character η since two choices would differ by a class group character (it depends only on ϵ and k which are fixed and on $D \to -\infty$). In the sequel we may and shall choose $\eta := \rho\psi^{2k-1}$ (see § 1.1.2 and § 1.1.3).

Remark 1. Under Lindelöf hypothesis one would have $L(1/2, \eta\chi) \ll_\epsilon (|D|\, qk)^\epsilon$. This is compatible with the value of the *main term* $L^{(q)}(1, \chi_D)$ on the right-hand-side of (1.4) which does not vary much with q and is independent of k (its order of magnitude is really $(q\,|D|)^\epsilon$).

Remark 2. The remainder term depends polynomially on k and q. We did not try to optimize the exponents (the 1/16 is far from optimal) but rather wanted to provide as simple proof as possible of an asymptotic with power saving in the D parameter.

1.3. **Average over a subgroup of class group characters.**

In [5], Kim, Masri and Yang consider an interesting variant (this came from the structure of the Villegas-Zagier formula):

$$\mathcal{L}_2(\eta) := \frac{h_2(D)}{h(D)} \sum_{\chi \in \widehat{\mathrm{Cl}}(D)^2} L(1/2, \eta\chi). \tag{1.5}$$

It is important to observe that unlike $\mathcal{L}(\eta)$, the value of $\mathcal{L}_2(\eta)$ may depend on the choice of η. More precisely we have the identity:

$$\mathcal{L}(\eta) = \frac{1}{h_2(D)} \sum_{\kappa \in \widehat{\mathrm{Cl}_2}(D)} \mathcal{L}_2(\eta\kappa). \tag{1.6}$$

They showed that the main term of the asymptotic as $D \to -\infty$ does not depend on the choice of η. We shall recover that result (in a slightly more general version) in section 3:

Theorem 2. *Under the same assumptions as in Theorem 1, one has:*

$$\mathcal{L}_2(\eta) = 2L^{(q)}(1, \chi_D) + O_{k,q}(|D|^{-1/64}). \tag{1.7}$$

Remark 3. In view of (1.6), Theorem 1 is a special case of Theorem 2 (except for the exponent of D in the remainder term).

Remark 4. It is possible (see also [8]), to address the case $\operatorname{sgn}\eta = -1$ for which $L(1/2, \eta\chi)$ is replaced by $L'(1/2, \eta\chi)$. The only change is to adapt the choice of G in the approximate functional equation method.

Remark 5. The results in [5] correspond to the following particular choice of Dirichlet character ϵ (this choice is made so that the Villegas-Zagier period formula is known [10]). Write $q = |d|$ for some odd fundamental discriminant d and $\epsilon := \chi_d$ (Kronecker symbol). It is then assumed that all prime factors of $2d$ are split in $\mathbb{Q}(\sqrt{D})$. In that case $(\mathcal{O}/(d))^\times \simeq (\mathbb{Z}/d\mathbb{Z})^\times \times (\mathbb{Z}/d\mathbb{Z})^\times$. , and the character ρ_f is the composition of that isomorphism with $\chi_d \times \chi_d$. The fact that the result of [5] coincides with (1.7) is inspected in section 4.

1.4. **Structure of the proofs.** The proof of Theorem 1 (§ 2) is rather straightforward, see also [9] for a nearby approach. After applying the approximate functional equation, a weighted character sum emerges and standard techniques reduce it to the Burgess estimate.

The proof of Theorem 2 (§ 3) presents some interesting features. Along the same line we arrive at a sum indexed by 2-torsion ideal classes. As we shall see only the trivial class contribute to the main term of the final asymptotic (1.7). To show this we make use of the explicit description of the 2-torsions ideal classes (genus theory going back to Gauss) which are parametrized by the factorizations $D = D_1 \cdot D_2$ of the discriminant.

The reason for the contribution of a non-trivial class to be negligible is not obvious as D_1 or D_2 could be very small compared to D. In other words non-trivial 2-torsion class could be "arbitrary close" to the principal class. Nevertheless the saving in (1.7) is independent of the presence of small prime factors of D, the key uniform tool is Lemma 3.1.

2. Proof of Theorem 1.

In this section and the next one the multiplicative "constants" involved in $\ll$ and O may depend on k and q. We do not display this dependence explicitly to ease notations. In other words we consider k and q as fixed.

2.1. **Functional equation.** First observe that η is a *primitive* Hecke character of conductor $\mathfrak{m} = (q\sqrt{D})$. By definition we have:

$$L(s,\eta) = \sum_{(\mathfrak{a},qD)=1} \eta(\mathfrak{a})\mathbf{N}\mathfrak{a}^{-s}, \tag{2.1}$$

and the functional equation is given, e.g., in [4, Theorem 3.8]. The Gamma factor is $(2\pi)^{-s}\Gamma(s+k-1/2)$ and the arithmetic conductor is $|D|\,\mathbf{N}\mathfrak{m} = q^2D^2$. The dual character $\overline{\eta}$ may be explicited as follows. Let σ be the nontrivial automorphism of $K/\mathbb{Q}$. We claim that $\overline{\eta} = \eta^\sigma \cdot \overline{\epsilon}^2 \circ \mathbf{N}$. Indeed, for all ideal $\mathfrak{a}$ with $(\mathfrak{a}, qD) = 1$:

$$\eta^\sigma\eta(\mathfrak{a}) = \eta(\overline{\mathfrak{a}})\eta(\mathfrak{a}) = \eta((\mathbf{N}\mathfrak{a})) = \eta_f(\mathbf{N}\mathfrak{a})\eta_\infty(\mathbf{N}\mathfrak{a}) = \epsilon(\mathbf{N}\mathfrak{a})^2. \tag{2.2}$$

Since ϵ is *real*[2], then $\overline{\eta} = \eta^\sigma$. In particular $L(s,\overline{\eta}) = L(s,\eta)$. The root number will be evaluated in the next section. Its value, denoted $\operatorname{sgn}\eta$ does not depend on the choice of η.

2.2. **Epsilon factors.** The root number of ψ^{2k-1} is $(-1)^{k+1}\chi_D(2)$, see [12]. The root number of $\rho\cdot\psi^{2k-1}$ is

$$\operatorname{sgn}\eta = (-1)^{k+1}\chi_D(2)\frac{\tau(\epsilon)^2}{q}, \tag{2.3}$$

where $\tau(\epsilon)$ is the Gauss sum of ϵ. This follows[3] by a similar computation as in [8, Appendix], see also [4, Exemple 5, § 3.8]. In particular note that $\operatorname{sgn}\eta \in \{\pm 1\}$. We assume from now on that $\operatorname{sgn}\eta = +1$.

2.3. **First moment.** By orthogonality of characters on a finite abelian group, we have:

$$\frac{1}{h(D)} \sum_{\chi\in\widehat{\mathrm{Cl}}(D)} L(s,\eta\chi) = \sum_{(\alpha,qD)=1} \eta((\alpha))\mathbf{N}\alpha^{-s}. \tag{2.4}$$

The sum is over elements $\alpha \in \mathcal{O}_D$ modulo units $\mathcal{O}_D^\times$, so that (α) runs over principal ideals. Observe that the value $\eta((\alpha)) = \eta_f(\alpha)\eta_\infty(\alpha)$ is explicit in terms of ϵ and k.

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[2] In that particular case $\rho = \epsilon \circ \mathbf{N}$, which is a base change from $GL_1(\mathbb{Q})$, is also a base change from $U_1(K/\mathbb{Q})$. The class group characters are base change from unramified automorphic characters on $U_1(K/\mathbb{Q})$. In general the property $\overline{\eta} \simeq \eta^\sigma$ characterizes base change from $U_1(K/\mathbb{Q})$.

[3] The reader should be careful that the root number of $\rho\cdot\psi^{2k-1}$ *is not* the product of the root numbers of ρ and ψ^{2k-1}

2.4. **Approximate functional equation.** Let $X := |qD|$. When $\operatorname{sgn}\eta = +1$, let $G(s)$ be fixed once and for all which satisfies:

- G is odd: $G(s) = -G(-s)$, $s \in \mathbb{C}^\times$,
- G is meromorphic with a single pole at $s = 0$ which is simple and of residue $\operatorname{Res}_0 G = 1$.
- G is of polynomial growth on vertical lines.

Introduce the cutoff function:

$$V(y) := \int_{(2)} y^{-s} \widehat{V}(s) \frac{ds}{2i\pi} \tag{2.5}$$

where:

$$\widehat{V}(s) := (2\pi)^{-s} \frac{\Gamma(s+k)}{(k-1)!} G(s). \tag{2.6}$$

We may express the critical value in the following way (see, e.g., [4, Theorem 5.3]):

$$L(\frac{1}{2}, \eta\chi) = 2 \sum_{(\mathfrak{a}, qD)=1} \eta(\mathfrak{a})\chi(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-1/2} V(\frac{\mathbf{N}\mathfrak{a}}{X}). \tag{2.7}$$

2.5. **Main term.** Averaging over $\chi \in \widehat{\mathrm{Cl}}(D)$ we thus obtain:

$$\mathcal{L}(\eta) = 2 \sum_{\substack{\alpha \in \mathcal{O}_D/\{\pm 1\} \\ (\alpha, qD)=1}} \epsilon(\mathbf{N}\alpha) (\frac{\alpha^2}{\mathbf{N}\alpha})^k \alpha^{-1} V(\frac{\mathbf{N}\alpha}{X}) \tag{2.8}$$

which is the starting point of our analysis.

A typical element $\alpha \in \mathcal{O}_D$ may be written as $\alpha = \dfrac{a + b\sqrt{D}}{2}$, with $a \equiv b(2)$. A *diagonal term* arises when $b = 0$, we set:

$$\mathcal{L}_0 = 2 \sum_{a \geqslant 1,\ (a, qD)=1} a^{-1} \chi_D(a) V(\frac{a^2}{X}). \tag{2.9}$$

(recall that ϵ is real so that $\epsilon(a^2) = 1$). Taking into account the Mellin transform (2.5) that defines V, we may write:

$$\mathcal{L}_0 = 2 \int_{(2)} L^{(q)}(s+1, \chi_D) \widehat{W}(s) \frac{ds}{2i\pi} \tag{2.10}$$

where

$$\widehat{W}(s) := \int_0^\infty V(\frac{y^2}{X}) y^s \frac{dy}{y} = \frac{1}{2} \widehat{V}(s/2) X^{s/2}. \tag{2.11}$$

In particular $\operatorname{Res}_0 \widehat{W} = \operatorname{Res}_0 \widehat{V} = 1$. It is classical to estimate $\mathcal{L}_0$ by a deformation of contour to $\Re e\, s = -1/2$ and then make use of Burgess estimate:

$$\mathcal{L}_0 = 2L^{(q)}(1, \chi_D) + O(|D|^{-1/16}). \tag{2.12}$$

2.6. **Remainder terms.** When $b \neq 0$, say even, we need to estimate an off-diagonal term. It is easy to check that the contribution from $a = 0$ is negligible and we set:

$$\mathcal{L}_b := \sum_{a \geqslant 1, a \equiv bD(2)} \epsilon(\frac{a^2 + b^2|D|}{4}) \chi_D(a) (\frac{a + b\sqrt{D}}{a - b\sqrt{D}})^{k-1/2} (a^2 + b^2|D|)^{-1/2} V(\frac{a^2 + b^2|D|}{4X}). \tag{2.13}$$

The rapid decay of the V-function [4, Proposition 5.4] shows that:

$$\mathcal{L}_b \ll_A |b|^{-A} \log|D|, \quad \text{for all } A > 0, \tag{2.14}$$

and we shall apply this estimate when $|b| \geqslant |D|^\eta$.

A similar estimate may be applied when $a \geqslant |D|^{1/2+\eta}$, so that we may suppose that the a-sum is of length at most $|D|^{1/2+\eta}$. The a-sum is a smoothly weighted sum of the character $\chi_D(a)$ twisted by a q-periodic function. We thus obtain by Burgess estimate:

$$\mathcal{L}_b \ll_\eta |D|^{-1/8+\eta} |b|^A, \quad \text{for some } A \text{ and all } \eta > 0. \tag{2.15}$$

Choosing η arbitrary small, we conclude that:

$$\sum_{b \neq 0} \mathcal{L}_b \ll |D|^{-1/8+\eta}. \tag{2.16}$$

This together with (2.12) concludes the proof of Theorem 1.

3. Proof of Theorem 2.

The first steps of the proof are similar. The sections 2.1, 2.2 and 2.4 are identical. As for § 2.3, averaging over the subgroup $\widehat{\mathrm{Cl}}(D)^2 \subset \widehat{\mathrm{Cl}}(D)$ yields in the present case:

$$\begin{aligned}\frac{h_2(D)}{h(D)} \sum_{\chi \in \widehat{\mathrm{Cl}}(D)^2} L(s, \eta\chi) &= \sum_{\substack{(\mathfrak{a}, qD)=1 \\ \mathfrak{a}^2 \text{ principal}}} \eta(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s} \\ &= \sum_{D = D_1 \cdot D_2} |D_1|^{-s} \sum_{\substack{\alpha \in \mathfrak{d}_1^{-1}/\{\pm 1\} \\ (\mathfrak{d}_1(\alpha), qD)=1}} \eta(\mathfrak{d}_1(\alpha)) \mathbf{N}\alpha^{-s}.\end{aligned} \tag{3.1}$$

The first sum is over all factorizations of D in product of two coprime (fundamental) discriminants[4]. The ideal $\mathfrak{d}_1$ is of norm $|D_1|$. More details on the structure of the 2-torsion $\mathrm{Cl}_2(D)$ are recalled in the next paragraph.

3.1. Gauss genus group.

It is a classical result by Gauss that $\mathrm{Cl}_2(D)$ is of cardinality $2^{\omega(D)-1}$. More precisely a 2-torsion ideal class contains exactly two ideals $\mathfrak{d}_1$ with $\mathbf{N}\mathfrak{d}_1 | D$. In other words:

$$\mathrm{Cl}_2(D) = \{[\mathfrak{d}_1], \mathbf{N}\mathfrak{d}_1 = |D_1|, D = D_1 \cdot D_2\}. \tag{3.2}$$

This justifies the identity (3.1) above.

In the context of (3.1) we may perform the following further manipulations. Recall that the sum is over $\alpha \in \mathfrak{d}_1^{-1}$ such that $(\mathfrak{d}_1(\alpha), qD) = 1$; fix such an element α_1. Then:

$$\eta(\mathfrak{d}_1(\alpha)) = \eta(\mathfrak{d}_1(\alpha_1)) \eta((\alpha\alpha_1^{-1})), \quad \text{for all such } \alpha \in \mathfrak{d}_1^{-1}. \tag{3.3}$$

Now the first term is fixed and the second term is explicit.

Let us introduce coordinates to be even more explicit:

$$\mathfrak{d}_1^{-1} = \{\frac{a + b\sqrt{D}/D_1}{2}, \quad a \equiv b(2)\}. \tag{3.4}$$

An element $\alpha = \frac{a+b\sqrt{D}/D_1}{2}$ satisfies $(\mathfrak{d}_1(\alpha), D) = 1$ if and only if $(b, D_1) = 1$ and $(a, D_2) = 1$. Put $\alpha_1 = \frac{A+B\sqrt{D}/D_1}{2}$. One clearly has:

$$4|D_1| \mathbf{N}\alpha = a^2 |D_1| + b^2 |D_2| \tag{3.5}$$

and it is not difficult to check that:

$$\chi_D(\alpha\alpha_1^{-1}) = \chi_D(D_1 \mathbf{N}\alpha_1) \chi_D(aAD_1 - bBD_2). \tag{3.6}$$

[4] we do identify the factorizations $D = D_1 \cdot D_2$ and $D = D_2 \cdot D_1$

3.2. **Main term.** From (3.1) we may split $\mathcal{L}_2(\eta)$ into:

$$\mathcal{L}_2(\eta) = \sum_{D=D_1\cdot D_2} \mathcal{L}_2(\eta, D_1) \tag{3.7}$$

where:

$$\mathcal{L}_2(\eta, D_1) := |D_1|^{-1/2} \sum_{\substack{\alpha\in\mathfrak{d}_1^{-1}/\{\pm 1\} \\ (\mathfrak{d}_1(\alpha), qD)=1}} \eta(\mathfrak{d}_1(\alpha))\mathbf{N}\alpha^{-1/2} V(\frac{|D_1|\,\mathbf{N}\alpha}{X}). \tag{3.8}$$

The term $\mathcal{L}_2(\eta, 1)$ is precisely equal to $\mathcal{L}(\eta)$ which has already been dealt with (Theorem 1). This yields the same main term in Theorem 2.

3.3. **Remainder terms.** Now we proceed to bound the remaining terms, that is we assume that $|D_1| > 1$ and $|D_2| > 1$. In general such a double sum may be difficult to handle. But we shall see that it is again possible to extract character sums and to apply Burgess estimate.

Up to multiplication by the number

$$\eta(\mathfrak{d}_1(\alpha_1))\chi_D(D_1\mathbf{N}\alpha_1)\bar{\epsilon}(A^2\,|D_1| + B^2\,|D_2|)(\frac{A\sqrt{D_1} - B\sqrt{D_2}}{A\sqrt{D_1} + B\sqrt{D_2}})^{k-1/2}, \tag{3.9}$$

which is of absolute value one, $\mathcal{L}_2(\eta, D_1)$ is equal to:

$$\sum_{\alpha=\frac{a+b\sqrt{D}/D_1}{2}} \epsilon(a^2\,|D_1| + b^2\,|D_2|)\chi_D(aAD_1 - bBD_2) \tag{3.10}$$
$$(\frac{a\sqrt{D_1} + b\sqrt{D_2}}{a\sqrt{D_1} - b\sqrt{D_2}})^{k-1/2}(a^2\,|D_1| + b^2\,|D_2|)^{-1/2} V(\frac{a^2\,|D_1| + b^2\,|D_2|}{4X}).$$

The first observation is that the term involving χ_D splits as the product of two smaller characters:

$$\chi_D(aAD_1 - bBD_2) = \chi_D(AD_1 - BD_2) \times \chi_{D_2}(a)\chi_{D_1}(b), \quad \forall a, b. \tag{3.11}$$

Put:

$$f(x, y) := (\frac{x+y}{x-y})^{k-1/2}(|x|^2 + |y|^2)^{-1/2} V(\frac{|x|^2 + |y|^2}{4}), \quad x, y \in \mathbb{C} \tag{3.12}$$

and $g(a, b) := f(\frac{a\sqrt{D_1}}{\sqrt{X}}, \frac{b\sqrt{D_2}}{\sqrt{X}})$. Observe that a and b are always non-zero because $(a, D_2) = (b, D_1) = 1$ and $|D_1|, |D_2| > 1$.

As before the value of the character ϵ depends only on a and b modulo q. The cases when a or b is negative are similar, so that we need to estimate:

$$X^{-1/2} \sum_{\substack{1\leqslant a\equiv a_0(q) \\ 1\leqslant b\equiv b_0(q)}} \chi_{D_2}(a)\chi_{D_1}(b) g(a, b). \tag{3.13}$$

We shall proceed by integration by parts. The following lemma summarizes the bounds to be applied for the summations on a and b. We fix a small exponent $\eta = 1/64$ which separates the contributions of the small (a, b) and the large (a, b).

Lemma 3.1. *(i) For all $\epsilon > 0$:*

$$X^{-1/2} \sum_{\substack{1\leqslant a<|D_2|^{1/2}|D|^{-\eta} \\ 1\leqslant b<|D_1|^{1/2}|D|^{-\eta}}} (\frac{a^2\,|D_1| + b^2\,|D_2|}{X})^{-1/2} \ll_\epsilon |D|^{-\eta+\epsilon}. \tag{3.14}$$

(ii) For all $\epsilon > 0$:

$$(3.15) \qquad |D|^{-1/2} \sum_{\substack{a,b\geqslant 1 \\ |D|^{1-2\eta}\leqslant a^2|D_1|+b^2|D_2|}} \frac{\partial^2 f}{\partial x \partial y}\left(\frac{a\sqrt{D_1}}{\sqrt{X}}, \frac{b\sqrt{D_2}}{\sqrt{X}}\right) \ll_\epsilon |D|^{\epsilon+7\eta}.$$

Proof. Recall $X = q\,|D|$. (i) follows by comparison with an integral. (ii) When differentiating twice the function f, several terms occur, each one we bound trivially:

$$(3.16) \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) \ll \max_{\substack{j\in\{0,1\}k\in\{0,1\} \\ l\in\{0,1,2\}m\in\{0,1,2\}}} |x|^j\,|y|^k\,(|x|^2+|y|^2)^{-l-1/2}(|x|+|y|)^{-m} V\left(\frac{|x|^2+|y|^2}{4}\right).$$

Inserting this into (3.15) one gets the result. ☐

Let us continue the proof. For notational simplicity we forget about the ϵ's. The terms with $a^2\,|D_1| + b^2\,|D_2| \leqslant |D|^{1-2\eta}$ are dealt with (i) of Lemma 3.1. They are bounded by $\ll |D|^{-1/64}$.

From Burgess estimate, we have:

$$(3.17) \qquad S_x := \sum_{\substack{1\leqslant a\leqslant x \\ a\equiv a_0(q)}} \chi_{D_1}(a) \ll |D_1|^{3/8}.$$

We use a similar notation for $S_y \ll |D_2|^{3/8}$ where $1 \leqslant b \leqslant y$ and $b \equiv b_0(q)$. By partial summation on (3.13), it is enough to estimate:

$$(3.18) \qquad X^{-1/2} \sum_{\substack{a\geqslant 1,b\geqslant 1 \\ |D|^{1-2\eta}\leqslant a^2|D_1|+b^2|D_2|}} |S_a S_b|\,|g(a+q,b+q) - g(a+q,b) - g(a,b+q) + g(a,b)|\,.$$

The second quantity is bounded by $\ll_q \frac{\sqrt{|D|}}{X}\frac{\partial^2 f}{\partial x\partial y}\left(\frac{a\sqrt{D_1}}{\sqrt{X}}, \frac{b\sqrt{D_2}}{\sqrt{X}}\right)$, so that we may apply (ii) of Lemma 3.1 which yields a bound $\ll |D|^{\epsilon+7\eta-1/8} \ll |D|^{-1/64}$.

4. Appendix – consistency of periods.

The aim of this section is to simplify the main term given by Kim-Masri-Yang in [5, Theorem 1.5] and then check that the result coincides with Theorem 1. To keep consistent notations put $\epsilon = \chi_d$, $q = d$ with all prime factors of d split in $\mathbb{Q}(\sqrt{D})$. Observe that $L^{(q)}(1,\chi_D) = L(1,\chi_D)\phi(d)/d$. Since D is assumed to be odd, one has $\chi_D(2) = (-1)^{\frac{D-1}{4}}$. In [5], $D \equiv 1(8)$ so that $\chi_D(2) = 1$ and $\epsilon = \chi_d$ so that $\tau(\epsilon)^2/q = \mathrm{sgn}(d)$. This is compatible with the fact that $\mathrm{sgn}(d) = (-1)^{k-1}$ is equivalent to $\mathrm{sgn}\,\eta = +1$, as claimed in [5, Theorem 1.5].

The main term in [5] is proportional to the L^2-norm of the θ-series occurring in the Villegas-Zagier formula. The following shows it coincides with the quantity $2L^{(q)}(1,\chi_D)$ in (1.7).

Lemma 4.1. *Let θ be the theta series whose definition is recalled below (its weight is $k-1/2$ and its level is $4d^2$) and let $<,>$ be the normalized Petersson inner product. Assume θ is a cusp form. Then:*

$$(4.1) \qquad \frac{(8\pi)^{k-1}}{(k-1)!} < \theta, \theta > = \frac{\phi(d)}{d}.$$

By normalized Petersson inner product it is meant that (where $\Gamma = \Gamma_0(4d^2)$):

$$(4.2) \qquad < \theta, \theta > := \mathrm{vol}(\Gamma\backslash\mathfrak{H})^{-1} \int_{\Gamma_0(4d^2)\backslash\mathfrak{H}} |\theta(x+iy)|^2\, y^{k-1/2}\frac{dxdy}{y^2}.$$

Remark 6. The identity holds without the assumption that θ is a cusp form. It is made to shorten the proof of the lemma. By [5], θ is a cusp form if and only if D has a prime factor congruent to 3 (mod 4).

4.1. **Orthogonal polynomials.** We set $H(z) := (8\pi)^{(1-k)/2} H_{k-1}\big(\sqrt{2\pi}z\big)$ which is equal to the polynomial in [5, Introduction]. Here H_k is the standard Hermite polynomial of degree k (see [1, Chapter 22] for instance):

Definition 4.1 (Hermite polynomial)**.**

$$H_n(X) := \sum_{0\leqslant j\leqslant n/2} \frac{n!}{j!(n-2j)!}(-1)^j(2X)^{n-2j}. \tag{4.3}$$

The orthogonality relation is:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \begin{cases} n!2^n\sqrt{\pi} & \text{if } n=m, \\ 0 & \text{else.} \end{cases} \tag{4.4}$$

4.2. **Real-analytic theta series.** The theta series θ occurring in Lemma 4.1 is:

$$\theta(z) := (2y)^{(1-k)/2} \sum_{n\geqslant 1,(n,d)=1} \chi_d(n)H\big(n\sqrt{2y}\big)e(n^2z). \tag{4.5}$$

4.3. **Proof of the Lemma.** To compute a Petersson inner product, an efficient method is to introduce Eisenstein series:

$$E(s,z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im\mathrm{m}\,\gamma z)^s, \quad \Re\mathrm{e}\, s > 1. \tag{4.6}$$

Recall that E admits an analytic continuation to $\mathbb{C}$ with a simple pole at $s=1$ with residue $1/\operatorname{vol}(\Gamma\backslash\mathfrak{H})^{-1}$. Since θ is a cusp form, $<\theta,\theta>$ is the residue at $s=1$ of

$$I(s) := \int_{\Gamma\backslash\mathfrak{H}} |\theta(z)|^2(\Im\mathrm{m}\, z)^{k-1/2}E(s,z)\frac{dxdy}{y^2}, \quad s\neq 1. \tag{4.7}$$

Unfolding, yields:

$$\begin{aligned} I(s) &= \int_{\Gamma_\infty\backslash\mathfrak{H}} |\theta(x+iy)|^2 y^{k-1/2}y^s\frac{dxdy}{y^2} \\ &= 2^{1-k}\sum_{(n,d)=1}\int_0^\infty \Big|H\big(n\sqrt{2y}\big)\Big|^2 y^{1/2+s}e^{-4\pi n^2 y}\frac{dy}{y^2}. \end{aligned}$$

After the change of variable $y \rightsquigarrow y/n^2$, we obtain:

$$I(s) = 2^{1-k}\zeta^{(d)}(2s-1)\int_0^\infty H\big(\sqrt{2y}\big)^2 y^{s-1/2}e^{-4\pi y}\frac{dy}{y}. \tag{4.8}$$

Thus from (4.4) we deduce (with the change $x := (4\pi y)^{1/2}$):

$$\begin{aligned} <\theta,\theta> &= 2^{1-k}\times 2\zeta_d(1)^{-1}\int_0^\infty H\big(\sqrt{2y}\big)^2y^{1/2}e^{-4\pi y}\frac{dy}{y} \\ &= 2^{1-k}\times 2\zeta_d(1)^{-1}\times(8\pi)^{1-k}\int_0^\infty H_{k-1}\big(\sqrt{4\pi y}\big)^2y^{1/2}e^{-4\pi y}\frac{dy}{y} \\ &= (16\pi)^{1-k}\times 2\zeta_d(1)^{-1}\times(4\pi)^{-1/2}\int_0^\infty H_{k-1}\big(x\big)^2e^{-x^2}2dx \\ &= (16\pi)^{1-k}\times 2\zeta_d(1)^{-1}\times(4\pi)^{-1/2}\times(k-1)!2^{k-1}\sqrt{\pi} \\ &= (8\pi)^{1-k}(k-1)!\zeta_d(1)^{-1}, \end{aligned}$$

which completes the proof of Lemma 4.1.

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Institute for Advanced Study, 08540 Princeton, NJ
E-mail address: `ntemplier@ias.edu`