HYBRID SUP-NORM BOUNDS FOR HECKE-MAASS CUSP FORMS

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Abstract. Let f be a Hecke–Maass cusp form of eigenvalue λ and square-free level N. Normalize the hyperbolic measure such that $\mathrm{vol}(Y_0(N)) = 1$ and the form f such that $\|f\|_2 = 1$. It is shown that $\|f\|_\infty \ll_\epsilon \lambda^{\frac{5}{24}+\epsilon} N^{\frac{1}{3}+\epsilon}$ for all $\epsilon > 0$. This generalizes simultaneously the current best bounds in the eigenvalue and level aspects.

1. Introduction

It is a classical problem to bound the L^∞-norm (or sup-norm) of Laplace eigenfunctions on manifolds. We shall establish new bounds for the well-studied modular surface $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{H}$ with its hyperbolic metric.

The total volume for the hyperbolic measure is asymptotically equal to $N^{1+o(1)}$. We work with a rescaled probability measure μ such that $\mu(Y_0(N)) = 1$. Hecke–Maass cuspidal newforms f are joint eigenfunctions of the Laplacian and Hecke operators. It shall be L^2-normalized, namely

$$\int_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z)|^2 \mu(dz) = 1.$$

It is interesting to bound the sup-norm $\|f\|_\infty$ in terms of the two basic parameters: the Laplacian eigenvalue λ and the level N.

In the λ-aspect the first nontrivial bound is due to Iwaniec–Sarnak [8] who establish

$$\|f\|_\infty \ll_{N,\epsilon} \lambda^{\frac{5}{24}+\epsilon} \tag{1.1}$$

for any $\epsilon > 0$. They find how to make use of the Hecke operators, through the method of amplification, in order to go beyond $\|f\|_\infty \ll_N \lambda^{\frac{1}{4}}$ which is valid on any compact Riemannian surface.

In the N-aspect the first non-trivial bound is due to Blomer-Holowinsky [2] who prove $\|f\|_\infty \ll_{\lambda,\epsilon} N^{\frac{216}{457}+\epsilon}$, for square-free N. In [14] we revisit the proof by making a systematic use of geometric arguments, and derive a stronger exponent $\|f\|_\infty \ll_{\lambda,\epsilon} N^{\frac{5}{11}+\epsilon}$. Helfgott–Ricotta (unpublished) improve some of the estimates in [14] and obtain $\|f\|_\infty \ll_{\lambda,\epsilon} N^{\frac{9}{20}+\epsilon}$. In [7] we introduce with Harcos a more efficient treatment of the counting problem and we derive the estimate $\|f\|_\infty \ll_{\lambda,\epsilon} N^{\frac{5}{12}+\epsilon}$. Optimizing the argument further we shall obtain:

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Theorem 1.1 (See also [6])**.** *Let f be a normalized Hecke–Maass cusp form of square-free level N. Then for any $\epsilon > 0$ the following holds:*

$$\|f\|_\infty \ll_{\lambda,\epsilon} N^{\frac{1}{3}+\epsilon}. \tag{1.2}$$

We now turn to uniform bounds in λ and N. It is not difficult to establish $\|f\|_\infty \ll_\epsilon \lambda^{\frac{1}{4}+\epsilon} N^{\frac{1}{2}+\epsilon}$. The details can essentially be found in Donnelly [3], Abbes–Ullmo [1] and Jorgenson–Kramer [10].

Hybrid bounds save a power simultaneously in the λ and N-aspects. The following hybrid bound is established by Blomer-Holowinsky [2, Theorem 2]:

$$\|f\|_\infty \ll (\lambda^{\frac{1}{2}} N)^{\frac{1}{2}-\frac{1}{2300}}. \tag{1.3}$$

It interpolates[1] between the following bound in the λ-aspect [2, §10] obtained by modifying the method of proof in [8]:

$$\|f\|_\infty \ll_\epsilon \lambda^{\frac{5}{24}+\epsilon} N^{\frac{1}{2}+\epsilon}, \tag{1.4}$$

and the following bound in the N-aspect [2, p.673]:

$$\|f\|_\infty \ll_\epsilon \lambda^{\frac{9979}{3658}+\epsilon} N^{\frac{216}{457}+\epsilon}. \tag{1.5}$$

The bound (1.3) isn't satisfactory for two reasons. First the quality of the exponents is not optimal. Second the method of proof is complicated because the proof of (1.5) is very different from the proof of (1.4), which explains the large value of the exponent in λ in (1.5). These issues will be resolved in the present paper.

We establish the following hybrid bound that generalizes the best known bounds in the λ-aspect and in the N-aspect simultaneously.

Theorem 1.2. *Let f be a normalized Hecke–Maass cusp form of eigenvalue λ and square-free level N. Then for any $\epsilon > 0$ the following holds:*

$$\|f\|_\infty \ll_\epsilon \lambda^{\frac{5}{24}+\epsilon} N^{\frac{1}{3}+\epsilon}. \tag{1.6}$$

Remarks.
- (i) In the context of L-functions, obtaining hybrid bounds that perfectly combine two aspect is also a difficult problem. There are few known cases, such as Jutila–Motohashi [11] who establish $L(\frac{1}{2}+it, f) \ll_\epsilon (\lambda^{\frac{1}{2}} + t)^{\frac{1}{3}+\epsilon}$.
- (ii) Inspecting the details of the proof below, the terms $(\lambda N)^\epsilon$ come from divisor bounds. Thus it could be slightly improved, for example into $\exp(\frac{\log(\lambda N)}{\log\log(\lambda N)})$.

The amplification method as used in [8] relates the sup-norm problem to an interesting lattice point counting. Our strategy is to produce an unified proof of both the λ and N-aspects: this seems to be the only way for establishing a hybrid bound that generalizes the best bounds in the two aspects. Our method is geometric, with refinements of the ideas from [7, 14]. In particular we shall provide a new treatment of the λ-aspect.

In the remarks following [14, Theorem 1] and [7, Theorem 1] we mention the possibility of studying hybrid bounds. The details of this project are achieved in the present paper to the point of obtaining uniform hybrid bounds. We continue this introduction with a discussion of the solution to the lattice point counting problem.

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[1] To be precise $\frac{1}{2300}$ in (1.3) could be replaced by $0.00044987... > \frac{1}{2223} > \frac{1}{2300}$.

1.1. **A lattice point counting problem.** For $z \in \mathfrak{H}$, $\delta > 0$ and two integers $\ell, N \geqslant 1$, let $\mathcal{M}(z,\ell,\delta,N)$ be the finite set of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $M_2(\mathbb{Z})$ such that

$$\det(\gamma) = \ell, \quad c \equiv 0(N), \quad u(\gamma z, z) \leqslant \delta. \tag{1.7}$$

Denote by $M(z,\ell,\delta,N)$ its cardinality.

We want an uniform upper bound for $M(z,\ell,\delta,N)$ in all parameters. The typical range that is of interest in the context of sup-norms is: ℓ is moderately large, $0 < \delta < 1$ and $N \to \infty$. The quality of the upper bound for $M(z,\ell,\delta,N)$ is directly related to that of the hybrid bound (1.6).

Previous results on the counting problem involve only one of the two parameters δ and N. We shall assume in this subsection that γ is generic in the sense that $c \neq 0$ and $(a+d)^2 \neq 4\ell$ and denote the associated count by $M_*(z,\ell,\delta,N)$.

If N is fixed and $\delta > 0$ is arbitrary, the original work of Iwaniec-Sarnak gives the bound $M_*(z,\ell,\delta,1) \preccurlyeq 1 + \ell\delta^{\frac{1}{4}}$. In fact one can verify [2, §10] that the same argument produces an uniform bound in N of the same quality:[2]

$$M_*(z,\ell,\delta,N) \preccurlyeq 1 + \ell\delta^{\frac{1}{4}}. \tag{1.8}$$

Of course this is not good enough in the N-aspect since we expect less solutions of (1.7) from the condition $c \equiv 0(N)$ when N grows.

An important idea in this paper is to establish an *improved estimate on average over* ℓ. A typical upper bound will be of the form:

$$\sum_{1 \leqslant \ell \leqslant L} M_*(z,\ell,\delta,1) \preccurlyeq \frac{L}{y} + L^{\frac{3}{2}}\delta^{\frac{1}{2}} + L^2\delta. \tag{1.9}$$

In fact we can reproduce the Iwaniec–Sarnak bound (1.1) from this estimate.

The Section 4 produces the bounds for the counting problem (1.7) that are uniform in δ and N. The typical estimate which generalizes (1.9) takes the form:

Lemma 1.3. *For any $z = x + iy \in \mathfrak{H}$, and any two integers $L, N \geqslant 1$ and $0 < \delta < 1$, we have*

$$\sum_{1 \leqslant \ell \leqslant L} M_*(z,\ell,\delta,N) \preccurlyeq \frac{L}{Ny} + \frac{L^{\frac{3}{2}}\delta^{\frac{1}{2}}}{N^{\frac{1}{2}}} + \frac{L^2\delta}{N}. \tag{1.10}$$

The progression from (1.8) to (1.10) via the estimate (1.9) is one key in obtaining an unified approach in all parameters.

1.2. **Proof of Lemma 1.3.** Without loss of generality we may assume that $z \in \mathcal{F}(N)$, the set of $z \in \mathfrak{H}$ such that $\Im\mathrm{m}(z) \geqslant \Im\mathrm{m}(\eta z)$ for all Atkin-Lehner operators η of level N. Indeed the count $M_*(z,\ell,\delta,N)$ is invariant under the Atkin-Lehner operators while the right-hand side of (1.10) is minimal if $z \in \mathcal{F}(N)$. We know [7, Lemma 2.2] that for all $z = x + iy \in \mathcal{F}(N)$, we have $Ny \gg 1$ and that for all $(a,b) \in \mathbb{Z}^2$ distinct from $(0,0)$ we have $|az+b|^2 \geqslant \frac{1}{N}$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\ell = ad - bc$ with $c \neq 0$, $(a+d)^2 \neq 4\ell$ and $1 \leqslant \ell \leqslant L$. The condition $u(\gamma z, z) \leqslant \delta$ yields in coordinates:

$$\left|-cz^2 + (a-d)z + b\right|^2 \leqslant 4L\delta y^2. \tag{1.11}$$

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[2]Here the notation $F \preccurlyeq G$ which stands for $F \ll_\epsilon G \cdot (\ell N)^\epsilon$ for any $\epsilon > 0$, see §2.1.

The left-hand side of (1.10) counts such matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying (1.11).

There are $\ll L^{\frac{1}{2}}/(Ny)$ possible values of c because we can verify that

$$cy \ll L^{\frac{1}{2}}. \tag{1.12}$$

Here is a proof of this inequality. The imaginary part of $-cz^2 + (a-d)z + b$ equals $(a-d-2cx)y$, hence

$$|a-d-2cx| \leqslant 2(L\delta)^{\frac{1}{2}}.$$

We can rewrite (1.11) as:

$$\left|\ell - |cz+d|^2 - (cz+d)(a-d-2cx)\right|^2 \leqslant \delta L c^2 y^2,$$

to which we apply the triangle inequality and obtain:

$$\left|\ell - |cz+d|^2\right| \ll (L\delta)^{\frac{1}{2}}(cy + |cz+d|) \ll (L\delta)^{\frac{1}{2}} |cz+d|.$$

It follows that $|cz+d| \ll L^{\frac{1}{2}}$. This implies (1.12) and also we have that $a+d \ll L^{\frac{1}{2}}$.

For each possible value of c we count the pairs of integers $(a-d, b)$ satisfying (1.11). This is equivalent to counting the number of lattice points in $\mathbb{Z} + z\mathbb{Z}$ that lie in an euclidean ball of radius $(L\delta)^{\frac{1}{2}} y$ and centered at cz^2. The lattice $\mathbb{Z} + z\mathbb{Z}$ has covolume y and shortest length at least $N^{-\frac{1}{2}}$. Hence that there are $\ll 1 + (LN\delta)^{\frac{1}{2}} y + L\delta y$ lattice points inside the ball [13, Lemma 2].

Since γ is determined by $c, a+d$ and $(a-d, b)$ and we have $cy \ll L^{\frac{1}{2}}$, $a+d \ll L^{\frac{1}{2}}$, we conclude that the total number of matrices satisfying (1.11) is

$$\ll \frac{L^{\frac{1}{2}}}{Ny} \cdot L^{\frac{1}{2}} \cdot (1 + (LN\delta)^{\frac{1}{2}} y + L\delta y).$$

1.3. **Structure of the paper.** The Section 2 provides some background on automorphic forms and the Selberg transform. The Section 3 gives a bound for $f(x+iy)$ via the Fourier expansion, which is good enough to cover the cuspidal region $y \gg T^{\frac{1}{6}} N^{-\frac{2}{3}}$. The complementary region is handled in the Section 6 which gathers all previous estimates to conclude the proof of Theorem 1.2. In Section 7 we review the proof of Iwaniec–Sarnak bound and provide a new treatment based on (1.9). Throughout the text we also introduce a number of improvements on existing techniques, which added together make the whole paper into a well-oiled machine for establishing sup-norm bounds of Hecke–Maass forms.

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2. Preliminaries

2.1. **Notation.** Without loss of generality when establishing Theorem 1.2, we may assume that λ and N are comparable in a logarithmic scale. Namely we can assume that for a given $A > 1$,

$$N^{1/A} \leqslant \lambda \leqslant N^A. \tag{2.1}$$

This is because the estimates in each of the λ and N aspects can be established with polynomial dependence on the other parameter. Similarly the amplifier will satisfy $1 \leqslant \ell \leqslant L \leqslant N^{O(1)}$.

The value of $\epsilon > 0$ may vary from line to line. For two functions $F(\lambda, N)$ and $G(\lambda, N)$ depending on the eigenvalue λ and the level N we adopt the notation

$$F \preccurlyeq G \text{ meaning that } |F(\lambda, N)| \ll_\epsilon G(\lambda, N)(\lambda N)^\epsilon$$

for all $\epsilon > 0$ (the multiplicative constant depends only on ϵ and is independent of λ and N). With (2.1) there is no loss in generality in replacing $(\lambda N)^\epsilon$ by N^ϵ or λ^ϵ. If F and G depend rather on $\ell \leqslant L$ as in §4-§7 then one should replace $(\lambda N)^\epsilon$ by L^ϵ.

2.2. **Hecke–Maass forms.** We let $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})^+$ act on the upper-half plane $\mathfrak{H} = \{x + iy,\ y > 0\}$ by fractional linear transformations. Denote by $u(w, z) = \frac{|w-z|^2}{4\,\Im\mathrm{m}(w)\,\Im\mathrm{m}(z)}$ the hyperbolic distance. Let $\Gamma_0(N)$ be the Hecke congruence subgroup of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ such that $c \equiv 0(N)$.

A cuspidal Hecke-Maass newform f of level N has a Nebentypus character χ of modulus N. It satisfies the automorphy condition

$$f(\gamma z) = \chi(d) f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ z \in \mathfrak{H},$$

is an eigenfunction of the Laplace operator $\Delta f = \lambda f$ and of the Hecke operators. It is also an eigenfunction of the Atkin–Lehner operators. Denote by $\lambda_f(n)$ the n-th Hecke eigenvalue, $n \geqslant 1$. We recall the Hecke relations:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda_f(\frac{mn}{d^2}), \quad \text{for } (mn, N) = 1.$$

2.3. **Selberg transform.** The book [9] is a classical reference on the Selberg transform. We identify the double quotient $\mathrm{SO}(2)\backslash \mathrm{SL}(2, \mathbb{R}) / \mathrm{SO}(2)$ with the interval $[0, \infty)$ using the hyperbolic distance. To a Paley-Wiener function h one associates a smooth function $k \in \mathcal{C}^\infty([0, \infty))$ with rapid decay as follows:

$$\begin{aligned} g(\xi) &:= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ir\xi} h(r) dr, \\ 2q(v) &:= g(2\log(\sqrt{v+1} + \sqrt{v})), \\ k(u) &:= -\frac{1}{\pi} \int_u^{+\infty} (v-u)^{-1/2} dq(v). \end{aligned}$$

A more direct construction is given by the spherical transform

$$k(u) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} F_{\frac{1}{2}+ir}(u) h(r) \tanh(\pi r) r dr,$$

where F is the spherical function:

$$F_s(u) = \frac{1}{\pi} \int_0^\pi (2u + 1 + 2\sqrt{u(u+1)}\cos\theta)^{-s} d\theta.$$

In particular we have the Plancherel formula

$$k(0) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(r) r \tanh(\pi r) dr. \tag{2.2}$$

2.4. **Choice of point-pair kernel.** As a preliminary step to establishing bounds in the eigenvalue aspect one needs to choose a suitable point-pair kernel.

Lemma 2.1. *For all $T \geqslant 1$ there is a point-pair kernel $k_T \in \mathcal{C}_c^\infty([0,\infty))$, supported on $[0,1]$ which satisfies the following properties:*

(i) The spherical transform $h_T(r)$ is positive for all $r \in \mathbb{R} \cup i\mathbb{R}$,
(ii) For all $T \leqslant r \leqslant T+1$, $h_T(r) \gg 1$,
(iii) For all $u \geqslant 0$, $|k_T(u)| \leqslant T$,
(iv) For all $T^{-2} \leqslant u \leqslant 1$, $|k_T(u)| \leqslant \dfrac{T^{\frac{1}{2}}}{u^{\frac{1}{4}}}$.

Remark. These conditions imply $k(0) \asymp T$, because of the Plancherel formula (2.2).

Proof. Let ϕ be a fixed positive Paley-Wiener function whose Fourier transform is supported on $(-\frac{1}{2}, \frac{1}{2})$. Adapting the method of Duistermaat–Kolk–Varadarajan [4] to the present setting we let

$$h_T(u) := |\phi(T-r) + \phi(T+r)|^2 .$$

Clearly the properties (i) and (ii) are satisfied. The property (iii) follows from the positivity of h_T and the fact that $\left|F_{\frac{1}{2}+ir}(u)\right| \leqslant 1$ for all $r \in \mathbb{R}$ and $u \geqslant 0$. The property (iv) follows by a direct computation. The fact that k_T is supported on $[0,1]$ follows from the compatibility with convolution of the Selberg transform. $\square$

Example 2.2. We review the explicit choice by Iwaniec-Sarnak of point-pair kernel [8, Lemma 1.1] which has similar properties except for the condition on the support. The spectral function is:

$$h_T(r) := \frac{4\pi^2 \cosh(\frac{\pi r}{2}) \cosh(\frac{\pi T}{2})}{\cosh(\pi r) + \cosh(\pi T)}, \quad r \in \mathbb{R} \cup i\mathbb{R}.$$

Clearly h is positive, $h(r) \geqslant 1$ for all $T \leqslant r \leqslant T+1$ and $h(r)$ decays rapidly when r is far from T. Then the Harish-Chandra transform is

$$g_T(\xi) = \frac{2\pi \cos(\xi T)}{\cosh(\xi)}.$$

and furthermore

$$q_T(v) = \pi(2v+1)^{-1} \Re\mathrm{e}\Big[\sqrt{v+1} + \sqrt{v})^{2iT}\Big],$$

from which it follows that for $u > 0$,

$$|k_T(u)| \leqslant 4T^{1/2} u^{-1/4} (u+1)^{-5/4}.$$

and for $0 \leqslant u \leqslant 1$,

$$k_T(u) = T + O(1 + u^{1/2} T^2).$$

2.5. **A weighted count.** The following simple estimate will be used repeatedly.

Lemma 2.3. *Let k_T be as in Lemma 2.1. Let $M : [0,1] \to \mathbb{R}_+$ be a non-decreasing function with finitely many discontinuities such that $M(\delta) \ll \delta^\alpha$ for some $\alpha > 0$. Then the following holds with $\beta := \max(\frac{1}{2}, 1-2\alpha)$:*

$$\int_0^1 k_T(\delta) dM(\delta) \ll T^\beta.$$

Example 2.4. If $\{\delta_i\} \subset (0,1)$ is a finite set and $M(\delta)$ is the counting function $\#\{i,\ \delta_i \leqslant \delta\}$, then

$$\int_0^1 k_T(\delta) dM(\delta) = \sum\nolimits_i k_T(\delta_i)$$

is a weighted count. We note that T^β is essentially an estimate for

$$k_T(0) \cdot M(T^{-2}) + |k_T(1)| \cdot M(1),$$

which can be interpreted as a trivial bound for the contributions near the two endpoints 0 and 1 respectively. By linearity the lemma extends to counting functions $M(\delta)$ which are sufficiently regular, e.g. bounded by a finite sum of monomials.

3. Bound via Fourier expansion

Proposition 3.1. *Let f be a Hecke–Maass cusp form of level N and eigenvalue $\lambda > 0$. Then for all $x + iy \in \mathfrak{H}$,*

$$f(x+iy) \preccurlyeq \frac{\lambda^{\frac{1}{4}}}{y^{\frac{1}{2}}} + \lambda^{\frac{1}{12}}.$$

Remarks. (i) When $N = 1$ is fixed, there was a mistake in the corresponding statement in [8, Lemma A.1]; a corrigendum is given in [12], see also [5, Eq.(41)].

(ii) When y is moderately large or bounded the first term dominates. The secondary term $\lambda^{\frac{1}{12}}$ occurs because of the transition range of K-Bessel functions (it is sharp because this is the true size of $f(x+iy)$ when $y \approx \lambda^{\frac{1}{2}}$).

(iii) Interestingly this type of estimate can be generalized to automorphic cusp forms on more general groups and to number fields. We shall return to this in a subsequent work.

Proof. For bounded λ, the estimate may be found in [2, §10] and [14, Lemma 3.1] (strictly speaking [2, §10] justify the estimate when $y > \lambda^{\frac{1}{6}}$). Without loss of generality we may assume $\lambda = \frac{1}{4} + r^2$ with $r \geqslant 1$. The Fourier expansion reads

$$f(x+iy) = y^{\frac{1}{2}} \sum_{n \neq 0} n^{\frac{1}{2}} \rho_f(n) K_{ir}(2\pi |n| y) e(nx)$$

where $\rho_f(n) = \rho_f(1) \dfrac{\lambda_f(n)}{\sqrt{n}}$. The estimate by Hoffstein–Lockhart yields $|\rho_f(1)| \preccurlyeq e^{\frac{\pi}{2} r}$.

We fix $\epsilon > 0$ arbitrary small. If $2\pi |n| y \geqslant r + r^{\frac{1}{3}+\epsilon}$, then the exponential decay of the K-Bessel function shows that the contribution is negligible.

In the range $2\pi y \leqslant r + r^{\frac{1}{3}+\epsilon}$ prior to the exponential decay, we shall use the following uniform estimate for the K-Bessel function

$$r^{\frac{1}{2}} e^{\frac{\pi}{2}r} K_{ir}(y) \ll \min\Big(r^{\frac{1}{6}}, \Big| \frac{y}{r} - 1 \Big|^{-\frac{1}{4}} \Big), \quad r \geqslant 1, y > 0. \tag{3.1}$$

This can be obtained from reference books on special functions, e.g. [15]. [3]

Taking absolute values we infer that

$$|f(x+iy)| \preccurlyeq \Big(\frac{y}{r}\Big)^{\frac{1}{2}} \sum_{n\neq 0} |\lambda(n)|\, r^{\frac{1}{2}} e^{\frac{\pi}{2}r} \left| K_{ir}(2\pi |n| y) \right|. \tag{3.2}$$

From the Cauchy–Schwarz inequality and Rankin–Selberg bounds for the mean square of $\lambda_f(n)$ we deduce

$$|f(x+iy)|^2 \preccurlyeq \sum_{n\neq 0} r e^{\pi r} \left| K_{ir}(2\pi |n| y) \right|^2.$$

It remains to plug-in the bound (3.1) and compare the sum over n to an integral. The maximum value is $r^{\frac{1}{3}}$ while the integral is

$$\int_0^{\frac{r+r^{\frac{1}{3}+\epsilon}}{y}} \Big| \frac{uy}{r} - 1 \Big|^{-\frac{1}{2}} du \ll \frac{r}{y}.$$

This concludes the proof of the proposition. $\square$

In certain ranges one can do slightly better using a bound $0 \leqslant \theta < \frac{1}{2}$ towards Ramanujan–Petersson, for which the current record by Kim–Sarnak–Shahidi reads $\theta = \frac{7}{64}$.

Proposition 3.2. *Let f be a Hecke–Maass cusp form of level N and eigenvalue $\lambda > 0$. Then for all $x+iy \in \mathfrak{H}$,*

$$f(x+iy) \preccurlyeq \frac{\lambda^{\frac{1}{4}}}{y^{\frac{1}{2}}} + \frac{\lambda^{\frac{1}{12}-\frac{\theta}{2}}}{y^{\theta}} \cdot \begin{cases} \lambda^{-\frac{1}{4}} y^{\frac{1}{2}}, & \text{if } \lambda^{\frac{1}{6}} \ll y \ll \lambda^{\frac{1}{2}}, \\ \lambda^{-\frac{1}{6}}, & \text{if } y \ll \lambda^{\frac{1}{6}}. \end{cases}$$

Furthermore if $y \gg \lambda^{\frac{1}{2}}$ then $f(x+iy)$ is exponentially small.

Proof. One can check that for n prior the transition range of the K-Bessel function, the application of Cauchy-Schwarz inequality is already optimal. But in the transition range we rather apply pointwise bounds towards Ramanujan–Petersson.

The term $\lambda^{\frac{1}{12}}$ when bounding (3.2) gets replaced by

$$\Big(\frac{y}{r}\Big)^{\frac{1}{2}} r^{\frac{1}{6}} \Big(1 + \frac{r^{\frac{1}{3}}}{y}\Big) \cdot \Big(\frac{r}{y}\Big)^{\theta}.$$

The estimate follows. $\square$

[3] Such estimates are usually stated in a different way in the literature, but we have found the expression (3.1) simple and practical for our purposes.

4. Counting lattice points

Our goal in this section is to achieve uppers bounds for $M(z,\ell,\delta,N)$ that are uniform in all parameters $z\in\mathfrak{H}$, $\ell\geqslant 1$, $0<\delta<1$ and $N\geqslant 1$. (see §1.1 for notation). We split the counting $M=M(z,\ell,\delta,N)$ of matrices $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ as

$$M=M_*+M_u+M_p$$

according to whether $c\neq 0$ and $(a+d)^2\neq 4\ell$ (generic), or $c=0$ and $a\neq d$ (upper-triangular), or $(a+d)^2=4\ell$ (parabolic).

Since we have given a detailed proof of Lemma 1.3 concerning M_* in §1.2 and the case $\delta=1$ may be found in [6, §2], it seems appropriate to omit the proofs here.[4]

Lemma 4.1. *For any $z=x+iy$ and $L\geqslant 1$,*

$$\sum_{\substack{1\leqslant\ell\leqslant L,\\ \ell\ square}} M_*(z,\ell,\delta,N)\preccurlyeq\frac{L^{\frac12}}{Ny}+\frac{L\delta^{\frac12}}{N^{\frac12}}+\frac{L^{\frac32}\delta}{N}.$$

Lemma 4.2. *For any $z=x+iy$ and $1\leqslant\ell_1\leqslant\Lambda\leqslant N^{O(1)}$,*

$$\sum_{1\leqslant\ell_2\leqslant\Lambda} M_*(z,\ell_1\ell_2^2,\delta,N)\preccurlyeq\frac{\Lambda^{\frac32}}{Ny}+\frac{\Lambda^3\delta^{\frac12}}{N^{\frac12}}+\frac{\Lambda^{\frac92}\delta}{N}.$$

Recall that $\mathcal{F}(N)$ is the set of $z\in\mathfrak{H}$ such that $\Im\mathrm{m}(z)\geqslant\Im\mathrm{m}(\eta z)$ for all Atkin-Lehner operators η of level N.

Lemma 4.3. *For any $z=x+iy\in\mathcal{F}(N)$ and $1\leqslant L,\Lambda\leqslant N^{O(1)}$, the following estimates holds where ℓ,ℓ_1,ℓ_2 run over primes:*

$$\sum_{1\leqslant\ell\leqslant L} M_u(z,\ell,\delta,N)\preccurlyeq 1+L^{\frac12}N^{\frac12}\delta^{\frac12}y+\frac{L\delta^{\frac12}}{N},$$
$$\sum_{1\leqslant\ell_1,\ell_2\leqslant\Lambda} M_u(z,\ell_1\ell_2,\delta,N)\preccurlyeq\Lambda+\Lambda^2N^{\frac12}\delta^{\frac12}y+\Lambda^3\delta^{\frac12}y,$$
$$\sum_{1\leqslant\ell_1,\ell_2\leqslant\Lambda} M_u(z,\ell_1\ell_2^2,\delta,N)\preccurlyeq\Lambda+\Lambda^{\frac52}N^{\frac12}\delta^{\frac12}y+\Lambda^4\delta^{\frac12}y,$$
$$\sum_{1\leqslant\ell_1,\ell_2\leqslant\Lambda} M_u(z,\ell_1^2\ell_2^2,\delta,N)\preccurlyeq 1+\Lambda^2N^{\frac12}\delta^{\frac12}y+\Lambda^4\delta^{\frac12}y.$$

Lemma 4.4 ([7, Lemma 4.1])**.** *For any $z=x+iy\in\mathcal{F}(N)$,*

$$M_p(z,\ell,\delta,N)\ll(1+\ell^{\frac12}\delta^{\frac12}y)\delta_\Box(\ell),$$

where $\delta_\Box(\ell)=1,0$ depending on whether ℓ is a perfect square or not.

[4] The present article precedes [6] in time, however it appears that [6] is already in print.

5. Amplifier

It is known that one can choose an amplifier sequence $y_\ell \in \mathbb{R}$ which satisfies

$$|y_\ell| \ll \begin{cases} \Lambda, & \ell = 1, \\ 1, & \ell = \ell_1 \text{ or } \ell_1\ell_2 \text{ or } \ell_1\ell_2^2 \text{ or } \ell_1^2\ell_2^2 \text{ with } \Lambda < \ell_1, \ell_2 < 2\Lambda \text{ primes}, \\ 0, & \text{otherwise.} \end{cases}$$

This section briefly recalls the details of the construction.

Let (x_ℓ) be a sequence supported on a finite set of primes powers. We define

$$y_\ell = \sum_{\substack{d|(\ell_1,\ell_2) \\ \ell=\ell_1\ell_2/d^2}} \chi(d) x_{\ell_1} \overline{x_{\ell_2}} = \sum_{\substack{d \geqslant 1 \\ \ell=\ell_1\ell_2}} \chi(d) x_{d\ell_1} \overline{x_{d\ell_2}}.$$

Then we have the inequalities

$$\sum_{\ell \geqslant 1} \ell^{-\frac{1}{2}} |y_\ell| \preccurlyeq \sum_{\ell \geqslant 1} |x_\ell|^2 \tag{5.1}$$

and

$$\sum_{\ell \geqslant 1} \ell^{\frac{1}{2}} |y_\ell| \preccurlyeq \Lambda^2 \Big(\sum_{\ell \geqslant 1} |x_\ell| \Big)^2. \tag{5.2}$$

We shall put $\mathcal{L} := \{\ell \text{ prime} : \ell \nmid N \text{ and } \Lambda \leqslant \ell \leqslant 2\Lambda\}$ and

$$x_\ell := \begin{cases} \operatorname{sgn}(\lambda_f(\ell)), & \text{if } \ell \in \mathcal{L} \cup \mathcal{L}^2 \\ 0, & \text{otherwise.} \end{cases}$$

The main property of this sequence is that

$$\Big| \sum_{\ell \geqslant 1} x_\ell \lambda_f(\ell) \Big| \gg_\epsilon \Lambda^{1-\epsilon}, \quad \text{for any } \epsilon > 0,$$

which follows from the Hecke relation $\lambda_f(\ell)^2 - \lambda_f(\ell^2) = \chi(\ell)$.

Bounds for Rankin–Selberg L-functions imply that $\sum |x_\ell|^2 \approx \Lambda$ and $\sum |x_\ell| \approx \Lambda$. While $\sum |y_\ell| \approx \Lambda^2$, $\sum \ell^{-\frac{1}{2}} |y_\ell| \approx \Lambda$ and $\sum \ell^{\frac{1}{2}} |y_\ell| \approx \Lambda^4$. Thus the above inequalities (5.1) and (5.2) are sharp for this choice of amplifier (also the main contribution to (5.1) comes from $\ell = 1$ and the main contribution to (5.2) comes from $\ell \asymp \Lambda^4$).

6. Conclusion of the Proof

Applying the amplification method of Friedlander–Iwaniec as in [8], [2, §10] and [7, §3] we have

$$\frac{\Lambda^2}{N} |f(z)|^2 \preccurlyeq \sum_{\ell \geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} K_T(z, \ell, N) \tag{6.1}$$

where the amplifier y_ℓ is chosen an in Section 5 and

$$K_T(z,\ell,N) := \sum_{\substack{\gamma\in M_2(\mathbb{Z})\\ \det(\gamma)=\ell,\ c\equiv 0(N)}} |k_T(z,\gamma z)|$$
$$= \int_0^1 |k_T(\delta)|\, dM(z,\ell,\delta,N).$$

Here the kernel k_T is chosen as in Lemma 2.1.

6.1. **Gathering estimates.** We begin with executing the sum over $\ell \geqslant 1$. Thus we estimate first the quantity

$$A(z,\delta,N) := \sum_{\ell\geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} M(z,\ell,\delta,N).$$

Since $M = M_* + M_u + M_p$ in Section 4 we decompose $A = A_* + A_u + A_p$ accordingly.

Proposition 6.1. *Let $N \geqslant 1$, $z = x+iy \in \mathcal{F}(N)$, $0<\delta<1$. Then*

$$A_*(z,\delta,N) \preccurlyeq \frac{\Lambda}{Ny} + \frac{\Lambda^{\frac{5}{2}}\delta^{\frac{1}{2}}}{N^{\frac{1}{2}}} + \frac{\Lambda^4\delta}{N},$$
$$A_u(z,\delta,N) \preccurlyeq \Lambda(1+N^{\frac{1}{2}}\delta^{\frac{1}{2}}y) + \Lambda^{\frac{5}{2}}\delta^{\frac{1}{2}}y$$
$$A_p(z,\delta,N) \ll \Lambda + \Lambda^2\delta^{\frac{1}{2}}y.$$

Proof. This follows from Lemma 1.3, the Lemmas 4.1, 4.2, 4.3, 4.4, and the properties of the amplifier sequence (y_ℓ) in Section 5. □

6.2. **Integration over δ.** We now execute the integration over δ in the Stieljes integral

$$\sum_{\ell\geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} K_T(z,\ell,N) = \int_0^1 |k_T(\delta)|\, dA(z,\ell,\delta,N).$$

From the Lemma 2.3 we see that we can make the substitutions $(\delta \rightsquigarrow T^{\frac{1}{2}})$, $(\delta^{\frac{1}{2}} \rightsquigarrow T^{\frac{1}{2}})$ and $(1 \rightsquigarrow T)$ starting from an upper-bound for $A(z,\ell,\delta,N)$ to obtain an upper bound for the Stieljes integral. Precisely a monomial δ^α for $A(z,\ell,\delta,N)$ becomes T^β after integrating where $\beta = \max(\frac{1}{2}, 1-2\alpha)$.

Altogether we obtain from Proposition 6.1 and after some simplifications using the fact that $Ny \gg 1$ and $T \geqslant 1$:

$$\sum_{\ell\geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} K_T(z,\ell,N) \preccurlyeq \Lambda T + \Lambda N^{\frac{1}{2}} T^{\frac{1}{2}} y + \Lambda^{\frac{5}{2}} T^{\frac{1}{2}} (N^{-\frac{1}{2}} + y) + \frac{\Lambda^4 T^{\frac{1}{2}}}{N}. \tag{6.2}$$

6.3. **Conclusion.** From the found via Fourier expansion in Proposition 3.1, we can assume without loss of generality when establishing Theorem 1.2 that

$$y \ll T^{\frac{1}{6}} N^{-\frac{2}{3}}.$$

Combining (6.1) with the estimate (6.2) we deduce

$$\frac{\Lambda^2}{N} |f(z)|^2 \preccurlyeq \Lambda T + \Lambda^{\frac{5}{2}} T^{\frac{1}{2}} y + \frac{\Lambda^{\frac{5}{2}} T^{\frac{1}{2}}}{N^{\frac{1}{2}}} + \frac{\Lambda^4 T^{\frac{1}{2}}}{N}.$$

We choose $\Lambda := T^{\frac{1}{6}}N^{\frac{1}{3}}$ in which case the first and fourth term are equal to $T^{\frac{7}{6}}N^{\frac{1}{3}}$ while the second and third terms are smaller. This yields

$$|f(z)| \preccurlyeq T^{\frac{5}{12}}N^{\frac{1}{3}},$$

as claimed in Theorem 1.2. □

7. Eigenvalue aspect

This section is partly for expository purposes and could serve as an introduction to the general argument. We shall review two different proofs of the Iwaniec-Sarnak bound: the first proof is an exposition of [8] with some simplification; the second proof uses the idea of averaged bounds on ℓ.

7.1. **Overview.** We want to establish (1.1), at least for z restricted to a bounded domain. Applying the amplification method as before we are reduced to the counting problem $M(z,\ell,\delta,1)$ of integral matrices described in §1.1. Precisely:

$$\Lambda^2\,|f(z)| \preccurlyeq \sum_{\ell\geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} K_T(z,\ell),$$

where $\Lambda^2 \geqslant 1$ is the amplifier length, (y_ℓ) is chosen as in Section 5 and we let

$$K_T(z,\ell) := \sum_{\substack{\gamma\in M_2(\mathbb{Z})\\ \det(\gamma)=\ell}} |k_T(z,\gamma z)| = \int_0^1 k_T(\delta)dM(z,\ell,\delta,1). \tag{7.1}$$

Depending on the relative position of the three parameters T, z, ℓ we need to use different techniques to obtain uniform upper bounds. The parameter $\delta > 0$ measures a distance, the displacement of z under the action of the Hecke operators of determinant ℓ. To ease the exposition we shall make the following reductions:

(i) By Lemma 2.1 we may assume that $0 < \delta < 1$. Both endpoints $\{0, 1\}$ play a key role in the final bound: in fact $k_T(0) \asymp T$ from the Plancherel formula produces the main contribution and $k_T(1) \asymp T^{\frac{1}{2}}$ produces the off-diagonal terms.

(ii) Without loss of generality we may assume that $z = x + iy$ lies in the standard fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. In particular $y \geqslant \frac{\sqrt{3}}{2}$. The case where y is large requires a separate treatment with Fourier expansion given in Section 3. Thus we focus on the case that $z \in \Omega$, a fixed compact subset of $\mathfrak{H}$.

(iii) The cases where $c = 0$ (upper-triangular) and where $(a + d)^2 = 4\ell$ (parabolic) don't play a role once we assume that $z \in \Omega$ (see Section 4 for the general case). Thus we focus on generic matrices: for $z \in \mathfrak{H}$, $0 < \delta < 1$ and an integer $\ell \geqslant 1$, recall that $M_*(z,\ell,\delta,1)$ be the number of matrices $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ in $M_2(\mathbb{Z})$ such that

$$\det(\gamma) = \ell, \quad u(\gamma z, z) \leqslant \delta, \quad c \neq 0, \quad (a+d)^2 \neq 4\ell.$$

Accordingly we denote the corresponding kernel in (7.1) by $K_{*T}(z,\ell)$.

7.2. **First proof (Iwaniec–Sarnak).**

Lemma 7.1 ([8, Eq. (A.9)]). *Uniformly on $z \in \mathfrak{H}$, $\ell \geqslant 1$ and $0 \leqslant \delta < 1$,*

$$M_*(z, \ell, \delta, 1) \preccurlyeq 1 + \ell \delta^{\frac{1}{4}}.$$

One immediately deduces the following bound using Lemma 2.3.

Corollary 7.2. *Uniformly on $T \geqslant 1$, $z \in \mathfrak{H}$ and $\ell \geqslant 1$ the following holds*

$$K_{*T}(z, \ell) \preccurlyeq T + \ell T^{\frac{1}{2}}.$$

Then using (5.1) and (5.2) we obtain:

$$\begin{aligned}\sum_{\ell \geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} K_{*T}(z, \ell) &\preccurlyeq T \sum_{\ell \geqslant 1} \ell^{-\frac{1}{2}} |y_\ell| + T^{\frac{1}{2}} \sum_{\ell \geqslant 1} \ell^{\frac{1}{2}} |y_\ell| \\ &\preccurlyeq T \sum_{\ell \geqslant 1} |x_\ell|^2 + T^{\frac{1}{2}} \Lambda^2 \Big(\sum_{\ell \geqslant 1} |x_\ell| \Big)^2.\end{aligned}$$

Thus for all $z \in \Omega$,

$$\Lambda^2 |f(z)|^2 \preccurlyeq T\Lambda + T^{\frac{1}{2}} \Lambda^4.$$

The choice $\Lambda = T^{\frac{1}{3}}$ finishes the sketch of the proof of (1.1).

7.3. **Second proof (averaged count).** In the second proof we start with an averaged bound for the counting problem from Section 4. The estimate (1.9) yields uniformly on $z \in \mathfrak{H}$, $\Lambda \geqslant 1$ and $0 < \delta < 1$,

$$A_*(z, \delta, 1) = \sum_{\ell \geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} M_*(z, \ell, \delta, 1) \preccurlyeq \Lambda + \Lambda^{\frac{5}{2}} \delta^{\frac{1}{2}} + \Lambda^4 \delta.$$

Corollary 7.3. *Uniformly on $T \geqslant 1$, $z \in \mathfrak{H}$ and $\Lambda \geqslant 1$ the following holds*

$$\sum_{\ell \geqslant 1} \frac{|y_\ell|}{\sqrt{\ell}} K_{*T}(z, \ell) \preccurlyeq T\Lambda + T^{\frac{1}{2}} \Lambda^4.$$

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