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Maximum likelihood estimation of a nonparametric signal in white noise by optimal control

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Abstract

We study extremal problems related to nonparametric maximum likelihood estimation (MLE) of a signal in white noise. The aim is to reduce these to standard problems of optimal control which can be solved by iterative procedures. This reduction requires a preliminary data smoothing; stability theorems are proved which justify such an operation on the data as a perturbation of the originally sought nonparametric (nonlinear) MLE. After this, classical optimal control problems appear; in the basic case of a signal with bounded first derivative one obtains the well-known problem of the optimal road profile. © 2001 Elsevier Science B.V. All rights reserved

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1. Introduction

Let us consider the model (cf. Ibragimov and Khasminskii, 1982)

$$\mathrm{d}a(t) = x(t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}w(t), \quad 0 \leqslant t \leqslant 1, \;\; a(0) = 0, \tag{1.1}$$

where $a(t)$ is observed, $x(t)$ is an unknown signal, $w(t)$ is a standard Wiener process and $\varepsilon > 0$ is a small parameter.

An extensive literature is devoted to various aspects of this model. We are concerned here with maximum likelihood estimation (MLE) of the unknown signal. Let ν be the probability measure in the space $\mathbf{C}[0,1]$ generated by the process $w(t)$. Then the likelihood function (cf. Ibragimov and Khasminskii, 1982) is

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equal to

$$\frac{\mathrm{d}P(a(\cdot)/\varepsilon)}{\mathrm{d}v} = \exp\left\{\frac{1}{\varepsilon^2}\int_0^1 x(t)\,\mathrm{d}a(t) - \frac{1}{2\varepsilon^2}\int_0^1 x^2(t)\,\mathrm{d}t\right\}.$$

When it is known that $x(\cdot)$ belongs to a class $\mathbf{K}$, then the maximum likelihood method of finding an estimate for $x(\cdot)$ leads to the problem

$$I = \frac{1}{2}\int_0^1 x^2(t)\,\mathrm{d}t - \int_0^1 x(t)\,\mathrm{d}a(t) \to \min_{x(\cdot)\in\mathbf{K}}. \tag{1.2}$$

A comprehensive theoretical analysis of this problem has been carried out by Tsirelson, 1982, 1985, 1986. Necessary and sufficient conditions for existence, uniqueness and consistency of the MLE are given there. The conditions are formulated in terms of some characteristics of the class $\mathbf{K}$. In Tsirelson, 1982, 1985, 1986 a number of properties of the maximum likelihood estimator $\hat{x}(a(\cdot))$ are considered as well. For example, probabilities of events $\|\hat{x}(a(\cdot)) - x\| \geqslant r$ are studied there. However, it seems that the problem of constructing estimators $\hat{x}(a(\cdot))$ has not been exhaustively treated in the statistical literature until now. For discrete (regression) data, nonparametric MLE has been thoroughly studied by Nemirovskii et al. (1984, 1985) and the methods developed there can be applied to model (1.1) after a suitable discretization. However, we prefer to give a direct solution of problem (1.2) for some important classes $\mathbf{K}$. By staying in the continuous framework, we are able to utilize some tools of optimal control theory for construction of nonparametric estimators. It seems that this aspect of nonparametric nonlinear MLE has escaped the attention of statisticians so far.

Suppose it is known that each function $x(\cdot)$ of the class $\mathbf{K}$ has a derivative $x'(\cdot)$ which is in $\mathbf{L}_2[0,1]$. In this case the functional (1.2) transforms to

$$I = \frac{1}{2}\int_0^1 x^2(t)\,\mathrm{d}t + \int_0^1 a(t)x'(t)\,\mathrm{d}t - a(1)x(1), \tag{1.3}$$

and problem (1.2) is equivalent to a problem of optimal control

$$I = \int_0^1 \left(\frac{1}{2}x^2 + a(t)u\right)\mathrm{d}t - a(1)x(1) \to \min, \quad x' = u, \tag{1.4}$$

in which restrictions on the control u and the phase variable x are present, given by the class $\mathbf{K}$.

If the signal $x(\cdot)$ is of Sobolev type, we treat a class $\mathbf{K}$ of form

$$\mathbf{K} = \left\{x(\cdot)\colon \exists x'(\cdot) = u(\cdot) \in \mathbf{L}_2[0,1], \frac{1}{2}\int_0^1 (\alpha x^2(t) + u^2(t))\,\mathrm{d}t \leqslant M,\ \alpha \geqslant 0, M > 0\right\}, \tag{1.5}$$

where α and M are known constants. In this class and in other Sobolev classes it is possible to obtain the MLE $\hat{x}(t)$ for the signal $x(t)$ up to a parameter which can be found from a transcendental equation (see Milstein and Nussbaum, 2000).

If the signal $x(\cdot)$ has a bounded derivative of order n, we treat a class $\mathbf{K}_n$ of form

$$\begin{aligned}\mathbf{K}_n = \{&x(\cdot)\colon \exists x^{(n-1)}(t) \text{ which is an absolutely continuous function,}\\ &|x^{(n)}(t)| \leqslant M_n,\ M_n > 0\},\end{aligned} \tag{1.6}$$

where M_n is a known constant. For the class (1.6), we modify problem (1.4) by replacing the observed data $a(t)$ which have bad analytical properties with slightly modified data $\bar{a}(t)$ such that there exists a piecewise continuous derivative $\bar{a}'(t)$. In Section 2, we show that the MLE with these modified data is close to the originally sought MLE. The results of Section 2 have not only an auxiliary meaning but are also of independent interest. They state stability of the maximum likelihood method with respect to certain fluctuations in the data.

After replacement of $a(t)$ by $\bar{a}(t)$ problem (1.4) can be reduced to

$$I = \frac{1}{2}\int_0^1 (x(t) - \bar{a}'(t))^2 \, \mathrm{d}t \to \min_{x(\cdot)\in \mathbf{K}_n}. \tag{1.7}$$

Problem (1.7) is well-known and has already been investigated by methods of optimal control in Gitenjova and Milstein (1967) and Boltyanskii (1961). We consider some iterative solution methods in Sections 3 and 4 and some generalizations in Section 5.

The preliminary data smoothing operation $a(t) \mapsto \bar{a}(t)$ which we propose, to pave the way for an application of optimal control theory, is similar in spirit to a discretization with respect to a regular grid $t_i = i/n$. With discrete data one could apply the nonparametric MLE theory of Nemirovskii et al. (1984, 1985). Our sensitivity theorems in the next section provide some formal rationale for preliminary data smoothing. Also, they are geared towards the subsequent application of continuous optimal control theory, which we believe furnishes useful insights on nonparametric (and nonlinear) MLE. In particular, the "optimal road profile" extremal problem described by Boltyanskii (1961) has a simple and appealing formulation (cf. Section 3), and to our knowledge the link to nonparametric MLE has not been made.

2. Sensitivity theorems for signals with bounded derivative

Let us consider the class

$$\mathbf{K}_1 = \{x(\cdot)\colon x(t) \text{ is absolutely continuous, } |x'(t)| \leqslant M,\ M > 0\} \tag{2.1}$$

and the minimization problem in this class

$$I = \frac{1}{2}\int_0^1 x^2(t)\,\mathrm{d}t + \int_0^1 a(t)x'(t)\,\mathrm{d}t - a(1)x(1) \to \min_{x(\cdot)\in\mathbf{K}_1}. \tag{2.2}$$

It is possible to prove that there exists a solution of the problem.

Theorem 2.1. *Let $\bar{a}(t)$ be a continuous function such that*

$$\bar{a}(0) = a(0) = 0, \quad \bar{a}(1) = a(1), \quad \int_0^1 |\bar{a}(s) - a(s)|\,\mathrm{d}s \leqslant \delta. \tag{2.3}$$

Let $x_0(\cdot)$ be a solution of the minimization problem (2.2) *and $\bar{x}_0(\cdot)$ be a solution of the problem*

$$\bar{I} = \frac{1}{2}\int_0^1 x^2(t)\,\mathrm{d}t + \int_0^1 \bar{a}(t)x'(t)\,\mathrm{d}t - \bar{a}(1)\cdot x(1) \to \min_{x(\cdot)\in\mathbf{K}_1}. \tag{2.4}$$

Then

$$0 \leqslant I(\bar{x}_0(\cdot)) - I(x_0(\cdot)) \leqslant 2\delta M, \tag{2.5}$$

$$\int_0^1 (\bar{x}_0(t) - x_0(t))^2\,\mathrm{d}t \leqslant 4\delta M, \tag{2.6}$$

and if $\delta \leqslant M/3$,

$$\max_{0\leqslant t\leqslant 1} |\bar{x}_0(t) - x_0(t)| \leqslant (24\delta M^2)^{1/3}. \tag{2.7}$$

Proof. Obviously

$$I(\bar{x}_0(\cdot)) - I(x_0(\cdot)) \geqslant 0, \quad \bar{I}(x_0(\cdot)) - \bar{I}(\bar{x}_0(\cdot)) \geqslant 0.$$

Furthermore

$$|I(\bar{x}_0(\cdot)) - \bar{I}(\bar{x}_0(\cdot))| = \left| \int_0^1 (a(t) - \bar{a}(t))\bar{x}_0'(t)\,\mathrm{d}t \right| \leqslant M\delta.$$

Analogously

$$|\bar{I}(x_0(\cdot)) - I(x_0(\cdot))| \leqslant M\delta.$$

Therefore

$$0 \leqslant I(\bar{x}_0(\cdot)) - I(x_0(\cdot)) \leqslant I(\bar{x}_0(\cdot)) - I(x_0(\cdot)) + \bar{I}(x_0(\cdot)) - \bar{I}(\bar{x}_0(\cdot))$$
$$\leqslant |I(\bar{x}_0(\cdot)) - \bar{I}(\bar{x}_0(\cdot))| + |\bar{I}(x_0(\cdot)) - I(x_0(\cdot))| \leqslant 2\delta M.$$

Thus the inequality (2.5) is proved.
For derivation of (2.6), let us note that

$$(1-\alpha)x_0(\cdot) + \alpha\bar{x}_0(\cdot) \in \mathbf{K}_1, \quad 0 \leqslant \alpha \leqslant 1,$$

and introduce the function $f(\alpha)$, $0 \leqslant \alpha \leqslant 1$ (see (1.2))

$$\begin{aligned} f(\alpha) &= I((1-\alpha)x_0(\cdot) + \alpha\bar{x}_0(\cdot)) \\ &= \frac{1}{2}\int_0^1 ((1-\alpha)x_0(s) + \alpha\bar{x}_0(s))^2\,\mathrm{d}s - \int_0^1 ((1-\alpha)x_0(s) + \alpha\bar{x}_0(s))\,\mathrm{d}a(s), \end{aligned}$$

which is a quadratic trinomial on α.
Clearly

$$f(0) = I(x_0(\cdot)) \leqslant f(\alpha), \qquad f'(0) \geqslant 0.$$

We have

$$f'(\alpha) = \int_0^1 ((1-\alpha)x_0(s) + \alpha\bar{x}_0(s))(\bar{x}_0(s) - x_0(s))\,\mathrm{d}s - \int_0^1 (\bar{x}_0(s) - x_0(s))\,\mathrm{d}a(s),$$
$$f''(\alpha) = \int_0^1 (\bar{x}_0(s) - x_0(s))^2\,\mathrm{d}s = \mathrm{const} = C > 0.$$

Further,

$$f'(\alpha) = f'(0) + \int_0^\alpha f''(\alpha)\,\mathrm{d}\alpha = f'(0) + C\alpha,$$
$$f(\alpha) = I(x_0(\cdot)) + f'(0)\alpha + C\frac{\alpha^2}{2}, \quad f(1) = I(\bar{x}_0(\cdot)) = I(x_0(\cdot)) + f'(0) + \frac{C}{2}.$$

From here and (2.5)

$$f'(0) + \frac{C}{2} \leqslant 2\delta M$$

and, as $f'(0) \geqslant 0$, we obtain the inequality (2.6).
Now let us prove the inequality (2.7). Let

$$m = \max_{0 \leqslant t \leqslant 1} |\bar{x}_0(t) - x_0(t)| = |\bar{x}_0(t^*) - x_0(t^*)|.$$

We take $x_0(t^*) = x_0^*$; then

$$x_0^* < \bar{x}_0(t^*) = x_0^* + m.$$

Since $|x_0'(t)| \leqslant M$ and $|\bar{x}_0'(t)| \leqslant M$ for $0 \leqslant t \leqslant 1$, it is clear that for $0 \leqslant t \leqslant t^*$

$$x_0(t) \leqslant x_0^* - M(t-t^*), \quad \bar{x}_0(t) \geqslant x_0^* + m + M(t-t^*)$$

and for $t^* \leqslant t \leqslant 1$

$$x_0(t) \leqslant x_0^* + M(t-t^*), \quad \bar{x}_0(t) \geqslant x_0^* + m - M(t-t^*).$$

Hence

$$\begin{aligned} 4\delta M &\geqslant \int_0^1 (\bar{x}_0(s) - x_0(s))^2 \,\mathrm{d}s \\ &\geqslant \int_{0\vee(t^*-m/2M)}^{t^*} (m + 2M(s-t^*))^2 \,\mathrm{d}s + \int_{t^*}^{1\wedge(t^*+m/2M)} (m - 2M(s-t^*))^2 \,\mathrm{d}s. \end{aligned} \tag{2.8}$$

We have to find the largest m for which this inequality can be valid. Clearly one can determine this quantity from (2.8) at $t^* = 0$.

We have

$$4\delta M \geqslant \int_0^{1\wedge m/2M} (m - 2Ms)^2 \,\mathrm{d}s = \begin{cases} m^3/6M, & m/2M \leqslant 1, \\ m(m-2M) + 4M^2/3, & m/2M > 1. \end{cases} \tag{2.9}$$

But for $\delta \leqslant M/3$ the second case in (2.9) is impossible and therefore

$$m^3 \leqslant 24\delta M^2.$$

Thus Theorem 2.1 is proved. □

Consider the class of functions

$$\mathbf{K}_2 = \{x(\cdot)\text{: } x'(t) \text{ is absolutely continuous and } |x''(t)| \leqslant M_2,\ M_2 > 0\}. \tag{2.10}$$

The functional (1.2) in the class $\mathbf{K}_2$ can be rewritten as

$$I(x(\cdot)) = \frac{1}{2}\int_0^1 x^2(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^t a(s)\,\mathrm{d}s\right) x''(t)\,\mathrm{d}t + x'(1)\int_0^1 a(s)\,\mathrm{d}s - a(1)x(1). \tag{2.11}$$

It is possible to prove that there exists a solution of the minimization problem for the functional (2.11) in the class $\mathbf{K}_2$.

Denote by $\bar{I}(x(\cdot))$ the functional which is similar to $I(x(\cdot))$ with $\bar{a}$ instead of a. The following theorem is true (see proof in Milstein and Nussbaum, 2000).

Theorem 2.2. *Let $\bar{a}(t)$ be a continuous function such that*

$$\bar{a}(0) = 0, \quad \bar{a}(1) = a(1), \quad \int_0^1 \bar{a}(s)\,\mathrm{d}s = \int_0^1 a(s)\,\mathrm{d}s, \tag{2.12}$$

$$\int_0^1 \left| \int_0^t \bar{a}(s)\,\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s \right| \mathrm{d}t \leqslant \delta, \tag{2.13}$$

where δ is sufficiently small. Let $x_0(\cdot)$ be a solution of the minimization problem for the functional $I(x(\cdot))$ in the class $\mathbf{K}_2$ and let $\bar{x}_0(\cdot)$ be a solution of the minimization problem for the functional $\bar{I}(x(\cdot))$ in the same class.

Then,

$$0 \leqslant I(\bar{x}_0(\cdot)) - I(x_0(\cdot)) \leqslant 2\delta M_2, \tag{2.14}$$

$$\int_0^1 (\bar{x}_0(t) - x_0(t))^2 \, \mathrm{d}t \leqslant 4\delta M_2, \tag{2.15}$$

$$|x_0(t)| \leqslant M_0, \quad |\bar{x}_0(t)| \leqslant M_0, \quad |x_0'(t)| \leqslant M_1, \quad |\bar{x}_0'(t)| \leqslant M_1, \qquad 0 \leqslant t \leqslant 1, \tag{2.16}$$

where M_0 and M_1 are certain constants depending only on M_2, $|a(1)|, |\int_0^1 a(s)\,\mathrm{d}s|$, and $\int_0^1 |\int_0^1 a(s)\,\mathrm{d}s|\,\mathrm{d}t$.

Further, there exist positive constants K_0, K_1, K_2 depending on M_1 and M_2 only such that

$$\int_0^1 (\bar{x}_0'(t) - x_0'(t))^2 \, \mathrm{d}t \leqslant K_0 \delta^{1/2}, \tag{2.17}$$

$$\max_{0 \leqslant t \leqslant 1} |\bar{x}_0(t) - x_0(t)| \leqslant K_1 \delta^{1/3}, \tag{2.18}$$

$$\max_{0 \leqslant t \leqslant 1} |\bar{x}_0'(t) - x_0'(t)| \leqslant K_2 \delta^{1/6}. \tag{2.19}$$

Remark 2.1. The principal results of Theorems 2.1 and 2.2 remain valid if the conditions $\bar{a}(1) = a(1)$ and $\int_0^1 \bar{a}(s)\,\mathrm{d}s = \int_0^1 a(s)\,\mathrm{d}s$ in (2.3) and (2.12) are replaced by the conditions

$$|\bar{a}(1) - a(1)| \leqslant \delta_1, \quad \left| \int_0^1 \bar{a}(s)\,\mathrm{d}s - \int_0^1 a(s)\,\mathrm{d}s \right| \mathrm{d}t \leqslant \delta_2$$

(see details in Milstein and Nussbaum, 2000).

3. Reduction of MLE to the problem of the optimal road profile

Let us return to the problem of finding the MLE $\hat{x}(t)$ in the class $\mathbf{K}_1$. This estimate can be found as a solution of the minimization problem (1.2) with $\mathbf{K} = \mathbf{K}_1$.

Consider also the following minimization problem:

$$\bar{I} = \frac{1}{2} \int_0^1 x^2(t)\,\mathrm{d}t - \int_0^1 x(t)\,\mathrm{d}\bar{a}(t) \to \min_{x(\cdot) \in \mathbf{K}_1}. \tag{3.1}$$

According to Theorem 2.1 if $\bar{a}(\cdot)$ is close to $a(\cdot)$, then the solution $\bar{x}(t)$ of problem (3.1) is close to the MLE $\hat{x}(t)$. There are extensive possibilities for a choice of the function $\bar{a}(t)$ such that conditions (2.3) are satisfied. For instance, the function $\bar{a}(t)$ can be easily found as a piecewise linear function, which has a piecewise constant derivative.

Let $\bar{a}(t)$ in (3.1) satisfy (2.3) and be piecewise differentiable. Denote $\bar{a}'(t)$ by $b(t)$. Then the functional (3.1) transforms to the functional

$$\bar{I} = \frac{1}{2} \int_0^1 (x - b(t))^2\,\mathrm{d}t - \frac{1}{2} \int_0^1 b^2(t)\,\mathrm{d}t,$$

and the following minimization problem appears (for the modified functional we use the initial notation I again without ambiguity):

$$I = \frac{1}{2} \int_0^1 (x - b(t))^2\,\mathrm{d}t \to \min_{|x'| \leqslant M}. \tag{3.2}$$

Problem (3.2) is a problem of mean-square approximation by functions with bounded derivative. It can be interpreted as a problem of building a road with profile $x(t)$ which cannot have steep ascents and descents and therefore $|x'(t)| \leqslant M$, $0 \leqslant t \leqslant 1$. The function $b(t)$ is interpreted as the profile of the locality and the integral I as the cost of building. The study of this problem in an optimal control formulation

$$I = \frac{1}{2}\int_0^1 (x - b(t))^2\,\mathrm{d}t \to \min_{u:\ |u| \leqslant M}, \tag{3.3}$$

$$x' = u \tag{3.4}$$

was initiated by Boltyanskii (1961). It has been investigated in more detail and in more general form in Gitenjova and Milstein (1967). In particular the sufficiency of Pontryagin's maximum principle is proved when in place of Eq. (3.4), one considers a general m-dimensional nonautonomous linear system with r-dimensional control and instead of a functional with quadratic integrand one considers a functional with convex function. Besides in Gitenjova and Milstein (1967) an iterative procedure is recommended for finding an optimal solution. Both Boltyanskii and Gitenjova Yu and Milstein (1967) made an assumption that $b(t)$ is piecewise differentiable. However, this assumption is not essential; we are interested in the case where $b(t)$ is only piecewise continuous, since the simplest method of approximating $a(t)$ is realized by means of piecewise linear functions. As a result, $b(t)$ will be piecewise constant. Therefore, but also for completeness of exposition we develop in Milstein and Nussbaum (2000) the required results from Gitenjova and Milstein (1967) with proofs, which are simplified substantially in the case considered. Here, we restrict ourselves to an iterative procedure which allows to obtain an approximate solution in a constructive manner.

Beforehand, let us remark that the solution to problem (3.3), (3.4) exists and is unique, which can be proved by standard methods of optimal control.

Let us write down necessary conditions for the optimal solution of problem (3.3), (3.4) (see Alexeev et al., 1979). Pontryagin's function H has the form

$$H(t,x,u,p) = pu - \frac{\lambda_0}{2}(x - b(t))^2.$$

It is not difficult to prove that $\lambda_0 \neq 0$ and hence we can put $\lambda_0 = 1$. The optimal solution $u(t)$, $x(t)$ satisfies the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p} = u, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial x} = x - b(t), \tag{3.5}$$

the conditions of transversality

$$p(0) = 0, \qquad p(1) = 0, \tag{3.6}$$

and the maximum condition

$$p(t)\,u(t) = \max_{|v| \leqslant M} p(t)v. \tag{3.7}$$

Sufficiency of the Pontryagin maximum principle (3.5)–(3.7) can be shown. So the solution of the problem (3.5)–(3.7) is optimal for problem (3.3), (3.4).

This solution can be found by the following iterative procedure. As a first approximation of the optimal control we take an arbitrary admissible control $u_1(t)$. The first approximation of the trajectory $x_1(t)$ and the function $p_1(t)$ are found as the unique solution of the boundary value problem (3.5), (3.6) under $u = u_1(t)$. Namely

$$x(t) = x(0) + \int_0^t u(s)\,\mathrm{d}s, \quad p(t) = \int_0^t \left[x(0) + \int_0^\tau u(s)\,\mathrm{d}s - b(\tau)\right]\mathrm{d}\tau,$$

$$x(0) = -\int_0^1 \left[\int_0^\tau u(s)\,\mathrm{d}s - b(\tau)\right]\mathrm{d}\tau, \tag{3.8}$$

where $u(t) = u_1(t)$, $0 \leqslant t \leqslant 1$.

Let the kth approximation be constructed: $u_k(t)$, $x_k(t)$, $p_k(t)$. Knowing $p_k(t)$, we find $v_k(t)$ from the condition

$$p_k(t)v_k(t) = \max_{|v| \leqslant M} p_k(t)v,$$

that is, in particular, one may put

$$v_k(t) = M \operatorname{sign} p_k(t) = \begin{cases} M, & p_k(t) > 0, \\ 0, & p_k(t) = 0, \\ -M, & p_k(t) < 0. \end{cases}$$

If $v_k(t) = u_k(t)$ a.s., then it is proved that $u_k(t)$ is the optimal control. Let $v_k(t) \neq u_k(t)$. Then we minimize the integral (3.3) on the one-parameter set of controls of the form

$$u(t;\alpha) = \alpha u_k(t) + (1-\alpha)v_k(t), \quad 0 \leqslant \alpha \leqslant 1.$$

It amounts to finding values x_k^0, α_k, which realize the minimal value of the function

$$G(x^0,\alpha) = \frac{1}{2}\int_0^1 \left(x^0 + \int_0^t u(\tau;\alpha)\,\mathrm{d}\tau - b(t)\right)^2 \mathrm{d}t$$

in the domain $-\infty < x_0 < +\infty$, $0 \leqslant \alpha \leqslant 1$. The values x_k^0, α_k can be found by the following rule.

Calculate the functions $\xi_k(t)$ and $\eta_k(t)$:

$$\xi_k(t) = \int_0^t (u_k(s) - v_k(s))\,\mathrm{d}s - \int_0^1 \int_0^\tau (u_k(s) - v_k(s))\,\mathrm{d}s\,\mathrm{d}\tau,$$

$$\eta_k(t) = \int_0^1 b(\tau)\,\mathrm{d}\tau + \int_0^t v_k(s)\,\mathrm{d}s - \int_0^1 \int_0^\tau v_k(s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

Calculate the constant β_k:

$$\beta_k = \frac{\int_0^1 \xi_k(t)(b(t) - \eta_k(t))\,\mathrm{d}t}{\int_0^1 \xi_k^2(t)\,\mathrm{d}t}.$$

Finally,

$$x_k^0 = -\int_0^1 \left[\int_0^t u(s;\alpha_k)\,\mathrm{d}s - b(t)\right] \mathrm{d}t, \qquad \alpha_k = \begin{cases} \beta_k & \text{if } 0 < \beta_k < 1, \\ 0 & \text{if } \beta_k \leqslant 0, \\ 1 & \text{if } \beta_k \geqslant 1. \end{cases} \tag{3.9}$$

Then the $(k+1)$th approximation of the control is chosen in the form

$$u_{k+1}(t) = \alpha_k u_k(t) + (1-\alpha_k)v_k(t), \tag{3.10}$$

and $x_{k+1}(t)$ and $p_{k+1}(t)$ are found as the unique solution of the boundary value problem (3.5), (3.6) due to formula (3.8), where $u(t) = u_{k+1}(t)$. In this manner we construct a sequence $\{u_n(t), x_n(t), p_n(t)\}$.

Let I_n be the value of the functional (3.2) at $x(\cdot) = x_n(t)$. By construction, the sequence I_n is nonincreasing and is bounded from below by the least value of the functional I: $I_1 \geqslant I_2 \geqslant \cdots \geqslant I_n \geqslant \cdots \geqslant I_0$.

The following theorem is true (see the proof in Milstein and Nussbaum, 2000).

Theorem 4.1. *The sequence $x_n(t)$ converges uniformly on $[0,1]$ to the optimal trajectory. The sequence $u_n(t)$ converges to the optimal control weakly.*

Remark 4.1. Since obviously $u_k(t)$ is a piecewise constant function, $x_k(t)$ is always a piecewise linear continuous function. Therefore, if $b(t)$ is a piecewise constant or continuous piecewise linear function, $p_k(t)$ is a quadratic spline (of defect 2 or 1). The knots of this spline are the switching points of $u_k(t)$ and the nonregular points of the function $b(t)$.

4. Inserting a parameter

In this section we present another approach to the constructive solution of the problem (3.3), (3.4). Let us consider the following problem of optimal control:

$$I = \frac{1}{2}\int_0^1 ((x - b(t))^2 + \lambda u^2)\,\mathrm{d}t \to \min_{|u| \leqslant M}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u, \tag{4.1}$$

which depends on the parameter $\lambda \geqslant 0$. For $\lambda = 0$ the problem coincides with (3.3), (3.4). Clearly the solution of problem (4.1) for small positive λ is close to the required solution of (3.3), (3.4).

Pontryagin's function H of problem (3.3), (3.4) has the form

$$H(t,x,u,p) = pu - \tfrac{1}{2}(x - b(t))^2 - \tfrac{1}{2}\lambda u^2.$$

The necessary conditions (it can be proved that they are sufficient as well) for the optimal solution under $\lambda > 0$ are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p} = u, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial x} = x - b(t), \quad p(0) = p(1) = 0, \tag{4.2}$$

$$p(t)u(t) - \tfrac{1}{2}\lambda u^2(t) = \max_{|v| \leqslant M}(p(t)v - \tfrac{1}{2}\lambda v^2). \tag{4.3}$$

Condition (4.3) gives the following expression for u:

$$u = u(p;\lambda) := \begin{cases} -M, & p < -\lambda M, \\ p/\lambda, & |p| \leqslant \lambda M, \\ M, & p > \lambda M. \end{cases} \tag{4.4}$$

Therefore to find the optimal solution we have to solve the boundary value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(p;\lambda), \quad \frac{\mathrm{d}p}{\mathrm{d}t} = x - b(t), \quad p(0) = p(1) = 0. \tag{4.5}$$

Below it is argued that for all sufficiently large λ the restriction $|p| \leqslant \lambda M$ is fulfilled and consequently problem (4.5) takes the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{\lambda}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = x - b(t), \quad p(0) = p(1) = 0. \tag{4.6}$$

Problem (4.6) has the following explicit solution:

$$x = \frac{1}{\sqrt{\lambda}}\left(\sinh\frac{1}{\sqrt{\lambda}}\right)^{-1}\cosh\frac{t}{\sqrt{\lambda}}\int_0^1 \cosh\frac{1-\tau}{\sqrt{\lambda}}b(\tau)\,\mathrm{d}\tau - \frac{1}{\sqrt{\lambda}}\int_0^t \sinh\frac{t-\tau}{\sqrt{\lambda}}b(\tau)\,\mathrm{d}\tau,$$

$$p = \left(\sinh\frac{1}{\sqrt{\lambda}}\right)^{-1}\sinh\frac{t}{\sqrt{\lambda}}\int_0^1 \cosh\frac{1-\tau}{\sqrt{\lambda}}b(\tau)\,\mathrm{d}\tau - \int_0^t \cosh\frac{t-\tau}{\sqrt{\lambda}}b(\tau)\,\mathrm{d}\tau.$$

Now it can be verified that the above mentioned restriction is in fact fulfilled if, for example,

$$\sqrt{\lambda}\exp\left(-\frac{1}{\sqrt{\lambda}}\right) > \frac{B}{M}, \quad B := \max_{0 \leqslant s \leqslant 1}|b(s)|.$$

Let us denote by $x(t;\lambda,x_0)$, $p(t;\lambda,x_0)$ the solution to the Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}t}=u(p;\lambda), \quad \frac{\mathrm{d}p}{\mathrm{d}t}=x-b(t), \quad x(0)=x_0, \quad p(0)=0. \tag{4.7}$$

To solve the boundary value problem (4.5) for a $\lambda>0$ it is necessary to find the root $x_0=x_0(\lambda)$ of the equation

$$p(1;\lambda,x_0)=0. \tag{4.8}$$

The transcendental equation (4.8) can be solved easily for every λ (for example, by the chord method) if an initial approximation for $x_0(\lambda)$ is known with sufficient accuracy.

Let $x_0(\lambda)$ for some $\lambda=\bar{\lambda}$ be known. Then $x_0(\bar{\lambda}-\Delta\lambda)$ for a small $\Delta\lambda>0$ can be found from the equation

$$p(1;\bar{\lambda}-\Delta\lambda,x_0)=0$$

if we take $x_0(\bar{\lambda})$ as an initial approximation for $x_0(\bar{\lambda}-\Delta\lambda)$. Thus, knowing the optimal solution for some $\lambda_1>0$ (fortunately, we do know it for some large $\lambda_1>0$), we can find it for $\lambda_2=\lambda_1-\Delta\lambda_1$ in a constructive manner. That is, knowing $x_0(\lambda_1)$, we find $x_0(\lambda_2)$. Analogously, knowing $x_0(\lambda_2)$, we find $x_0(\lambda_2-\Delta\lambda_2)=x_0(\lambda_3)$ and so on. With these ideas it is not difficult to construct a numerical procedure for solving problem (4.1) for a sufficiently small $\lambda>0$ and consequently for approximate solution of the problem (3.3), (3.4).

5. Some generalizations

The problem of finding a maximum likelihood estimate $\hat{x}(t)$ in each class $\mathbf{K}_n$ (see (1.6)), $n\geqslant 2$, is solved analogously. After substituting $a(t)$ by a nearby $\bar{a}(t)$ such that there exists the piecewise continuous derivative $\bar{a}'(t)=b(t)$, this problem is also reduced to the "problem of finding the optimal road profile." For example, in the case $n=2$, we obtain the problem of minimization of the functional

$$I=\frac{1}{2}\int_0^1 (x_1-b(t))^2\,\mathrm{d}t \to \min_{|u|\leqslant M}, \quad \frac{\mathrm{d}x_1}{\mathrm{d}t}=x_2, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t}=u. \tag{5.1}$$

Problem (5.1) can be solved with using Gitenjova and Milstein (1967) as problem (3.3), (3.4) was done above, and due to Theorem 2.2 the optimal solution $\bar{u}(t)$, $\bar{x}_1(t)$, $\bar{x}_2(t)$ of the problem is such that $\bar{x}_1(t)$ is close to $\hat{x}(t)$.

The same approach is possible also in the case of stronger information on the unknown signal. For instance, it may be known that the signal is a nondecreasing function with first derivative bounded from above. Then

$$\mathbf{K}_1^*=\{x(\cdot)\colon x(t) \text{ is absolutely continuous and } 0\leqslant x'(t)\leqslant M\},$$

and the optimal control problem takes the form

$$I=\frac{1}{2}\int_0^1 (x-b(t))^2\,\mathrm{d}t \to \min_{0\leqslant u\leqslant M}, \quad x'=u. \tag{5.2}$$

Let us introduce a new control v and a new phase variable y:

$$v=u-\frac{M}{2}, \quad y=x-\frac{M}{2}t.$$

Then problem (5.2) transforms to

$$I=\frac{1}{2}\int_0^1 (y-c(t))^2\,\mathrm{d}t \to \min_{|v|\leqslant M/2}, \quad y'=v,$$

where $c(t)=b(t)-Mt/2$, which coincides with the problem (3.3), (3.4).

Now consider the class

$$\mathbf{K}_2^* = \{x(\cdot)\colon x'(t) \text{ is absolutely continuous and } 0 \leqslant x''(t) \leqslant M\}$$

which corresponds to information on the signal being a convex function with bounded second derivative. As above it can be reduced to the problem

$$I = \frac{1}{2}\int_0^1 (y_1 - c(t))^2\,\mathrm{d}t \to \min_{|v| \leqslant M/2}, \quad \frac{\mathrm{d}y_1}{\mathrm{d}t} = y_2, \quad \frac{\mathrm{d}y_2}{\mathrm{d}t} = v,$$

where

$$v = u - \frac{M}{2}, \quad y_1 = x - \frac{Mt^2}{4}, \quad c(t) = b(t) - \frac{Mt^2}{4},$$

which coincides with the problem (5.1).

Analogously one treats the case where it is known that there exists absolutely continuous $x^{(n-1)}(t)$, and $0 \leqslant x^{(n)}(t) \leqslant M$. Such a class appears if it is known that the signal does not have more than n pieces of monotonicity (and, of course, if it is sufficiently smooth and its nth derivative is subject to the bounds indicated).

Another quite natural information on the signal would be

$$\mathbf{K} = \{x(\cdot)\colon A \leqslant x(t) \leqslant B, \quad x^{(n-1)}(t) \text{ is absolutely continuous and } 0 \leqslant x^{(n)}(t) \leqslant M\},$$

i.e. besides the fact that the signal does not have more than n pieces of monotonicity it is known that it is in a certain band. This problem can also be reduced to a typical optimal control problem but this time with bounded phase variables. To find a constructive solution of such a problem is a more complicated task.

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References

Alexeev, V.M., Tichomirov, V.M., Fomin, S.V., 1979. Optimal Control. Nauka, Moscow.

Boltyanskii, V.G., 1961. Application of the theory of optimal processes to problems of approximation of functions. Trudy Steklov Math. Inst. 60, 82–95.

Gitenjova, Yu.A., Milstein, G.N., 1967. Minimization of convex functional by free trajectories of linear system. Differential Equations 3 (12), 2081–2093.

Ibragimov, I.A., Khasminskii, R.Z., 1982. Statistical Estimation: Asymptotic Theory. Springer, New York.

Milstein, G.N., Nussbaum, M., 2000. Maximum likelihood estimate for nonparametric signal in white noise by optimal control. WIAS-Preprint No. 596, Berlin.

Nemirovskii, A.S., Polyak, B.T., Tsybakov, A.B., 1984. Signal processing via the nonparametric maximum likelihood method. Problemy Peredachi Informatsii 20, 29–46.

Nemirovskii, A.S., Polyak, B.T., Tsybakov, A.B., 1985. Rate of convergence of nonparametric estimates of maximum likelihood type. Problemy Peredachi Informatsii 21, 258–272.

Tsirelson, B.S., 1982, 1985, 1986. A geometrical approach to maximum likelihood estimation for infinite-dimensional Gaussian location. Theory Probab. Appl. I: 27, 411–418; II: 30, 820–828; III: 31, 470–483.