Effective dimension for weighted function spaces

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Abstract

This paper introduces some notions of effective dimension for weighted function spaces. A space has low effective dimension if the smallest ball in it that contains a function of variance 1, has no functions with large values of certain ANOVA mean squares. For a Sobolev space of periodic functions defined by product weights we get explicit formulas describing effective dimension in terms of those weights. In particular, for a space with product weights it is possible to compute truncation and superposition dimensions directly from the weight sequence. For weights $\gamma_j = \gamma_1 j^{-q}$ with $q > 1$, and $\gamma_j \leqslant 1$, the result is a low superposition dimension though high truncation dimensions are possible.

1 Introduction

Quadrature of high dimensional functions is a fundamental numerical task with implications for every branch of science and engineering. Of particular prominence recently are quasi-Monte Carlo rules which approximate $\mu = \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ by an equal weight rule $(1/n)\sum_{i=1}^n f(\boldsymbol{x}_i)$ for carefully chosen points $\boldsymbol{x}_i \in [0,1]^d$. See Niederreiter (1992) for an introduction and Dick and Pillichshammer (2010) for recent developments.

It has been known since Bakhvalov (1959) that there is a curse of dimensionality for quadrature. For any quadrature rule of the form $\sum_i w_i f(\boldsymbol{x}_i)$ with $w_i \in \mathbb{R}$ and $\boldsymbol{x}_i \in [0,1]^d$, there is a function with all r'th order partial derivatives below 1 in absolute value that has a quadrature error above $kn^{-r/d}$ where $k = k(r,d) > 0$. For randomized rules, such as Monte Carlo methods, the root mean squared error has a lower bound proportional to $n^{-1/2-r/d}$.

Despite the curse of dimension, good results are often observed for high dimensional quadrature problems. For instance Paskov and Traub (1995) report successful results from quasi-Monte Carlo sampling on some 360 dimensional problems.

There have been several explanations proposed for such results. One is the notion of effective dimension (Caflisch et al., 1997), which is based on the ANOVA decomposition of $L^2[0,1]^d$. A function is of low effective dimension if it is dominated by its low dimensional ANOVA components as described in Section 2. A function of effective dimension $s < d$ might only depend strongly on the first s components of $\boldsymbol{x}$ (truncation sense) or alternatively it may be nearly a sum of interaction functions that each depend on s or fewer of the components in $\boldsymbol{x}$ (superposition sense). Integrals and equal weight quadrature rules are both linear in f, and when a rule is accurate on all of the important low dimensional parts of f, it is not surprising to find that it is accurate on f itself.

A second approach is to model the integrands as members of certain weighted Sobolev spaces. Using weighted function spaces, in which higher order interactions (Hickernell, 1996), or successive dimensions (Sloan and Woźniakowski, 1998) are relatively less important, it is possible to study the tractability of high dimensional integration. If the importance of higher order terms decays quickly enough then there is no curse of dimensionality and quadrature is said to be tractable for those spaces.

This paper looks at a model for such Sobolev spaces due to Sloan and Woźniakowski (1998) and introduces notions of effective dimension for a space. Let $\mathcal{B}$ be the unit ball in one such space. That space is said to be of low effective dimension (truncation sense) if $f \in \mathcal{B}$ precludes f from having any significant dependence on components after the $s+1$'st. Similarly, if membership in $\mathcal{B}$ rules out significantly large interactions among $s+1$ or more inputs to the function, then the ball is effectively of dimension s in the superposition sense.

The superposition and truncation dimensions can be computed explicitly from the weights defining the space. The main finding is that in those cases where quadrature is tractable, the spaces themselves are of low superposition dimension though possibly high truncation dimension. In some cases of interest the superposition dimension is 1.

The mathematical approach used is to bound the ANOVA variance components by multiples of the mean squared partial derivatives that define the Sobolev norms. Our starting point is an observation by Sobol' (1963). Let f be a function on $[0,1]$ with first derivative f' in $L^2[0,1]$ and whose integral $\int_0^1 f(x)\,\mathrm{d}x$ is 0. Then the variance $\sigma^2 = \int_0^1 f(x)^2\,\mathrm{d}x$ of $f(X)$ for $X \sim \mathbf{U}[0,1]$ satisfies

$$\sigma^2 \leqslant \frac{1}{\pi^2}\int_0^1 |f'(x)|^2\,\mathrm{d}x. \tag{1}$$

The bound (1) is attained by scalar multiples of $f(x) = \cos(\pi x)$.

Equation (1) allows us to bound the variance of a function by a multiple of the seminorm $\int_0^1 |f'(x)|^2\,\mathrm{d}x$. It is not possible to get an upper bound on $\int_0^1 |f'(x)|^2\,\mathrm{d}x$ as a multiple of σ^2 in this setting. We will generalize (1) to bound variance components of functions in terms of the weights defining their function classes. Sobol' points to calculus of variations for the result (1). We will find it easier to generalize his observation via Fourier series. By working with periodic functions f we are able to interchange differentiation with the taking of Fourier series.

An outline of this paper is as follows. Section 2 introduces our notation, describing a certain weighted Hilbert space with product weights, and the ANOVA decomposition of $L^2[0,1]^d$. Section 3 presents bounds on ratios of variances to function norms, derived via a Fourier series representation. Section 4 defines effective dimension for function spaces in terms of the bounds from Section 3 and computes effective dimensions for some specific function spaces. Those spaces have product weights $\gamma_j = \gamma_1 j^{-\alpha}$. If $\alpha > 1$, as is required for tractability, and $\gamma_1 \leqslant 1$, as is required to make higher order effects less important than their strict sub-effects, then the space necessarily has a low effective dimension. Section 5 has conclusions.

We finish up this introduction by surveying related literature.

Equation (1) and the results here are Poincaré inequalities. Sobol' and Kucherenko (2009, 2010) generalize equation (1) to some multidimensional problems motivated by global sensitivity analysis. They bound certain sums of ANOVA variances by some practically estimable quantities involving integrals of squared first order partial derivatives of a multidimensional function. The bounds presented here are different, being derived through higher order mixed partial derivatives. Lamboni et al. (2012) independently obtain similar inequalities for Sobol' indices. That problem requires only first order partial derivatives, and their techniques require Boltzmann distributions for the inputs to f.

A notion of effective dimension for Sobolev spaces was previously considered by Wang and Fang (2003). They define what they call a typical function in the space, which takes the form of a product of one dimensional functions involving the second Bernoulli polynomial. They express the effective dimension of their typical function in terms of the weights which define the space and they compute some examples.

The approach taken here is uniform over a ball in the Sobolev space and does not rely on typical functions. The ball in this Sobolev space includes functions that are not products and may not resemble typical functions especially when their norm is small. The latter functions need not have low effective dimension in the sense of Caflisch et al. (1997), yet their good quadrature outcomes are covered by results in Sloan and Woźniakowski (2002). These functions are covered by the present notion in which the space as a whole has low effective dimension.

This article makes use of some recent results in quasi-Monte Carlo integration, particularly the approach via lattice rules, and some related notions of complexity and tractability. An excellent survey appears in Kuo et al. (2012). A comprehensive treatment is available in Novak and Woźniakowski (2010).

2 Notation

We consider real-valued periodic functions on the domain $[0,1]^d$ for $1 \leqslant d < \infty$. A typical point in the domain is $\boldsymbol{x} = (x_1, x_2, \dots, x_d)$. The components of $\boldsymbol{x}$ have indices in the set $1{:}d \equiv \{1, 2, \dots, d\}$. Let $u \subseteq 1{:}d$. If $u = \{j_1, j_2, \dots, j_r\}$ then $\boldsymbol{x}_u$ is the point $(x_{j_1}, \dots, x_{j_r})$. We use $-u$ for the complement $1{:}d - u$.

The cardinality of u is $|u|$. For $u \neq \varnothing$, the highest indexed element in u is $\lceil u \rceil \equiv \max\{j \mid j \in u\}$. The point $\boldsymbol{z} = \boldsymbol{x}_u{:}\boldsymbol{y}_{-u}$ is the one with $z_j = x_j$ for $j \in u$ and $z_j = y_j$ for $j \notin u$.

Given weights $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_d > 0$, we define an inner product between functions f and g via

$$\langle f, g \rangle_{W_{d,\gamma}} = \sum_{u \subseteq 1:d} \gamma_u^{-1} \int \frac{\partial^{|u|} f(\boldsymbol{x})}{\partial \boldsymbol{x}_u} \frac{\partial^{|u|} \bar{g}(\boldsymbol{x})}{\partial \boldsymbol{x}_u} \,\mathrm{d}\boldsymbol{x}, \tag{2}$$

where

$$\gamma_u = \prod_{j \in u} \gamma_j. \tag{3}$$

Integrals with no explicit domain are over $[0,1]^d$. We use $\|f\|_{W_{d,\gamma}}$ for $\sqrt{\langle f, \bar{f} \rangle_{W_{d,\gamma}}}$. Though the functions we study are real-valued, some of the expressions for them contain complex numbers and so we use complex conjugates in the definition of the inner product.

The inner product (2) with the product weights (3) defines the first Hilbert space described in Sloan and Woźniakowski (2002) (called there $W_{d,\gamma,1}$). It is often useful to replace the product weights (3) by more general quantities.

The function classes we consider take the form

$$W_{d,\gamma} = \{f : [0,1]^d \to \mathbb{R} \mid \|f\|_{W_{d,\gamma}} < \infty\}. \tag{4}$$

We study subsets of the form

$$\mathcal{B}(d, \gamma, \rho) = \{f \in W_{d,\gamma} \mid \|f\|^2_{W_{d,\gamma}} \leqslant \rho^2\}$$

where $\rho > 0$ is the radius of the ball $\mathcal{B}(d, \gamma, \rho)$.

Let $\mathcal{P}_d$ be the set of real-valued functions on $[0,1]^d$ whose partial derivatives taken at most once with respect to each of the d components of $\boldsymbol{x}$ have a continuous 1-periodic extension from $[0,1]^d$ to $\mathbb{R}^d$. We work with the subspace $W_{d,\gamma} \cap \mathcal{P}_d$ and the ball $\mathcal{B}(d, \gamma, \rho) \cap \mathcal{P}_d$.

We will also make use of the ANOVA decomposition of $L^2[0,1]^d$. For a derivation and some history see Liu and Owen (2006). If $f \in L^2[0,1]^d$ we may write

$$f(\boldsymbol{x}) = \sum_{u \subseteq 1:d} f_u(\boldsymbol{x}) \tag{5}$$

where the function $f_u(\boldsymbol{x})$ only depends on $\boldsymbol{x}$ through $\boldsymbol{x}_u$, that is, $f_u(\boldsymbol{x}) = f_u(\boldsymbol{x}_u{:}\boldsymbol{y}_{-u})$ for all $\boldsymbol{y} \in [0,1]^d$. The decomposition satisfies $\int_0^1 f_u(\boldsymbol{x}) \,\mathrm{d}x_j = 0$ whenever $j \in u$. It follows that the orthogonality $\int f_u(\boldsymbol{x}) g_v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0$ holds for $f, g \in L^2[0,1]^d$ when $u \neq v$. The variance of f satisfies

$$\sigma^2(f) = \sum_{u \subseteq 1:d} \sigma^2(f_u) = \sum_{u \neq \varnothing} \int f_u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x}. \tag{6}$$

We use σ_u^2 as a shorthand for $\sigma^2(f_u)$. For the empty set, $f_\varnothing(\boldsymbol{x}) = \int f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ (a constant function) and $\sigma_\varnothing^2 = 0$.

3 Bounds on ratios of norms

Given d and γ we would like to know how important some subset of variables might be. We formulate this concept by finding the smallest radius $\rho \geqslant 0$ such that the $f \in \mathcal{B}(d, \gamma, \rho)$ contains a function of variance 1. Then we maximize $\sigma_u^2(f)$ over functions $f \in \mathcal{B}(d, \gamma, \rho)$. If that quantity takes a small value, such as 0.01 or 0.0001, then we infer that the joint effect of components x_j for $j \in u$ is not important in the space $W_{d,\gamma}$.

Variance is one of many measures of a function's magnitude. It is relevant here because we are interested in the improvement that quasi-Monte Carlo sampling offers over plain Monte Carlo sampling. When a portion of the variance is trivially small, it does not make an important contribution to the Monte Carlo error, and there is little scope for more sophisticated quadrature methods, such as quasi-Monte Carlo, to make an improvement. By contrast, larger variance components point to places where quasi-Monte Carlo might make a material improvement over Monte Carlo.

Instead of measuring the contribution of σ_u^2, we might look at the superset importance $\Upsilon_u^2 = \sum_{v \supseteq u} \sigma_v^2$ (Liu and Owen, 2006) or more generally a measure like $\sum_{v \in \mathcal{V}} \sigma_v^2$ where $\mathcal{V}$ is a non-empty subset of non-empty subsets of $1{:}d$. For instance $\mathcal{V}_{\mathrm{S}} = \{v \mid |v| > s\}$ and $\mathcal{V}_{\mathrm{T}} = \{v \mid \lceil v \rceil > s\}$ are appropriate collections to study superposition and truncation dimension respectively. In the weighted spaces we study, the maximum of $\sum_{v \in \mathcal{V}} \sigma_v^2$ over the ball $\mathcal{B}(d, \gamma, \rho)$ reduces to the maximum of σ_u^2 over that ball, for some least penalized subset $u \in \mathcal{V}$. For superposition this set is $u = 1{:}(s+1)$, while for truncation it is $u = \{s+1\}$. More details are given near Definitions 1 and 2 below.

The elements of $W_{d,\gamma}$ have Fourier representation

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \lambda_{\boldsymbol{k}} \exp(2\pi i \boldsymbol{k}^{\mathsf{T}} \boldsymbol{x}) \tag{7}$$

where

$$\lambda_{\boldsymbol{k}} = \int f(\boldsymbol{x}) \exp(-2\pi i \boldsymbol{k}^{\mathsf{T}} \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{8}$$

The partial derivative of $f \in W_{d,\gamma} \cap \mathcal{P}_d$ taken once with respect to each component variable is in $L^2[0,1]^d$, and hence has a Fourier representation that converges almost everywhere.

For f given by (7)

$$\mathrm{Var}(f) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d - \{\boldsymbol{0}\}} |\lambda_{\boldsymbol{k}}|^2.$$

Next we find the squared norm of f.

Proposition 1. *Let $f \in W_{d,\gamma} \cap \mathcal{P}_d$ have representation* (7)*, where the space $W_{d,\gamma}$ is defined through the product weights* (3)*. Then*

$$\|f\|_{W_{d,\gamma}}^2 = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\lambda_{\boldsymbol{k}}|^2 \prod_{j=1}^d \Bigl(1 + \frac{4\pi^2 k_j^2}{\gamma_j}\Bigr).$$

Proof. First for $u \subseteq 1{:}d$,

$$\frac{\partial^{|u|} f(\boldsymbol{x})}{\partial \boldsymbol{x}_u} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \nu_{\boldsymbol{k}} \exp(2\pi i \boldsymbol{k}^\mathsf{T}\boldsymbol{x}),$$

where $\nu_{\boldsymbol{k}} = \lambda_{\boldsymbol{k}}(2\pi i)^{|u|}\prod_{j\in u} k_j$. Then

$$\begin{aligned}
\|f\|^2_{W_{d,\gamma}} &= \sum_u \prod_{j\in u}\gamma_j^{-1}\int \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\sum_{\boldsymbol{k}'\in\mathbb{Z}^d} \nu_{\boldsymbol{k}}\exp(2\pi i\boldsymbol{k}^\mathsf{T}\boldsymbol{x})\overline{\nu_{\boldsymbol{k}'}}\exp(-2\pi i\boldsymbol{k}'^\mathsf{T}\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&= \sum_u \prod_{j\in u}\gamma_j^{-1}\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\nu_{\boldsymbol{k}}|^2\\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\lambda_{\boldsymbol{k}}|^2\sum_u(4\pi^2)^{|u|}\prod_{j\in u}\gamma_j^{-1}\prod_{j\in u}k_j^2\\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\lambda_{\boldsymbol{k}}|^2\prod_{j=1}^d(1+4\pi^2k_j^2/\gamma_j). \qquad \square
\end{aligned}$$

Corollary 1. *For dimension $d \geqslant 1$, weights $\gamma_1, \dots, \gamma_d$, and space $W_{d,\gamma}$ defined through product weights* (3)*, the smallest radius $\rho > 0$ for which $\mathcal{B}(d,\gamma,\rho)$ contains a function f with* $\mathrm{Var}(f) = 1$ *has $\rho^2 = 1 + 4\pi^2/\gamma_1$.*

Proof. We choose $\lambda_{\boldsymbol{k}}$ to minimize $\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\lambda_{\boldsymbol{k}}|^2\prod_{j=1}^d(1+4\pi^2k_j^2/\gamma_j)$ subject to $\sum_{\boldsymbol{k}\in\mathbb{Z}^d-\{\boldsymbol{0}\}}|\lambda_{\boldsymbol{k}}|^2 = 1$. We accomplish this by taking $\lambda_{\boldsymbol{k}} = 0$ unless $\boldsymbol{k} = (\pm 1, 0, \dots, 0)$ and then choosing $|\lambda_{(-1,0,\dots,0)}|^2 + |\lambda_{(1,0,\dots,0)}|^2 = 1$. Any such function has $\|f\|^2_{W_{d,\gamma}} = 1 + 4\pi^2/\gamma_1$. $\square$

Proposition 2. *Let u be a non-empty subset of $1{:}d$. For dimension $d \geqslant 1$, weights $\gamma_1, \dots, \gamma_d$, and space $W_{d,\gamma}$ defined through product weights* (3)*,*

$$\sup_{f\in\mathcal{B}(d,\gamma,\rho)\cap\mathcal{P}_d}\sigma_u^2(f) = \rho^2\prod_{j\in u}(1+4\pi^2/\gamma_j)^{-1}.$$

Proof. The ANOVA component f_u is

$$\sum_{\boldsymbol{k}_u\in(\mathbb{Z}-\{0\})^{|u|}}\lambda_{\boldsymbol{k}_u:\boldsymbol{0}_{-u}}\exp(2\pi i\boldsymbol{k}_u^\mathsf{T}\boldsymbol{x}_u)$$

which has variance

$$\sigma_u^2 = \sum_{\boldsymbol{k}_u\in(\mathbb{Z}-\{0\})^{|u|}}|\lambda_{\boldsymbol{k}_u:\boldsymbol{0}_{-u}}|^2.$$

We maximize σ_u^2 subject to $f \in \mathcal{B}(d,\gamma,\rho)\cap\mathcal{P}_d$ by taking $\lambda_{\boldsymbol{k}_u:\boldsymbol{0}_{-u}} = 0$ unless all elements of $\boldsymbol{k}_u$ are ± 1. For such a function

$$\|f\|^2_{W_{d,\gamma}} = \Bigl(\prod_{j\in u}(1+4\pi^2/\gamma_j)\Bigr)\sum_{\boldsymbol{k}_u\in\{-1,1\}^{|u|}}|\lambda_{\boldsymbol{k}_u:\boldsymbol{0}_{-u}}|^2, \quad \text{and}$$

$$\sigma_u^2 = \sum_{\boldsymbol{k}_u \in \{-1,1\}^{|u|}} |\lambda_{\boldsymbol{k}_u:\boldsymbol{0}_{-u}}|^2.$$

That is $\sigma_u^2 = \|f\|_{W_{d,\gamma}}^2 \prod_{j\in u}(1+4\pi^2/\gamma_j)^{-1}$. □

Corollary 2. *For dimension $d \geqslant 1$, weights $\gamma_1, \dots, \gamma_d$, and space $W_{d,\gamma}$ defined through product weights* (3)*, let ρ be the smallest radius for which $W_{d,\gamma}$ contains a function f with* $\mathrm{Var}(f) = 1$*. Let u be a nonempty subset of* $1{:}d$*. Then*

$$\sup_{f\in\mathcal{B}(d,\gamma,\rho)\cap\mathcal{P}_d} \sigma_u^2 = \frac{1+4\pi^2/\gamma_1}{\prod_{j\in u}(1+4\pi^2/\gamma_j)}.$$

Proof. Combine Proposition 2 with ρ^2 from Corollary 1. □

4 Effective dimension

In this section we define notions of effective dimension for weighted spaces. To motivate our results we considered balls just large enough to contain a function of unit variance. But because both the variance and the norm squared are homogeneous of degree 2 we can simply use the unit ball in $W_{d,\gamma}$ in our definitions.

Definition 1. Let W be a product weighted subspace (4) of $L^2[0,1]^d$ with norm $\|\cdot\|_W$ and let $\mathcal{B}$ be the unit ball in W. The space W is said to be of effective dimension $s \in \{1, 2, \dots, d-1\}$ in the superposition sense (at level $\varepsilon \in (0,1)$) if

$$\frac{\sup_{f\in\mathcal{B}} \sum_{|u|>s} \sigma_u^2(f)}{\sup_{f\in\mathcal{B}} \sigma^2(f)} \leqslant \varepsilon.$$

For product weights, this definition yields the same answer if we replace $\sum_{|u|>s}$ by $\sup_{|u|>s}$, because

$$\sup_{f\in\mathcal{B}} \sum_{|u|>s} \sigma_u^2 = \sup_{f\in\mathcal{B}} \sum_{\|\boldsymbol{k}\|_0>s} |\lambda_{\boldsymbol{k}}|^2$$

and that supremum is attained (non-uniquely) by a function f with only one nonzero $\lambda_{\boldsymbol{k}}$ where $\boldsymbol{k}$ has exactly $s+1$ nonzero elements all of which are ± 1. The same reduction happens using $\sup_{|u|>s}$ in the numerator. Using $\sup_u$ makes for simpler computations.

Definition 2. Let W be a product weighted subspace (4) of $L^2[0,1]^d$ with norm $\|\cdot\|_W$ and let $\mathcal{B}$ be the unit ball in W. The space W is said to be of effective dimension $s \in \{1, 2, \dots, d-1\}$ in the truncation sense (at level $\varepsilon \in (0,1)$) if

$$\frac{\sup_{f\in\mathcal{B}} \sum_{\lceil u\rceil>s} \sigma_u^2(f)}{\sup_{f\in\mathcal{B}} \sigma^2(f)} \leqslant \varepsilon.$$

Similarly to the superposition case, we can replace $\sup_{f\in\mathcal{B}}\sum_{\lceil u\rceil>s}$ in the numerator by $\sup_{f\in\mathcal{B}}\sup_{\lceil u\rceil>s}$ which makes computations easier.

For periodic functions and product weights, the weighted space has effective dimension s in the superposition sense if

$$\frac{1+4\pi^2/\gamma_1}{\prod_{j=1}^{s+1}(1+4\pi^2/\gamma_j)} = \frac{1}{\prod_{j=2}^{s+1}(1+4\pi^2/\gamma_j)} \leqslant \varepsilon. \tag{9}$$

The periodic weighted space defined through product weights has effective dimension s in the truncation sense if

$$\frac{1+4\pi^2/\gamma_1}{1+4\pi^2/\gamma_{s+1}} \leqslant \varepsilon.$$

Effective dimension s in the truncation space implies effective dimension s in the superposition sense, at the same level ε.

These notions of effective dimension are different from the ones in Caflisch et al. (1997). The definitions there apply to a single function which then has low effective dimension if it is dominated by its own low order or low index ANOVA terms, $\sum_{|u|>s}\sigma_u^2 \leqslant \epsilon\sum_u \sigma_u^2$, or, $\sum_{\lceil u\rceil>s}\sigma_u^2 \leqslant \epsilon\sum_u \sigma_u^2$, respectively. Here a space has low effective dimension if in any ball, the variance of one function dominates the high index or high order parts of *every function* in that ball.

A quadrature problem is tractable if the effort to attain accuracy ε (one way to measure accuracy is described below) is at most polynomial in ε^{-1} and the dimension d. It is strongly tractable if the complexity bound is independent of d. The problem of quadrature is known to be strongly tractable in the weighted function space of this paper, if

$$\sum_{j=1}^{\infty}\gamma_j < \infty \tag{10}$$

holds for a sequence of quadrature problems with increasing dimension d. In this case, the sample size n required to reduce the initial error (that of an $n=0$ rule) by a factor $\epsilon>0$ grows as ϵ^{-2} uniformly in dimension. This is essentially the Monte Carlo rate, and the original proof (Sloan and Woźniakowski, 1998) was non-constructive. An improved but non-constructive rate with error $O(n^{-1+\delta})$ for arbitrary $\delta>0$ was obtained by Hickernell and Woźniakowski (2000) under the condition that

$$\sum_{j=1}^{\infty}\gamma_j^{1/2} < \infty. \tag{11}$$

There are now constructive coordinate-by-coordinate algorithms (Sloan et al., 2002) that find lattice rules attaining the better rate, as shown by both Kuo (2003) and Dick (2004) when (11) holds. Fast search algorithms based on the FFT are available (Nuyens and Cools, 2006a,b).

ε	Truncation dimensions			Superposition dimensions		
	$q = 1.01$	$q = 2$	$q = 3$	$q = 1.01$	$q = 2$	$q = 3$
0.01	97	10	4	2	1	1
0.0001	9,357	101	21	3	2	2

Table 1: Superposition and truncation dimensions of some periodic weighted spaces. The weights are $\gamma_j = j^{-q}$ and the threshold is $\varepsilon \in \{0.01, 0.0001\}$.

ε	Truncation dimensions		Superposition dimensions	
	$q = 1$	$q = 2$	$q = 1$	$q = 2$
0.01	161	5	5	2

Table 2: Superposition and truncation dimensions from Wang and Fang (2003) for dimension $d = 500$ and threshold $\varepsilon = 0.01$, with $\gamma_j = j^{-q}$.

We can illustrate effective dimension with some examples for γ_j. Suppose for example that $\gamma_j = Aj^{-q}$ where $q > 1$. Then the truncation dimension is the first s for which

$$s \geqslant \left(\frac{4\pi^2 + A(1-\varepsilon)}{4\pi^2\varepsilon}\right)^{1/q} - 1 \geqslant \varepsilon^{-1/q} - 1.$$

For instance, with $\varepsilon = 0.01$ the space must have truncation dimension of about 100 or more if q is close to its lower bound of 1. When $q = 2$ (almost large enough for the better rate), then the truncation dimension must be larger than 9 but will only be much larger than that if $A \gg 1$. This example reinforces the insight that it is not just the rate of weight decay that matters, but also the constant of proportionality. If weights are too small, then high index components may be ignored in component-by-component constructions of lattice rules. Anecdotally those algorithms sometimes choose lattices that duplicate components in their generating vector, producing quadrature points $\boldsymbol{x}_i \in [0,1]^d$ with $x_{ij} = x_{ik}$ for some $j \neq k$ and all i.

The effective dimension in the superposition sense is the first $s \geqslant 1$ for which (9) holds. It is easy to compute given a sequence of weights. Table 1 reports the effective dimension at levels $\varepsilon = 0.01$ and $\varepsilon = 0.0001$ for some spaces with product weights. We choose $A = 1$ here in order to keep all $\gamma_j \leqslant 1$ so that $\gamma_u \leqslant \gamma_v$ whenever $u \supseteq v$.

The definition based on typical functions in Wang and Fang (2003) leads to qualitatively similar but not identical answers. The present approach gives one number for the infinite sequence of weights. Their approach gives an answer which depends on the nominal dimension d. At their highest investigated dimension, $d = 500$, they obtain effective dimensions in Table 2.

5 Discussion

This paper develops an approach to measure effective dimension for a space of integrands. A space of periodic functions with product weights was used for illustration, but the analysis should be applicable to other spaces. In the particular weighted spaces investigated, integration was tractable. Some of those spaces had high truncation dimension, but all had low superposition dimension. It is possible to construct weighted spaces in which the superposition dimension is higher, and the $O(n^{-1+\delta})$ rate still holds. For example with $\gamma_j = 100 \times j^{-2.01}$, the superposition dimension rises to $s = 4$ at level $\varepsilon = 0.01$, but the first 9 γ_j are larger than one, and $\gamma_{\{1,2,\dots,9\}}$ becomes the most important interaction. It remains to see how such large weights affect the constants in the complexity bounds for integration.

Several alternatives to product weights have been proposed. Among these are finite order weights (Sloan et al., 2004) where the coefficient γ_u in the inner product (2) vanishes for $|u| > s$. Such a space will necessarily have superposition dimension at most s. More recently, the product and order dependent (POD) weights were studied by Kuo et al. (2012). The POD weights take the form $\gamma_u = \alpha(|u|)\prod_{j\in u}\gamma_j$ for some function α acting on the cardinality (order) of u alone. Having the extra flexibility to choose $\alpha(|u|)$ may allow researchers to find tractable spaces with higher effective dimensions.

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