

DIFFERENTIAL HARNACK ESTIMATES FOR BACKWARD HEAT EQUATIONS WITH POTENTIALS UNDER THE RICCI FLOW

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ABSTRACT. In this paper, we derive a general evolution formula for possible Harnack quantities. As a consequence, we prove several differential Harnack inequalities for positive solutions of backward heat-type equations with potentials (including the conjugate heat equation) under the Ricci flow. We shall also derive Perelman's Harnack inequality for the fundamental solution of the conjugate heat equation under the Ricci flow.

1. Introduction

In [LY86], P. Li and S.-T. Yau proved a differential Harnack inequality by developing a grading estimate for positive solutions of the heat equation (with fixed metric). More precisely, they proved that, for any positive solution f of the heat equation

$$\frac{\partial f}{\partial t} = \Delta f$$

on Riemannian manifolds with nonnegative Ricci curvature, then

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \frac{n}{2t} = \Delta \ln f + \frac{n}{2t} \geq 0.$$

The idea was brought to study general geometric evolution equations by R. Hamilton. The differential Harnack estimate has since become an important technique in the studies of geometric evolution equations.

For the Ricci flow, R. Hamilton [Ham93a] proved a Harnack estimate on Riemannian manifolds with weakly positive curvature operator. The Harnack quantities are also known as curvatures of a degenerate metric in space-time thanks to the work of B. Chow and S.-C. Chu [CC95]. In dimension two, R. Hamilton [Ham88] proved a Harnack estimate for the scalar curvature when it is positive, the Harnack estimate in the general case was proved by B. Chow in [Cho91b]. B. Chow and R. Hamilton generalized their results for the heat equation and for the Ricci flow on surfaces in [CH97].

For other geometric flows, R. Hamilton proved a Harnack estimate for the mean curvature flow in [Ham95]. B. Chow proved Harnack estimates for Gaussian curvature flow in [Cho91a] and for Yamabe flow in [Cho92]. H.-D. Cao [Cao92] proved a Harnack estimate and L. Ni [Ni07] proved a matrix Harnack estimate (of the forward conjugate heat equation) for the Kähler-Ricci flow.

2000 *Mathematics Subject Classification.* Primary 53C44.

*Research partially supported by the Jeffrey Sean Lehman Fund from Cornell University.

For the heat equation, besides the classical result of P. Li and S.-T. Yau, R. Hamilton proved a matrix Harnack estimate for the heat equation in [Ham93b]. C. Guenther [Gue02] studied the fundamental solution and Harnack inequality of time-dependent heat equation. In [CN05], H.-D. Cao and L. Ni proved a matrix Harnack estimate for the heat equation on Kähler manifolds. In an earlier paper [CH07], R. Hamilton and the author proved several Harnack estimates for positive solutions of the heat-type equation with potential when the metric is evolving by the Ricci flow (a more detailed discussion about the literature of Harnack estimates can also be found in that paper). In [Per02], G. Perelman proved a Harnack estimate for the fundamental solution of the conjugate heat equation under the Ricci flow. Namely, let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold, f be the positive *fundamental* solution to the conjugate heat equation

$$\frac{\partial}{\partial t} f = -\Delta f + Rf,$$

$\tau = T - t$, and $u = -\ln f - \frac{n}{2} \ln(4\pi\tau)$. Then for $t \in [0, T)$, G. Perelman proved that

$$2\Delta u - |\nabla u|^2 + R + \frac{u}{\tau} - \frac{n}{\tau} \leq 0$$

(see [Ni06] or [CCG⁺07, Chapter 16] for a detailed proof).

In the present paper, we will first derive a general evolution equation for possible Harnack quantities, we will then prove Harnack estimates for all positive solutions of the backward heat-type equation with potentials when the metric is evolving under the Ricci flow.

Suppose $(M, g(t))$, $t \in [0, T]$, is a solution to the Ricci flow on a closed manifold. Let f be a positive solution of the backward heat equation with potential $2R$, i.e.,

$$(1.1) \quad \frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

$$(1.2) \quad \frac{\partial f}{\partial t} = -\Delta_{g(t)} f + 2Rf.$$

Our first main theorem is the following,

Theorem 1.1. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold. Let f be a positive solution to the backward heat-type equation (1.2), $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + 2R - \frac{2n}{\tau}.$$

Then for all time $t \in [0, T)$,

$$H \leq 0.$$

If we further assume that our solution to the Ricci flow is of Type I, i.e.,

$$|Rm| \leq \frac{d_0}{T-t}$$

for some constant d_0 , here T is the blow-up time, then we shall prove the following theorem.

Theorem 1.2. *Let $(M, g(t))$, $t \in [0, T)$, be a type I solution to the Ricci flow on a closed manifold. Let f be a positive solution to the backward heat equation (1.2), $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + 2R - d\frac{n}{\tau},$$

here $d = d(d_0, n) \geq 2$ is some constant such that $H(\tau) < 0$ for small τ . Then for all time $t \in [0, T)$,

$$H \leq 0.$$

We will consider the conjugate heat equation under the Ricci flow. In this case, we also assume that our initial metric $g(0)$ has nonnegative scalar curvature, it is well-known that this property is preserved by the Ricci flow. We shall prove

Theorem 1.3. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow (1.1) on a closed manifold, and suppose that $g(t)$ has nonnegative scalar curvature. Let f be a positive solution to the conjugate heat equation*

$$(1.3) \quad \frac{\partial f}{\partial t} = -\Delta_{g(t)} f + Rf,$$

$u = -\ln f$, $\tau = T - t$ and

$$H = 2\Delta u - |\nabla u|^2 + R - 2\frac{n}{\tau}.$$

Then for all time $t \in [0, T)$,

$$H \leq 0.$$

Remark 1.1. *S. Kuang and Q. Zhang [KZ07] have also proved a similar estimate as in Theorem 1.3 (see Remark 3.1). Our proof follows from a direct calculation of more general evolution equation in Lemma 2.1.*

The rest of this paper is organized as follows. In section 2, we will first derive a general evolution equation of Harnack quantity H , for backward heat-type equation with potentials, then we prove Theorem 1.1 and Theorem 1.2. We will also prove an integral version of the Harnack inequality (Theorem 2.3). In section 3, we will prove Theorem 1.3, then we will derive a general evolution formula for a Harnack quantity similar to Perelman's, as a consequence, we will prove Perelman's Harnack estimate. In section 4, we will define two entropy functionals and prove that they are monotone. In section 5, we will prove a gradient estimate for the backward heat equation (without the potential term). This is also a consequence of the general evolution formula of our Harnack quantity.

Acknowledgement: The author would like to thank Laurent Saloff-Coste for general discussion on Harnack inequalities. He would also like to thank Leonard Gross and Richard Hamilton for helpful suggestions.

2. General Evolution Equation and Proof of Theorem 1.1

Let us first consider positive solutions of general evolution equations

$$\frac{\partial}{\partial t} f = -\Delta f - cRf$$

for all constant c which we will fix later. Let $f = e^{-u}$, then $\ln f = -u$. We have

$$\frac{\partial}{\partial t} \ln f = -\frac{\partial}{\partial t} u,$$

and

$$\nabla \ln f = -\nabla u, \quad \Delta \ln f = -\Delta u.$$

Hence u satisfies the following equation,

$$(2.1) \quad \frac{\partial}{\partial t} u = -\Delta u + |\nabla u|^2 + cR.$$

Let $\tau = T - t$, then we have

$$(2.2) \quad \frac{\partial f}{\partial \tau} = \Delta f + cRf,$$

and u satisfies

$$(2.3) \quad \frac{\partial u}{\partial \tau} = \Delta u - |\nabla u|^2 - cR.$$

We can now define a general Harnack quantity and derive its evolution equation.

Lemma 2.1. *Let $(M, g(t))$ be a solution to the Ricci flow, and u satisfies (2.3). Let*

$$H = \alpha \Delta u - \beta |\nabla u|^2 + aR + b \frac{u}{\tau} + d \frac{n}{\tau},$$

where α, β, a, b and d are constants that we will pick later. Then H satisfies the following evolution equation,

$$\begin{aligned} \frac{\partial H}{\partial \tau} = & \Delta H - 2\nabla H \cdot \nabla u - (2\alpha - 2\beta) |\nabla \nabla u|^2 + \frac{\alpha}{2\alpha - 2\beta} R_{ij} - \frac{\lambda}{2\tau} |g_{ij}|^2 - \frac{2\alpha - 2\beta}{\alpha} \frac{\lambda}{\tau} H \\ & - 2(\alpha - 2\beta) R_{ij} u_i u_j + 2(a + \beta c) \nabla R \cdot \nabla u + (2\alpha - 2\beta) \frac{n\lambda^2}{4\tau^2} \\ & + (b - \frac{2\alpha - 2\beta}{\alpha} \lambda \beta) \frac{|\nabla u|^2}{\tau} - (\alpha c + 2a) \Delta R + (\frac{\alpha^2}{2\alpha - 2\beta} - 2a) |Rc|^2 \\ & + (\frac{2\alpha - 2\beta}{\alpha} \lambda - 1) b \frac{u}{\tau^2} + (\frac{2\alpha - 2\beta}{\alpha} \lambda - 1) d \frac{n}{\tau^2} + (\frac{2\alpha - 2\beta}{\alpha} a \lambda - \alpha \lambda - bc) \frac{R}{\tau}, \end{aligned}$$

where λ is also a constant that we will pick later.

Proof. The proof follows from a direct computation. We first calculate the first two terms in H ,

$$\frac{\partial(\Delta u)}{\partial \tau} = \Delta(\Delta u) - \Delta(|\nabla u|^2) - c\Delta R - 2R_{ij}u_{ij},$$

and

$$\begin{aligned} \frac{\partial(|\nabla u|^2)}{\partial\tau} &= 2\nabla u \cdot \nabla \Delta u - 2Rc(\nabla u, \nabla u) - 2\nabla u \cdot \nabla(|\nabla u|^2) - 2c\nabla u \cdot \nabla R \\ &= \Delta(|\nabla u|^2) - 2|\nabla \nabla u|^2 - 2\nabla u \cdot \nabla(|\nabla u|^2) - 2c\nabla u \cdot \nabla R - 4R_{ij}u_i u_j, \end{aligned}$$

here we used

$$\Delta(|\nabla u|^2) = 2\nabla u \cdot \Delta \nabla u + 2|\nabla \nabla u|^2,$$

and

$$\Delta \nabla u = \nabla \Delta u + Rc(\nabla u, \cdot).$$

Using the evolution equation of R ,

$$\frac{\partial R}{\partial \tau} = -\Delta R - 2|Rc|^2,$$

and (2.3), we have

$$\begin{aligned} \frac{\partial}{\partial \tau} H &= \Delta H - \alpha \Delta(|\nabla u|^2) - \alpha c \Delta R - 2\alpha R_{ij}u_i u_j + 2\beta |\nabla \nabla u|^2 + 2\beta \nabla u \cdot \nabla(|\nabla u|^2) \\ &\quad + 2\beta c \nabla u \cdot \nabla R + 4\beta R_{ij}u_i u_j - 2a \Delta R - 2a|Rc|^2 - b \frac{|\nabla u|^2}{\tau} - b \frac{cR}{\tau} - b \frac{u}{\tau^2} - d \frac{n}{\tau^2} \\ &= \Delta H - 2\nabla H \cdot \nabla u - 2(\alpha - 2\beta)R_{ij}u_i u_j - (2\alpha - 2\beta)|\nabla \nabla u|^2 + 2(a + \beta c)\nabla R \cdot \nabla u \\ &\quad + b \frac{|\nabla u|^2}{\tau} - (\alpha c + 2a)\Delta R - 2\alpha R_{ij}u_i u_j - 2a|Rc|^2 - b \frac{u}{\tau^2} - d \frac{n}{\tau^2} - b \frac{cR}{\tau} \\ &= \Delta H - 2\nabla H \cdot \nabla u - 2(\alpha - 2\beta)R_{ij}u_i u_j - (2\alpha - 2\beta)|\nabla \nabla u|^2 + \frac{\alpha}{2\alpha - 2\beta}R_{ij} - \frac{\lambda}{2\tau}g_{ij}|^2 \\ &\quad + 2(a + \beta c)\nabla R \cdot \nabla u + (2\alpha - 2\beta)\frac{n\lambda^2}{4\tau^2} - (2\alpha - 2\beta)\frac{\lambda}{\tau}\Delta u - \frac{\alpha\lambda}{\tau}R \\ &\quad + b \frac{|\nabla u|^2}{\tau} - (\alpha c + 2a)\Delta R + \left(\frac{\alpha^2}{2\alpha - 2\beta} - 2a\right)|Rc|^2 - b \frac{u}{\tau^2} - d \frac{n}{\tau^2} - b \frac{cR}{\tau} \\ &= \Delta H - 2\nabla H \cdot \nabla u - 2(\alpha - 2\beta)R_{ij}u_i u_j - (2\alpha - 2\beta)|\nabla \nabla u|^2 + \frac{\alpha}{2\alpha - 2\beta}R_{ij} - \frac{\lambda}{2\tau}g_{ij}|^2 \\ &\quad - \frac{2\alpha - 2\beta}{\alpha} \frac{\lambda}{\tau} H + 2(a + \beta c)\nabla R \cdot \nabla u + (2\alpha - 2\beta)\frac{n\lambda^2}{4\tau^2} \\ &\quad + \left(b - \frac{2\alpha - 2\beta}{\alpha}\lambda\beta\right)\frac{|\nabla u|^2}{\tau} - (\alpha c + 2a)\Delta R + \left(\frac{\alpha^2}{2\alpha - 2\beta} - 2a\right)|Rc|^2 \\ &\quad + \left(\frac{2\alpha - 2\beta}{\alpha}\lambda - 1\right)b \frac{u}{\tau^2} + \left(\frac{2\alpha - 2\beta}{\alpha}\lambda - 1\right)d \frac{n}{\tau^2} + \left(\frac{2\alpha - 2\beta}{\alpha}a\lambda - \alpha\lambda - bc\right)\frac{R}{\tau}. \end{aligned}$$

□

In the above lemma, let us take $\alpha = 2$, $\beta = 1$, $a = 2$, $c = -2$, $\lambda = 2$, $b = 0$, $d = -2$, as a consequence of the above lemma, we have

Corollary 2.2. *Let $(M, g(t))$ be a solution to the Ricci flow, f be a positive solution of the following backward heat equation*

$$\frac{\partial}{\partial t} f = -\Delta f + 2Rf,$$

let $u = -\ln f$, $\tau = T - t$ and

$$H = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau}.$$

Then we have

$$(2.4) \quad \frac{\partial}{\partial \tau} H = \Delta H - 2\nabla H \cdot \nabla u - \frac{2}{\tau} H - \frac{2}{\tau} |\nabla u|^2 - 2|Rc|^2 - 2|\nabla_i \nabla_j u + R_{ij} - \frac{1}{\tau} g_{ij}|^2.$$

Now we can finish the proof of Theorems 1.1 and 1.2.

Proof. (Proof of Theorems 1.1 and 1.2) To prove Theorem 1.1, it is easy to see that for τ small enough then $H(\tau) < 0$. It follows from (2.4) and maximum principle that

$$H \leq 0$$

for all time τ .

From the above proof, we can easily see that if the solution to the Ricci flow is of type I and $d \geq 2$ is large enough, such that $H(\tau) < 0$ for τ small, then Theorem 1.2 is true for all time $t < T$. \square

We now can integrate the inequality

$$2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau} \leq 0$$

along a space-time path and get a classical Harnack inequality.

Theorem 2.3. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold. Let f be a positive solution to the equation*

$$\frac{\partial}{\partial t} f = -\Delta f + 2Rf.$$

Assume that (x_1, t_1) and (x_2, t_2) , $0 \leq t_1 < t_2 < T$, are two points in $M \times [0, T]$. Let

$$\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} \left(\frac{1}{2} |\dot{\gamma}|^2 + R \right) dt,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2) . Then we have

$$f(x_2, t_2) \leq f(x_1, t_1) \left(\frac{T - t_1}{T - t_2} \right)^n \exp^{\Gamma}.$$

Proof. Since $H \leq 0$, $\tau = T - t$ and $u = -\ln f$ satisfies

$$\frac{\partial u}{\partial \tau} = \Delta u - |\nabla u|^2 + 2R,$$

we have

$$2\frac{\partial u}{\partial \tau} + |\nabla u|^2 - 2R - \frac{2n}{\tau} \leq 0.$$

If we pick a space-time path $\gamma(x, t)$ joining (x_1, τ_1) and (x_2, τ_2) with $\tau_1 > \tau_2 > 0$. Along γ , we have

$$\begin{aligned} \frac{du}{d\tau} &= \frac{\partial u}{\partial \tau} + \nabla u \cdot \dot{\gamma} \\ &\leq -\frac{1}{2}|\nabla u|^2 + R + \frac{n}{\tau} + \nabla u \cdot \dot{\gamma} \\ &\leq \frac{1}{2}(|\dot{\gamma}|^2 + 2R) + \frac{n}{\tau}. \end{aligned}$$

Hence

$$u(x_1, \tau_1) - u(x_2, \tau_2) \leq \frac{1}{2} \inf_{\gamma} \int_{\tau_2}^{\tau_1} (|\dot{\gamma}|^2 + 2R) d\tau + n \ln\left(\frac{\tau_1}{\tau_2}\right).$$

Or we can write this as

$$u(x_1, t_1) - u(x_2, t_2) \leq \frac{1}{2} \inf_{\gamma} \int_{t_1}^{t_2} (|\dot{\gamma}|^2 + 2R) dt + n \ln\left(\frac{T - t_1}{T - t_2}\right).$$

If we denote $\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} (\frac{1}{2}|\dot{\gamma}|^2 + R) dt$, then we have

$$f(x_2, t_2) \leq f(x_1, t_1) \left(\frac{T - t_1}{T - t_2}\right)^n \exp^{\Gamma},$$

this finishes the proof. □

3. On the Conjugate Heat Equation

In this section, we consider positive solutions of general evolution equations

$$\frac{\partial}{\partial t} f = -\Delta f - cRf$$

on $[0, T]$, for the special case $c = -1$, which is the case of conjugate heat equation. In Lemma 2.1, let us take $\alpha = 2$, $\beta = 1$, $a = 1$, $c = -1$, $\lambda = 2$, $b = 0$, $d = -2$, and we arrive at

Corollary 3.1. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow, f be a positive solution of the conjugate heat equation*

$$(3.1) \quad \frac{\partial}{\partial t} f = -\Delta f + Rf,$$

let $u = -\ln f$, $\tau = T - t$ and

$$H = 2\Delta u - |\nabla u|^2 + R - 2\frac{n}{\tau}.$$

Then we have

$$(3.2) \quad \frac{\partial}{\partial \tau} H = \Delta H - 2\nabla H \cdot \nabla u - 2|u_{ij} + R_{ij} - \frac{1}{\tau}g_{ij}|^2 - \frac{2}{\tau}H - \frac{2}{\tau}|\nabla u|^2 - 2\frac{R}{\tau}.$$

Now we can finish the proof of Theorem 1.3.

Proof. (Proof of Theorem 1.3) It is easy to see that for τ small enough then $H(\tau) < 0$. It follows from (3.2) and maximum principle that

$$H \leq 0$$

for all time τ . □

As in section 2, we can see that if the solution to the Ricci flow is of type I, i.e.,

$$|Rm| \leq \frac{d_0}{T-t},$$

here T is the blow-up time, then the Harnack estimate is true for all time $t < T$.

Theorem 3.2. *Let $(M, g(t))$, $t \in [0, T)$, be a type I solution to the Ricci flow on a closed manifold with nonnegative scalar curvature. Let f be a positive solution to the conjugate heat equation (3.1), $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + R - d\frac{n}{\tau},$$

here $d = d(d_0, n) \geq 2$ is some constant such that $H(\tau) < 0$ for small τ . Then for all time $t \in [0, T)$,

$$H \leq 0.$$

We can also derive a classical Harnack inequality by integrating along a space-time path.

Theorem 3.3. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold with nonnegative scalar curvature. Let f be a positive solution to the conjugate heat equation*

$$\frac{\partial}{\partial t} f = -\Delta f + Rf.$$

Assume that (x_1, t_1) and (x_2, t_2) , $0 \leq t_1 < t_2 < T$, are two points in $M \times [0, T]$. Let

$$\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} (|\dot{\gamma}|^2 + R) dt,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2) . Then we have

$$f(x_2, t_2) \leq f(x_1, t_1) \left(\frac{T-t_1}{T-t_2}\right)^n \exp^{\Gamma/2}.$$

In the rest of this section, we will derive Perelman's Harnack inequality for the conjugate heat equation (3.1). First let $f = (4\pi\tau)^{-n/2} e^{-v}$, then $\ln f = -\frac{n}{2} \ln(4\pi\tau) - v$. We have

$$\frac{\partial}{\partial t} \ln f = -\frac{\partial}{\partial t} v + \frac{n}{2\tau},$$

and

$$\nabla \ln f = -\nabla v, \quad \Delta \ln f = -\Delta v.$$

Hence v satisfies the following equation,

$$(3.3) \quad \frac{\partial}{\partial t} v = -\Delta v + |\nabla v|^2 + cR + \frac{n}{2\tau}.$$

Or if we write the equation in backward time τ , we have

$$(3.4) \quad \frac{\partial f}{\partial \tau} = \Delta f + cRf,$$

and v satisfies

$$(3.5) \quad \frac{\partial v}{\partial \tau} = \Delta v - |\nabla v|^2 - cR - \frac{n}{2\tau}.$$

We can now define a general Harnack quantity and derive its evolution equation.

Lemma 3.4. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow, and v satisfies (3.5). Let*

$$P = \alpha \Delta v - \beta |\nabla v|^2 + aR + b\frac{v}{\tau} + d\frac{n}{\tau},$$

where α , β , a , b and d are constants that we will pick later. Then P satisfies the following evolution equation,

$$\begin{aligned} \frac{\partial P}{\partial \tau} = & \Delta P - 2\nabla P \cdot \nabla v - (2\alpha - 2\beta)|\nabla \nabla v|^2 + \frac{\alpha}{2\alpha - 2\beta} R_{ij} - \frac{\lambda}{2\tau} g_{ij}|^2 - \frac{2\alpha - 2\beta}{\alpha} \frac{\lambda}{\tau} P \\ & - 2(\alpha - 2\beta) R_{ij} v_i v_j + 2(a + \beta c) \nabla R \cdot \nabla v + (2\alpha - 2\beta) \frac{n\lambda^2}{4\tau^2} \\ & + (b - \frac{2\alpha - 2\beta}{\alpha} \lambda \beta) \frac{|\nabla v|^2}{\tau} - (\alpha c + 2a) \Delta R + (\frac{\alpha^2}{2\alpha - 2\beta} - 2a) |Rc|^2 \\ & + (\frac{2\alpha - 2\beta}{\alpha} \lambda - 1) b \frac{v}{\tau^2} + (\frac{2\alpha - 2\beta}{\alpha} \lambda - 1) d \frac{n}{\tau^2} - \frac{bn}{2\tau^2} + (\frac{2\alpha - 2\beta}{\alpha} a\lambda - \alpha\lambda - bc) \frac{R}{\tau}, \end{aligned}$$

here λ is also a constant that we will pick later.

Proof. The proof again follows from the same direct computation as in the proof of Lemma 2.1. Notice that the only extra term $-\frac{bn}{2\tau^2}$ comes from the evolution of v . \square

In the above lemma, let us take $\alpha = 2$, $\beta = 1$, $a = 1$, $c = -1$, $\lambda = 1$, $b = 1$, $d = -1$, as a consequence of Lemma 3.4, we have Perelman's Harnack inequality.

Theorem 3.5. (Perelman) *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow, f be the positive fundamental solution of the conjugate heat equation*

$$\frac{\partial}{\partial t} f = -\Delta f + Rf,$$

let $v = -\ln f - \frac{n}{2} \ln(4\pi\tau)$, $\tau = T - t$ and

$$P = 2\Delta v - |\nabla v|^2 + R + \frac{v}{\tau} - \frac{n}{\tau}.$$

Then we have

$$(3.6) \quad \frac{\partial}{\partial \tau} P = \Delta P - 2\nabla P \cdot \nabla v - 2|v_{ij} + R_{ij} - \frac{1}{2\tau} g_{ij}|^2 - \frac{1}{\tau} P.$$

Moreover,

$$P \leq 0$$

on $[0, T)$.

In the same spirit of searching Perelman's Harnack inequality, we shall also take $\alpha = 2$, $\beta = 1$, $a = 1$, $c = -1$, $\lambda = 2$, $b = 0$, $d = -2$ in the above Lemma 3.4. We have a Harnack inequality for all positive solutions of the conjugate heat equation (3.1).

Theorem 3.6. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow, suppose that $g(t)$ has nonnegative scalar curvature. Let f be a positive solution of the conjugate heat equation*

$$\frac{\partial}{\partial t} f = -\Delta f + Rf,$$

let $v = -\ln f - \frac{n}{2} \ln(4\pi\tau)$, $\tau = T - t$ and

$$P = 2\Delta v - |\nabla v|^2 + R - \frac{2n}{\tau}.$$

Then we have

$$(3.7) \quad \frac{\partial}{\partial \tau} P = \Delta P - 2\nabla P \cdot \nabla v - 2|v_{ij} + R_{ij} - \frac{1}{\tau}g_{ij}|^2 - \frac{2}{\tau}P - 2\frac{|\nabla v|^2}{\tau} - 2\frac{R}{\tau}.$$

Moreover, for all time $t \in [0, T]$,

$$P \leq 0.$$

Proof. It is easy to see that for τ small enough then $P(\tau) < 0$. It follows from (3.7) and maximum principle that

$$P \leq 0$$

for all time τ , hence for all t . □

Remark 3.1. *Theorem 3.6 can be deduced from Theorem 1.3 directly since $v = u - \frac{n}{2} \ln(4\pi\tau)$ and $b = 0$. But as we can see in the direct calculation, it will also lead to Perelman's Harnack inequality (by choosing different coefficients). S. Kuang and Q. Zhang proved this estimate in [KZ07].*

Remark 3.2. *We can prove similar estimate for P if we have a Type I solution to the Ricci flow. We can also prove a classical Harnack inequality by integrating along a space-time path.*

4. Entropy Formulas and Monotonicities

In this section, we will define two entropies which are similar to Perelman's entropy functionals as in [Per02], and we will show that both of them are monotone under the Ricci flow. Let $(M, g(t))$ be a solution to the Ricci flow on a close manifold, we shall also assume that $g(t)$ has nonnegative scalar curvature, we first prove

Theorem 4.1. *Assume that $(M, g(t))$, $t \in [0, T]$, is a solution to the Ricci flow on a Riemannian manifold with nonnegative scalar curvature. Let f be a positive solution of*

$$\frac{\partial}{\partial t} f = -\Delta f + 2Rf,$$

$u = -\ln f$ and $\tau = T - t$. Define

$$H = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau}$$

and

$$F = \int_M \tau^2 H e^{-u} d\mu,$$

then $\forall t \in [0, T)$, we have $F \leq 0$ and

$$\frac{d}{dt} F \geq 0.$$

Proof. The fact that $F \leq 0$ follows directly from $H \leq 0$. We calculate its time derivative, using (2.4) in Lemma 2.1 and $\frac{\partial}{\partial t} d\mu = -Rd\mu$, we have

$$\begin{aligned} \frac{d}{dt} F &= -\frac{dF}{d\tau} = - \int_M (2\tau H e^{-u} + \tau^2 e^{-u} \frac{\partial}{\partial t} H + \tau^2 H \frac{\partial}{\partial \tau} e^{-u} + R\tau^2 H e^{-u}) d\mu \\ &= - \int_M [\Delta(\tau^2 H e^{-u}) - 2\tau^2 e^{-u} |u_{ij} + R_{ij} - \frac{1}{t} g_{ij}|^2 - 2\tau e^{-u} |\nabla u|^2 \\ &\quad - \tau^2 e^{-u} (R + 2|Rc|^2)] d\mu \geq 0. \end{aligned}$$

□

We now define an entropy associate with the conjugate heat equation.

Theorem 4.2. *Assume that $(M, g(t))$, $t \in [0, T]$, is a solution to the Ricci flow on a closed Riemannian manifold with nonnegative scalar curvature. Let f be a positive solution of*

$$\frac{\partial}{\partial t} f = -\Delta f + Rf,$$

$u = -\ln f$ and $\tau = T - t$. Let

$$H = 2\Delta u - |\nabla u|^2 + R - 2\frac{n}{\tau}$$

and

$$W = \int_M \tau^2 H e^{-u} d\mu,$$

then $\forall t \in [0, T)$, we have $W \leq 0$ and

$$\frac{d}{dt} W \geq 0.$$

Proof. The fact that $W \leq 0$ follows directly from $H \leq 0$. To calculate its time derivative, using (3.2) and $\frac{\partial}{\partial t} d\mu = -Rd\mu$, we have

$$\begin{aligned} \frac{d}{dt} W &= -\frac{dW}{d\tau} = - \int_M (2\tau H e^{-u} + \tau^2 e^{-u} \frac{\partial}{\partial t} H + \tau^2 H \frac{\partial}{\partial \tau} e^{-u} + R\tau^2 H e^{-u}) d\mu \\ &= - \int_M [\Delta(\tau^2 H e^{-u}) - 2\tau^2 e^{-u} |u_{ij} + R_{ij} - \frac{1}{t} g_{ij}|^2 - 2\tau e^{-u} |\nabla u|^2 \\ &\quad - 2\tau R e^{-u}] d\mu \geq 0. \end{aligned}$$

□

Remark 4.1. *As noted in [Per02], if we consider the system*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j u), \\ \frac{\partial}{\partial t} u = -\Delta u - R, \end{cases}$$

then the measure $dm = e^{-u} d\mu$ is fixed. This system differs from the original system of the conjugate heat equation under the Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \\ \frac{\partial}{\partial t} u = -\Delta u + |\nabla u|^2 - R \end{cases}$$

by a diffeomorphism.

5. Gradient Estimate for the Backward Heat Equation

In this section, we consider a gradient estimate for positive solutions f to the backward heat equation

$$(5.1) \quad \frac{\partial}{\partial t} f = -\Delta f.$$

Since the equation is linear, without loss of generality, we may assume that $0 < f < 1$. Let $f = e^{-u}$, then u satisfies

$$(5.2) \quad \frac{\partial}{\partial t} u = -\Delta u + |\nabla u|^2,$$

and $u > 0$.

In the proof of Lemma 2.1, let take $\alpha = 0$, $\beta = -1$, $a = c = 0$, $b = -1$ and $d = 0$, then

$$H = |\nabla u|^2 - \frac{u}{\tau},$$

and we have

$$(5.3) \quad \frac{\partial}{\partial \tau} H = \Delta H - 2\nabla H \cdot \nabla u - 4Rc(\nabla u, \nabla u) - \frac{1}{\tau} H - 2|\nabla \nabla u|^2.$$

This leads to the following theorem.

Theorem 5.1. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold with nonnegative curvature operator. Let $f(< 1)$ be a positive solution to the backward heat equation (5.1), $u = -\ln f$, $\tau = T - t$ and*

$$H = |\nabla u|^2 - \frac{u}{\tau}.$$

Then for all time $t \in [0, T)$,

$$H \leq 0,$$

i.e.,

$$|\nabla f|^2 \leq \frac{f^2 \ln(1/f)}{T-t}.$$

Proof. Notice that as τ small enough, $H < 0$, now the proof follows from (5.3) and the maximum principle. \square

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