Some of the Combinatorics Related To Michael's Problem

by
J. Tatch Moore

Abstract

We present some new methods for constructing a Michael space, a regular Lindelöf space which has a non-Lindelöf product with the space of irrationals. The central result is a combinatorial statement about the irrationals which is a necessary and sufficient condition for the existence of a certain class of Michael spaces. We also show that there are Michael spaces assuming $\mathfrak{d} = \text{cov}(\mathcal{M})$ and that it is consistent with $\text{cov}(\mathcal{M}) < \mathfrak{b} < \mathfrak{d}$ that there is a Michael space. The influence of Cohen reals on Michael's Problem is discussed as well. Finally we present an example of a Michael space of weight less than $\mathfrak{b}$ under the assumption that $\mathfrak{b} = \mathfrak{d} = \text{cov}(\mathcal{M}) = \aleph_{\omega+1}$ (whose product with the irrationals is necessarily linearly Lindelöf).

0. Introduction

Our goal in this paper is to provide some necessary and sufficient conditions for the existence of a Michael space, a regular Lindelöf space whose product with the irrationals is not Lindelöf. The (almost) necessary condition is the focus of Section 1. In Section 2 we will present two constructions of Michael spaces: one from the combinatorial statement $\mathfrak{d} = \text{cov}(\mathcal{M})$ and the other from the addition of many Cohen reals. Section 3 will attempt to show how these constructions can be generalized and where problems can arise. The final section will be devoted to a discussion of Michael spaces whose product with the irrationals is linearly Lindelöf.

Before we begin, we will fix some notation and review some definitions. The space of irrationals will be denoted P and we will sometimes use their representation as functions from N to N, denoted ${}^N N$, equipped with the product topology. Through out this paper we will view the Cantor set C as being a compactification of P by adding a countable set Q_C. Ordinals will be taken to be the set of their predecessors and cardinals will be viewed as the first ordinal of a given size. If A is any set, we will denote its cardinality by $\#(A)$ and its power set by $P(A)$. Ordinals will be given the order topology when viewed as topological spaces and products will always be given the standard product topology.

The relation $\leq$ refers to the coordinate-wise order whenever it is used to compare two elements of a countable product of ordered sets: $f \leq g$ iff $f(n) \leq g(n)$ for every n. Similarly $f \leq_* g$ refers to the statement that for all but finitely many n $f(n) \leq g(n)$. The cardinals $\mathfrak{b}$ and $\mathfrak{d}$ refer to the smallest sizes of unbounded and dominating families in $({}^N N, \leq_*)$ respectively. The number of meager sets it takes to cover the irrationals will be denoted by $\text{cov}(\mathcal{M})$. If $\mathbf{V}$ is a model of ZFC, we say that a real number is Cohen (random) over $\mathbf{V}$ if it is not in any meager (measure 0) set coded in $\mathbf{V}$. For more information on the combinatorics of P and Cohen and random reals the reader is referred to [BaJ].

If X is a non-Lindelöf topological space, we will let $L(X)$ denote the minimum cardinality of an uncountable open cover of X with no countable subcover. Note that $L(X)$ is either regular or of countable cofinality. A space is linearly Lindelöf if every increasing open cover has a countable subcover. The weight of a space X will be denoted by $w(X)$.

The author would like to thank S. Todorčević for his insightful comments and for his beautiful example in Theorem 3.1. He also wishes to thank D. Burke for introducing him to this wonderful problem.

1. Toward A Standard Form

In this section we will show that if there is a Michael space M, then there is one which is a subspace of $(\mathfrak{d}+1)\times C$, provided that $L(M\times P)$ is a regular cardinal.

First it will be helpful to make a definition.

1.1. Definition. A sequence of distinct subsets $\{X_\xi\}_{\xi\leq\theta}$ of C is said to be a *θ-Michael sequence* if the following conditions hold:

(i). $C\supseteq X_\eta\supseteq X_\xi\supseteq X_\theta=Q_C$ for every $\eta<\xi<\theta$.

(ii). For every compact subset K of P the ordinal $\delta=\min\{\xi\leq\theta : X_\xi\cap K=\emptyset\}$ does not have uncountable cofinality.

A θ-Michael sequence is said to be *reduced* if it also satisfies the following condition:

(iii). For every subset A of P which analytic (a continuous image of P) the ordinal $\delta=\min\{\xi\leq\theta : X_\xi\cap A=\emptyset\}$ is either θ or does not have uncountable cofinality.

What is somewhat surprising is that very little information about a Michael space is actually necessary to construct a Michael space inside of $(\mathfrak{d}+1)\times C$. This is evident in the proof of the next theorem (see Corollaries 1.3 and 1.4). Note that no separation axioms need to be assumed of the space X.

1.2. Theorem. The following are equivalent for any regular cardinal θ:

a). There is a Lindelöf space X such that $X\times P$ is not Lindelöf and $L(X\times P)=\theta$.

b). There is a reduced θ-Michael sequence $\{X_\xi\}_{\xi\leq\theta}$.

c). There is a subspace M of $(\theta+1)\times C$ which is a Michael space and $w(M\times P)=L(M\times P)=\theta$.

Proof. (a⇒b) Fix an enumeration $\{U_\xi\}_{\xi<\theta}$ of an open cover $\mathcal{U}$ of $X\times P$ witnessing $L(X\times P)$. If $\xi\leq\theta$ let $X_\xi=Q_C\cup\{p\in P : X\times\{p\}\not\subseteq\bigcup_{\eta<\xi}U_\eta\}$. We may assume, by going to a subsequence if necessary, that $X_\eta\neq X_\xi$. Suppose that K is a compact subset of P. Note that if $\delta=\min\{\xi\leq\theta : X_\xi\cap K=\emptyset\}$ has uncountable cofinality, $\{U_\eta\}_{\eta<\xi}$ is an uncountable open cover of $X\times K$ with no countable subcover. This is impossible since the product of a Lindelöf spaces and a compact space is Lindelöf. Thus $\{X_\xi\}_{\xi\leq\theta}$ is a θ-Michael sequence.

To see that $\{X_\xi\}_{\xi\leq\theta}$ is reduced, let $A\subseteq P$ be analytic. We may find a closed subset E of $P\times P$ such that A is equal to $\pi_1[E]$ and E is homeomorphic to P. Note that $\{U_\eta\}_{\eta<\xi}$ covers $X\times A$ iff $\{U_\eta\times P\}_{\eta<\xi}$ covers $X\times E$. Since $L(X\times E)=L(X\times P)=\theta$, we can deduce using methods from an earlier portion of this proof that $\delta=\min\{\xi\leq\theta : X_\xi\cap A=\emptyset\}$ is either θ or does not have uncountable cofinality.

(b⇒c) For any $\xi\leq\theta$, define $M_\xi=\bigcup_{\eta\leq\xi}\{\eta\}\times X_\eta$. We will now show that M_ξ is Lindelöf by induction on ξ. The only non-trivial stages of the induction are if the cofinality of ξ is uncountable. In this case fix an open cover $\mathcal{U}$ of M_ξ. We may choose a rectangular open set $V=(\xi+1)\times H$ which contains $(\xi+1)\times X_\xi$ and is covered by countably many elements $\mathcal{U}_0$ of $\mathcal{U}$. Note that $K = C \setminus H$ is disjoint from X_ξ and $E = M_\xi \setminus V$ is contained in $(\xi+1) \times K$. Since the cofinality of ξ is uncountable, there is a $\delta < \xi$ such $K \cap X_\delta = \emptyset$. It follows that E is contained in M_δ. By our induction hypothesis we can find countably many elements $\mathcal{U}_1$ of $\mathcal{U}$ which cover M_δ. We now have that $\mathcal{U}_0 \cup \mathcal{U}_1$ is countable subcover of M_ξ and therefore M_ξ is Lindelöf for all $\xi \leq \theta$.

Now let $M = M_\theta$. We leave it to the reader to check that $w(M \times P) = \theta$. To see that $M \times P$ is not Lindelöf note that $\Delta = \{(\xi, p, p) : (\xi, p) \in M\}$ is closed in $M \times P$. It is easy to verify that $\{M_\xi \times P\}_{\xi<\theta}$ is an increasing open cover of Δ with no countable subcover.

To finish the proof we need to show that $L(M \times P) = \theta$. Fix an open cover $\mathcal{U}$ of $M \times P$ of cardinality less than θ. It can be assumed without loss of generality that $\mathcal{U}$ is a cover of $M \times P$ by open subsets of $(\theta+1) \times C \times P$ and that $\mathcal{U}$ is closed under the operation of taking finite unions. We must show that $\mathcal{U}$ has a countable subcover of $M \times P$.

Before we proceed, we will first show that $M_\xi \times P$ is Lindelöf for all $\xi < \theta$. The proof is done by induction on ξ. Again the only non-trivial stages of the induction are when ξ is an ordinal of uncountable cofinality. Let $\mathcal{V}$ be an open cover of $M_\xi \times P$. First pick a rectangular open set $V = (\xi+1) \times H$ about $(\xi+1) \times X_\xi \times P$ which is covered by countably many elements $\mathcal{V}_0$ of $\mathcal{V}$. Now let $A = \pi_C[C \times P \setminus H]$ be the projection of H onto C. Since A is analytic and $X_\xi \cap A = \emptyset$, it follows from condition (iii) in the definition of a reduced Michael sequence that there is a $\delta < \xi$ such that $X_\delta \cap A = \emptyset$. Thus $(M_\xi \times P) \setminus V$ is contained in $M_\delta \times P$ which is Lindelöf by our inductive assumption. Pick $\mathcal{V}_1$ to be a countable collection of elements of $\mathcal{V}$ which cover $M_\delta \times P$. Then $\mathcal{V}_0 \cup \mathcal{V}_1$ is a countable subcover of $M_\xi \times P$ and hence M_ξ is Lindelöf for all $\xi < \theta$.

Let $\{B_n\}_{n<\omega}$ be an enumeration of a countable base for $C \times P$. For every U in $\mathcal{U}$, define $U[B_n]$ to be the union of all rectangular open subsets of U of the form $\alpha \times B_n$, for some ordinal α. We now wish to show that for every $x = (\alpha, p_1, p_2)$ in $M \times P$ there is a U in $\mathcal{U}$ and a $n < \omega$ such that x is in $U[B_n]$. Since $A_x = (\alpha+1) \times \{(p_1, p_2)\}$ is compact, there is a U in $\mathcal{U}$ such that $A_x \subseteq U$. Applying a standard theorem of topology (the Tube Lemma) there is a $n < \omega$ such that $A_x \subseteq (\alpha+1) \times B_n \subseteq U$. It follows that x is in $U[B_n]$.

Observe that for each n we can pick a $\xi_n \leq \theta+1$ such that $\xi_n \times B_n = \cup\{U[B_n] : U \in \mathcal{U}\}$. If $\xi_n < \theta$, then $M_{\xi_n} \times P$ is a Lindelöf space by previous observation and we may choose $\mathcal{U}_n$ to be a countable subcover of $(\xi_n \times B_n) \cap (M \times P) \subseteq M_{\xi_n} \times P$. If $\xi_n \geq \theta$ then note that since $\#(\mathcal{U}) < \theta$, there is a U in $\mathcal{U}$ such that $U[B_n] = \xi_n \times B_n$. In this case let $\mathcal{U}_n = \{U\}$. Since $M \times P \subseteq \bigcup_{n<\omega} \xi_n \times B_n$, the collection $\bigcup_{n<\omega} \mathcal{U}_n$ is a countable subcover of $\mathcal{U}$ and $L(M \times P)$ must be θ.

(c$\Rightarrow$a) Trivial.

Observe that if M is a Michael space, then any open cover $\mathcal{U}$ of $M \times P$ has a subcover $\mathcal{U}_0$ of cardinality at most $\mathfrak{d}$. The reason for this is that P can be covered by a collection $\mathcal{K}$ of $\mathfrak{d}$ many compact subsets. For every K in $\mathcal{K}$, there is countable subcollection $\mathcal{U}_K$ of $\mathcal{U}$ which covers $M \times K$. If we let $\mathcal{U}_0 = \bigcup_{K\in\mathcal{K}} \mathcal{U}_K$ then $\mathcal{U}_0$ has the desired properties.

Note that the following corollaries were essentially proved in Theorem 1.2.

1.3. Corollary. If X is a Lindelöf space and $\mathcal{U}$ is an open cover of $X \times P$ with no subcover of smaller cardinality, then there is a $\#(\mathcal{U})$-Michael sequence.

1.4. Corollary. If there is a θ-Michael sequence and the cofinality of θ is uncountable, then there is a Michael space.

It is perhaps worth remarking here that most of the Michael spaces which have been constructed in the history of the problem "live" inside of the product of an ordinal and C. For instance if $A = \{a_\xi\}_{\xi<\omega_1}$ is a set which is concentrated about Q_C, the "standard" construction of a Michael Space by isolating A (see [M1] and [M2]) is the same as the space $[\{\omega_1\} \times Q_C] \cup \{(\xi+1, a_\xi) : \xi < \omega_1\} \subseteq (\omega_1 + 1) \times C$. K. Alster's example [A] can be embedded in a similar way inside of $(\mathfrak{c}+1) \times Y$, where Y is the metric compactification of P used in his construction.

2. Building A New Michael Space

W. Fleissner noted that the only portion of Martin's Axiom which K. Alster used in his construction could be summarized by the statement $\mathfrak{b} = \mathfrak{d} = \text{cov}(\mathcal{M})$ [1]. We will begin this section by showing that $\mathfrak{d} = \text{cov}(\mathcal{M})$ is enough to imply the existence of a Michael space. The following lemma will prove useful in this as well as later constructions. A proof can be found in [S].

2.1. Lemma. If A is an analytic subset of P and $\mathcal{F}$ is a cover of A by closed sets then either $\mathcal{F}$ has a countable subcover or there is a nonempty G_δ set G contained in A such that $F \cap G$ is nowhere dense in G for every F in $\mathcal{F}$ and G is homeomorphic to P.

Here is our first construction.

2.2. Theorem. ($\mathfrak{d} = \text{cov}(M)$) There is a Michael space.

Proof. First note that there is a sequence of compact sets $\{D_\xi\}_{\xi<\mathfrak{d}}$ in P such that for every compact subset K of P, there is a $\xi < \mathfrak{d}$ such that K is contained in D_ξ. Let $X_\xi = C \setminus \cup_{\eta<\xi} D_\eta$. To see that $\{X_\xi\}_{\xi\leq\mathfrak{d}}$ is a $\mathfrak{d}$-Michael sequence, fix a compact subset K of P. Suppose that $K \cap X_\xi$ is empty and $\xi < \mathfrak{d}$ has uncountable cofinality. Then $\{D_\eta\}_{\eta<\xi}$ is a cover of K by fewer than $\text{cov}(\mathcal{M})$ many closed sets and thus must have a countable subcover. It follows that there is a $\delta < \xi$ such that $X_\delta \cap K$ is empty. By Corollary 1.4 there is a Michael space.

Since Cohen reals are very closely connected to questions about Baire category and the cardinal $\text{cov}(\mathcal{M})$, it is natural to ask if Michael spaces can be constructed from Cohen reals. The naive answer is "yes, of course," since $\mathfrak{b} = \aleph_1$ holds in any model obtained by adding uncountably many Cohen reals. Not only is this true, but if $\theta = \mathfrak{d}^{\mathbf{V}}$ many Cohen reals are added to a model $\mathbf{V}$, we also get the equality $\mathfrak{d} = \text{cov}(\mathcal{M})$ in the extension $\mathbf{V}^{\mathcal{C}_\theta}$.

As it turns out though, we get much more than just an arbitrary Michael space from adding many Cohen reals. The following Lemma plays a crucial role in the proofs of Theorems 2.4 and 3.2. A proof can be found in [BaJ; ch. 3, §1].

[1] W. Fleissner actually claimed that all that was needed in K. Alster's line of proof was the statement $\mathfrak{b} \leq \text{cov}(\mathcal{M})$ [V, p. 206], though he has since retracted this claim. The author will also note here that B. Lawrence has also withdrawn his assertion that $\mathfrak{b} = \mathfrak{d}$ implies the existence of a Michael space.

2.3. Lemma. Suppose $\{r_\alpha\}_{\alpha<\theta}$ is a collection of reals which are either Cohen or random over a model $\mathbf{V}$ and r is a real in $\mathbf{V}[r_\alpha : \alpha < \theta]$. Then there is a countable subset B of θ such that r is in $\mathbf{V}[r_\alpha : \alpha \in B]$.

2.4. Theorem. If $\mathbf{V}$ is a model of ZFC and $\{c_\xi\}_{\xi<\theta}$ is a collection of θ reals which are Cohen over $\mathbf{V}$ for some regular uncountable cardinal θ, then in $\mathbf{V}[c_\alpha : \alpha < \theta]$ there is a Michael space M such that $L(M \times P)$ is θ.

Proof. Define X_ξ to be the the union of Q_C and all of the elements of $({}^N N, \leq_*)$ not bounded in $\mathbf{V}[c_\alpha : \alpha < \xi] \cap ({}^N N, \leq_*)$. It now suffices to show that $\{X_\xi\}_{\xi<\theta}$ is a reduced θ-Michael sequence. Let A be an analytic subset of $P = {}^N N$ and $\delta \leq \theta$ be the first ordinal such that $X_\delta \cap A$ is empty. We now need to show that if $\delta < \theta$ then the cofinality of δ is not uncountable.

Since $\delta < \theta$, every element of A is bounded in $\mathbf{V}[c_\alpha : \alpha < \delta] \cap ({}^N N, \leq)$. For every p in $\mathbf{V}[c_\alpha : \alpha < \delta] \cap ({}^N N, \leq)$, let E_p be the elements of ${}^N N$ bounded by p under the relation $\leq$. Each E_p is closed and we may apply Lemma 2.1 to $\mathcal{F} = \{E_p : p \in \mathbf{V}[c_\alpha : \alpha < \delta] \cap {}^N N\}$ and A. If there is a G_δ subset G of A as described in Lemma 2.1, we may pick a real c which is Cohen over $\mathbf{V}[c_\alpha : \alpha < \xi][G]$ and construct an element in G which is not in any E_p. Thus there must be a countable collection of reals S such that K is covered by $\{E_p : p \in S\}$. It follows from Lemma 2.3 that δ can not have uncountable cofinality.

3. Some Subtleties

This section is intended to shed light on just how far we might hope to expand on the construction in Theorem 2.2. It is not initially clear that $\mathrm{cov}(\mathcal{M})$ can be witnessed in a way which would cause difficulty in this line of proof. After a great deal of unsuccessful thought on the part of the author, S. Todorčević came up with the following example over a cup of coffee.

3.1. Theorem. There is a compact set $K \subseteq {}^N N$ and a collection $\{p_\xi\}_{\xi<\mathrm{cov}(\mathcal{M})}$ of elements of ${}^N N$ such that each p_ξ bounds only a nowhere dense subset of K under $\leq$ and for every p in K, there is a $\xi < \mathrm{cov}(\mathcal{M})$ such that $p \leq p_\xi$.

Proof. Fix an enumeration $\{t_n\}_{n\in N}$ of $2^{<\omega}$ and a cover $\{E_\xi\}_{\xi<\mathrm{cov}(\mathcal{M})}$ of 2^ω by closed nowhere dense sets. Let $\mathcal{A}$ denote the collection of all subsets A of N such that there is a single element r in 2^ω which has t_n as an initial segment for every n in A. It is easy to verify that $\mathcal{A}$ is closed and compact as a subspace of $P(N)$ with the product topology. Similarly let B_ξ be the set of all n in N such that there a $r \in E_\xi$ which contains t_n as an initial segment. Now define $\Phi : P(N) \to {}^N N$ by $[\Phi(A)](n) = \sum_{i\in A\cap n} 2^i$. It is easy to check that Φ is continuous and has the property that if $A \subseteq B$ then $\Phi(A) \leq \Phi(B)$. Let K be the image of $\mathcal{A}$ under Φ and let $p_\xi = \Phi(B_\xi)$. It is now routine to verify that K and $\{p_\xi\}_{\xi<\mathrm{cov}(\mathcal{M})}$ have the desired properties.

What the previous theorem tells us is that we can get into trouble mimicking the proof of Theorem 2.2 if $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$, provided we take no further care in choosing our compact sets D_ξ. In some instances, however, we can still survive.

3.2. Theorem. There is a model of $\mathrm{ZFC} + \mathrm{cov}(\mathcal{M}) < \mathfrak{b} < \mathfrak{d}$ in which there is a Michael space.

Proof. Start with a ground model $\mathbf{V}$ satisfying $\aleph_1 < \mathfrak{b} < \mathfrak{d} = \mathrm{cov}(\mathcal{M})$. Let M be the Michael space constructed in Theorem 2.2 and let $\{D_\xi\}_{\xi<\mathfrak{d}}$ be the collection of compact sets used in the construction of M. Now add $\aleph_1$ random reals to $\mathbf{V}$ to obtain $\mathbf{V}[r_\alpha : \alpha < \omega_1]$. The statement $\aleph_1 = \mathrm{cov}(\mathcal{M}) < \mathfrak{b} = \mathfrak{b}^{\mathbf{V}} < \mathfrak{d} = \mathfrak{d}^{\mathbf{V}}$ [2] now holds in $\mathbf{V}[r_\alpha : \alpha < \omega_1]$ and we need only to show that M remains a Michael space in this extension. Suppose that K is a compact subset of P in $\mathbf{V}[r_\alpha : \alpha < \omega_1]$. Since K is coded by a real, by Lemma 2.3 there is a $\beta < \omega_1$ such that K is in $\mathbf{V}[r_\alpha : \alpha < \beta]$. In $\mathbf{V}[r_\alpha : \alpha < \beta]$, the cardinal $\mathrm{cov}(\mathcal{M})$ is still equal to $\mathfrak{d}$ (see [BaJ; ch. 3, §2.E]) and we can use Lemma 2.1 to show that, in $\mathbf{V}[r_\alpha : \alpha \in A]$, there is a countable set of ordinals $B \subseteq \mathfrak{d}$ such that K is contained in $\cup_{\xi\in B} D_\xi$. Thus the first ordinal δ for which $K \cap X_\delta$ is empty has countable cofinality. Since B is countable, we can pick an enumeration $\{\xi_n\}_{n<\omega}$ of B. The statement that K is covered by $\{D_{\xi_n}\}_{n<\omega}$ can be written $\forall x \in K \exists n < \omega (x \in D_{\xi_n})$. This is a Π^1_1 statement and is therefore absolute (see [K; ch. 13]). It follows that in $V[r_\alpha : \alpha < \omega_1]$, δ is still the least ordinal such that $X_\delta \cap K$ is empty.

Another approach one can take in improving upon Theorem 2.2 is as follows. Suppose that $\{X_\xi\}_{\xi\leq\mathfrak{d}}$ is as given in the construction in Theorem 2.2 under the possible assumption $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$. A natural question to ask is whether there is a club $A \subseteq \mathfrak{d}$ such that $\{X_\xi\}_{\xi\in A\cup\{\mathfrak{d}\}}$ is a Michael sequence. It is unknown to the author whether such a club can be found just on the basis of ZFC.

4. Countable Limit Cardinals And Michael Spaces

The linearly Lindelöf non-Lindelöf pathology has been around for some time in the study of topological spaces (see [Ru; p. 190] and [Mi]). Until now it was uncertain whether this pathology could appear in the context of Michael's problem (see [AG]). It is the intention of this section to show how to build Michael spaces who's product with the irrationals is linearly Lindelöf from a reduced Michael sequence of regular length. A byproduct of this construction is the consistency of the statement "there is a Michael space of weight less than $\mathfrak{b}$" which answers a few open questions.

The following theorem of Shelah (essentially proved in [BuMa; §§2-3]) is at the heart of our proofs in this section.

4.1. Theorem. If θ is a cardinal of countable cofinality then there is an increasing sequence $\{\theta_n\}_{n<\omega}$ of regular cardinals which is cofinal in θ and a scale $\{f_\xi\}_{\xi<\theta^+}$ on $(\prod_{n<\omega} \theta_n, \leq_*)$.

The following theorem shows us how to "step down" to cardinals of countable cofinality. Note that if X is a non-Lindelöf space and $w(X) = L(X)$ then X is linearly Lindelöf.

4.2. Theorem. If θ is a cardinal of countable cofinality and there is a reduced θ^+-Michael sequence, then there is a Michael space M such that $w(M \times P) = L(M \times P) = \theta$.

Proof. Let θ_n and $\{f_\xi\}_{\xi<\theta^+}$ be the cardinals and scales as guaranteed by Theorem 4.1. Fix a θ^+-Michael sequence $\{X_\xi\}_{\xi<\theta^+}$. Let Z denote the space $\prod_{n<\omega} \theta_n + 1$ and $Z_0 =$

[2] In [BaJ; ch. 3, §2.E] it is actually proved that after adding $\aleph_1$ random reals, the statement $\mathrm{non}(\mathcal{N}) = \aleph_1$ holds. Since $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{N})$, the we also have $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

$\{f \in Z : \exists m \forall n > m(f(n) < \theta_n)\}$ (see [Mi]). We will now define M_ξ for $\xi \leq \theta^+$ to be the set $\{(f,p) : \exists \eta \leq \xi[(f \leq_* f_\eta) \& (p \in X_\eta)]\} \subseteq Z \times C$ with the subspace topology (for convenience we define $f_{\theta^+}(n) = \theta_n$).

We will now prove that M_ξ is Lindelöf by induction on ξ. Define $Z_f = \{g \in Z : g \leq_* f\}$. Note that $M_\xi = [Z_{f_\xi} \times X_\xi] \cup [\bigcup_{\eta<\xi} M_\eta]$. Since Z_{f_ξ} is σ-compact, it is easy to see that the only case in the induction where troubles arise is when the cofinality of ξ is uncountable. Suppose that this is so and pick an open cover $\mathcal{U}$ of M_ξ. We may choose a G_δ set $H \subseteq C$ such that there is a countable subcollection $\mathcal{U}_0$ of $\mathcal{U}$ which covers $Z_{f_\xi} \times H$ and X_ξ is contained in H. Since $K = C \setminus H$ is σ-compact there is a $\delta < \xi$ such that K is disjoint from X_δ. Thus $M_\xi \setminus \cup\mathcal{U}_0$ is contained in M_δ and our induction hypothesis applies. If $\mathcal{U}_1$ is a countable cover for M_δ, then $\mathcal{U}_0 \cup \mathcal{U}_1$ is a countable subcover of M_ξ.

Let $M = M_{\theta^+}$. It is again left to the reader to check that $w(M \times P) = \theta$. We need to show that $L(M \times P)$ is θ. It is easy to check that $\Delta = \{(f,p,p) : (f,p) \in M\}$ is closed in $M \times P$. Also, if we define $V_{n,\alpha}$ to be the set $\{f \in Z : f(n) < \alpha\}$ for $\alpha < \theta_n$, then the collection $\{V_{n,\alpha} \times C \times P : \alpha < \theta_n\}$ is an open cover of Δ of cardinality θ with no countable subcover.

To finish the proof of our theorem, we need to show that any open cover $\mathcal{U}$ of $M \times P$ of cardinality less than θ has a countable subcover. For convenience we assume that the elements of $\mathcal{U}$ are actually open sets in $Z \times C \times P$ and that $\mathcal{U}$ is closed under the operation of taking finite unions. Pick a countable base $\{B_n\}_{n<\omega}$ for $C \times P$. Define $U[B_n]$ to be the union of all rectangular open subsets $W \times B_n$ of U which have the following properties:

(a). If f is in W and $g \leq f$ then g is in W.

(b). W does not depend on coordinates greater than n.

Suppose that $x = (f, p_1, p_2)$ is a point in $M \times P$. We wish to find a U in $\mathcal{U}$ and an $n < \omega$ such that x is in $U[B_n]$. Since $A_x = \{(g, p_1, p_2) \in Z \times C \times P : g \leq f\}$ is compact and contained in $M \times P$, there is a U in $\mathcal{U}$ which contains it. It follows from a basic theorem of topology (the Tube Lemma) that there is a rectangular open set $W \times B_n$ satisfying (a) and (b) such that $A_x \subseteq W \times B_n \subseteq U$ and thus x is in $U[B_n]$.

Since $(\prod_{i\leq n} \theta_i + 2, \leq)$ has only finite antichains, we can partition $U[B_n]$ into finitely many open sets $\{U_j[B_n]\}_{j<k}$ with the following property: for every $j < k$ there is a t_j in $\prod_{i\leq n} \theta_i + 2$ such that $f \in U_j[B_n]$ iff $f(i) < t_j(i)$ for all $i \leq n$. Define $\mathcal{U}^r$ be the collection of all $U_j[B_n]$ such that U is in $\mathcal{U}$. We have already verified that $\mathcal{U}^r$ is a cover for $M \times P$. Since $\mathcal{U}^r$ refines $\mathcal{U}$, it suffices to show that $\mathcal{U}^r$ has a countable subcover.

Let $\mathcal{U}_0$ be the collection of all $U_j[B_n]$ such that for the corresponding t_j we have $t_j(i) \geq \theta_i$ for all $i \leq n$. Note that $\mathcal{U}_0$ is countable. Also, if U is in $\mathcal{U}^r \setminus \mathcal{U}_0$, there is a (k_U, α_U) such that $U \subseteq V_{k_U,\alpha_U} \times B_n$ and $\alpha_U < \theta_{k_U}$. Since $\mathcal{U}^r$ has cardinality less than θ, there is a f in Z_0 such that for every U in $\mathcal{U}^r$, $\alpha_U < f(k_U)$.

Now pick a $\xi < \theta^+$ such that $f \leq_* f_\xi$ and let $E = \{p_1 \in C : \exists g \in Z \exists p_2 \in P((g, p_1, p_2) \in M \times P \setminus \cup\mathcal{U}_0)\}$. We will show that $X_\xi \cap E$ is empty. If this is not the case, choose a $(g, p_1, p_2) \in M \times P \setminus \cup\mathcal{U}_0$ such that p_1 is in X_ξ. It follows from the definition of M that there is a h in Z such that $f, g \leq h \leq_* f_\xi$ and thus (h, p_1, p_2) is in $M \times P$. Note that (h, p_1, p_2) is not in $\cup\mathcal{U}_0$ because $g \leq h$ and therefore is also not in $\cup\mathcal{U}^r$ by definition of f, a contradiction. Thus $X_\xi \cap E$ is empty and $M \times P \setminus \cup\mathcal{U}_0$ is contained in $M_\xi \times P$.

It can be seen that $M_\xi \times P$ is Lindelöf for all $\xi < \theta^+$ using the same techniques used to prove that M_ξ is Lindelöf (as in the proof of Theorem 1.2, we are essentially substituting the use of condition (iii) in the definition of a reduced Michael sequence for the use of condition (ii)). Therefore there are countably many elements $\mathcal{U}_1$ of $\mathcal{U}^r$ which cover $M_\xi \times P$. We now have that $\mathcal{U}_0 \cup \mathcal{U}_1$ is a countable subcover of $M \times P$ and we are done.

The following corollary answers questions posed in [AG], [L], and [A].

4.3. Corollary. Under $\mathfrak{b} = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \aleph_{\omega+1}$ there is a Michael space M such that $w(M) = L(M \times P) = \aleph_\omega < \mathfrak{b}$.

From Theorem 4.2 we can also prove a characterization of the statement that $\mathfrak{c} > \aleph_\omega$.

4.4. Corollary. The cardinality of the continuum is greater than $\aleph_\omega$ iff there is a regular Lindelöf space X and a separable metric space Y such that $X \times Y$ is linearly Lindelöf but not Lindelöf.

Proof. ($\Rightarrow$) Since $\mathfrak{c} > \aleph_\omega$, there is a subset A of P of cardinality $\aleph_{\omega+1}$ such that A contains no uncountable compact subsets. Let $\{a_\xi\}_{\xi<\omega_{\omega+1}}$ be an enumeration of A and define $X_\xi = C \setminus \{a_\eta : \eta < \xi\}$ for all $\xi \leq \omega_{\omega+1}$. Let Z, Z_0, and $\{f_\xi\}_{\xi \leq \omega_{\omega+1}}$ be as defined in Theorem 4.2 for $\theta = \aleph_\omega$. Then define $M = \{(f,p) \in Z \times C : \exists \xi \leq \omega_{\omega+1}[(f \leq_* f_\xi)\&(p \in X_\xi)]\}$ and note that it follows from the proof of Theorem 4.2 that M is Lindelöf and $w(M) = L(M \times A) = \aleph_\omega$ as desired.

($\Leftarrow$) Assume that $\mathfrak{c} < \aleph_\omega$ ($\mathfrak{c}$ can never be equal $\aleph_\omega$ by König's Lemma). It is well known that if Y is a separable metric space then $\#(Y) \leq \mathfrak{c}$. Also note that if $X \times Y$ is linearly Lindelöf but not Lindelöf then $L(X \times Y) \geq \aleph_\omega$. Since $L(X \times Y) \leq \min\{\#(X), \#(Y)\}$ (see the remark at the end of Theorem 1.2), the product of a Lindelöf space and a separable metric space is Lindelöf iff it is linearly Lindelöf.

We leave the reader with some open questions.

Question 1. If $\{D_\xi\}_{\xi<\mathfrak{d}}$ is a collection of compact sets as described in Theorem 3.2, is there a club (in ZFC) $A \subseteq \mathfrak{d}$ such that $\{C \setminus \cup_{\eta<\xi}\}_{\xi \in A \cup \{\mathfrak{d}\}}$ is a Michael sequence?

REFERENCES

[A] K. Alster, *The Product Of A Lindelöf Space With The Space Of Irrationals Under Martin's Axiom*, Proc. Amer. Math. Soc., **110** n2 (1990), pp. 543-547.

[AG] K. Alster, G. Gruenhage, *Products Of Lindelöf Spaces And GO-Spaces*, Topology Appl., **64** n1 (1995), pp. 23-36.

[BaJ] T. Bartoszyński, H. Judah, *Set Theory: On The Structure Of The Real Line*, A. K. Peters (1995).

[BuMa] M. Burke, M. Magidor, *Shelah's pcf Theory And It's Applications*, Ann. Pure. Appl. Logic, **50** (1990), pp. 207-254.

[K] A. Kanamori, *The Higher Infinite. Large Cardinals In Set Theory From Their Beginnings*, Springer-Verlag (1994).

[L] B. Lawrence, *The Influence Of A Small Cardinal On The Product Of A Lindelöf Space And The Irrationals*, Proc. Amer. Math. Soc., **110** n2 (1990), pp. 535-542.

[M1] E. Michael, *The Product Of A Normal Space And A Metric Space Need Not Be Normal*, Bull. Amer. Math. Soc., **69** (1963), pp. 375-376.

[M2] E. Michael, *Paracompactness And The Lindelöf Property In Finite And Countable Cartesian Products*, Composito Math., **23** (1971), pp. 199-214.

[Mi] Miščenko, *On Finally Compact Spaces*, Soviet Math. Doklady, n2 (1962), pp. 1199-1202.

[Ru] M. E. Rudin, Some Conjectures, in: *Open Problems In Topology*, J. van Mill, G. M. Reed (editors), North Holland (1990), pp. 185-193.

[S] Solecki, *Covering Analytic Sets By Families Of Closed Sets*, J. Symbolic Logic, **59** (1994), pp. 1022-1031.

[V] J. Vaughan, Small Uncountable Cardinals And Topology, in: *Open Problems In Topology*, J. van Mill, G. M. Reed (editors), North Holland (1990) pp. 197-216.

Justin T. Moore
Department of Mathematics
University of Toronto
Toronto, Canada M5S 1A1
justin@math.toronto.edu