A LINEARLY FIBERED SOUSLINEAN SPACE UNDER MA

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Abstract. Under Martin's Axiom a c.c.c. nonseparable compact space is constructed which maps continuously into $[0,1]$ with linear fibers. Such a space can not, for instance, map onto $[0,1]^{\aleph_1}$.

1. Introduction.

In 1920, Souslin asked whether the countable chain condition is a sufficient restriction on linear compacta to imply that they are metrizable [6]. Although the answer was shown to be independent of ZFC in the 1960's, this now classic question has led to a prominent theme of modern set theoretic topology: c.c.c. versus separability. There are several results, for example, which state that under Martin's Axiom and the negation of the Continuum hypothesis c.c.c. nonseperable compacta must be, in some sense, complex. However until very recently it was unknown whether such compacta must always map onto $[0,1]^{\aleph_1}$ even if one assumes some strong assumption such as PFA. This question was asked by S. Todorčević at the North Bay Summer Topology Conference in August 1997. The purpose of this note is to present a general method for constructing a c.c.c. nonseparable compactum which does not map onto $[0,1]^{\aleph_1}$. This answers the question of S. Todorčević but leaves the following version of the same question (also due to S. Todorčević) open: Is it possible that every c.c.c. compactum without a σ-linked base maps onto $[0,1]^{\aleph_1}$?

The construction of this paper can be considered a general way of associating a compact space to a gap in quotient algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$. For the all of the desired properties to be present in the space we need that the gap be linear, have both sides countably directed, and at least one side $\aleph_1$ directed. This method is a generalization of the special case $\mathcal{I} = \text{fin}$ which has been considered already in [2], the original version of [8], [10], and other papers. The trouble with using gaps in $\mathcal{P}(\mathbb{N})/\text{fin}$ is that OCA destroys all of the gaps in this algebra which are useful in constructing Souslinean spaces (see [10]). It was rather surprising when I. Farah discovered that such gaps do exist only on the basis of ZFC in the algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for some F_σ P-ideal $\mathcal{I}$ [3]. Thus while the natural partition one associates to such gaps (see [10, §8.6]) is still open and therefore subject to the consequences of OCA, neither of the two alternatives of OCA yield a contradiction.

I would like to emphasize that while at first sight the ideals and the gap which will be constructed differ from those I. Farah originally built in [3], the basic idea behind the construction presented in this paper is essentially the same. It should also be pointed out that the final version of the paper [8] contains a construction of a c.c.c. nonseperable compact space that does not map onto $[0,1]^{\aleph_1}$ which is difference from the one in this paper (and does not require the use of MA). While this space is more optimal in that it does not require additional set theoretic assumptions I feel that it is worth presenting an alternate construction, particularly as it fits into the more general framework of constructing a topological space from a gap.

The compact space of interest will be the Stone space of a certain Boolean algebra. Consequently the bulk of this paper will focus on the construction of a Boolean algebra and will be inherently algebraic in nature. Rather than give a technical outline of how to modify existing constructions, I will focus first on building the Boolean algebra from a gap which is given as a parameter. Section one will introduce a template for building a Boolean algebra from a gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. In sections two and three I will prove different assertions about the Boolean algebra and its Stone space, each at the cost of a restriction which must be placed on the gap I use as a parameter. Section three will also contain some rationalization for considering the notion of being linearly fibered. The final section will close the paper with the construction of a gap which has all the attributes required for the claims in the previous sections.

It should be remarked here that, particularly in sections one and three, I am only modifying existing techniques. The main advance is in showing how to get an algebra to satisfy the c.c.c. in the general setting. For those readers interested in a broader discussion of associating c.c.c. compact spaces with certain structures in $\mathcal{P}(\mathbb{N})/\mathcal{I}$, I recommend [8]. This source also contains an extensive list of related references. I would like to thank S. Todorčević for his thoughts and insights related to this problem and also the referee for offering some suggestions on how to improve this paper.

2. A template for building a Boolean algebra from a gap.

Since both gaps and ideals will be a recurring theme throughout this paper I will first take the time to review some of the associated definitions. The notation $A \subseteq_* B$ is the usual abbreviation for "$A \setminus B$ is finite," where A and B are sets of integers. The set $\mathcal{P}(\mathbb{N})$ is given the standard product topology when viewed as the set $2^{\mathbb{N}}$.

Definition 2.1. An *ideal* $\mathcal{I}$ on $\mathbb{N}$ is a subset of $\mathcal{P}(\mathbb{N})$ which is closed under taking finite unions and subsets. In addition $\mathcal{I}$ is said to be a P-ideal if $(\mathcal{I}, \subseteq_*)$ is σ-directed (every countable set has an upper bound in $\mathcal{I}$). An ideal on $\mathbb{N}$ is dense if every infinite set contains an infinite subset in the ideal.

Throughout the rest of this paper I will write "ideal" when I really mean "ideal on some countable set."

Definition 2.2. If A and B are subsets of $\mathbb{N}$ and $\mathcal{I}$ is an ideal on $\mathbb{N}$ then $A \subseteq_{\mathcal{I}} B$ abbreviates $A \setminus B \in \mathcal{I}$ and $A \perp_{\mathcal{I}} B$ abbreviates $A \cap B \in \mathcal{I}$. A pair $(\mathcal{A}, \mathcal{B})$ of subsets of $\mathcal{P}(\mathbb{N})$ is said to be *orthogonal* modulo an ideal $\mathcal{I}$ on $\mathbb{N}$ (or orthogonal in $\mathcal{P}(\mathbb{N})/\mathcal{I}$) if $A \perp_{\mathcal{I}} B$ whenever A is in $\mathcal{A}$ and B is in $\mathcal{B}$.

Definition 2.3. A subset C of $\mathbb{N}$ is a said to *split* a set $S \subseteq \mathcal{A} \times \mathcal{B}$ modulo an ideal $\mathcal{I}$ if $A \subseteq_{\mathcal{I}} C$ and $B \perp_{\mathcal{I}} C$ whenever (A, B) is in S. If there does not exist a C which splits a subset S of $\mathcal{A} \times \mathcal{B}$ and $\mathcal{A}$ is orthogonal to $\mathcal{B}$ then S is said to be a *gap* modulo $\mathcal{I}$ (or a gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$). If $S = \mathcal{A} \times \mathcal{B}$ then I will simply say $(\mathcal{A}, \mathcal{B})$ is a gap modulo $\mathcal{I}$.

It is worth noting here that this is a more general definition than the one which is frequently given to the term gap (usually both sides are required to be well ordered by $\subseteq_*$). The advantage of this definition is that is allows us to examine gaps which are definable — something which will be of use to us later. The typical definability restriction on ideals is that they are *analytic*, i.e. the continuous image of a Polish space. All of the ideals mentioned in this paper are either F_σ or $F_{\sigma\delta}$ subsets of $\mathcal{P}(\mathbb{N})$. I will remark more on this in section three.

Of course all of the above definitions make sense if $\mathbb{N}$ is replaced by some other countable set. For technical reasons which will become apparent in the final section it will be useful to view the following objects as existing in $\mathcal{P}(R)$ for some infinite symmetric subset R of $\mathbb{N} \times \mathbb{N}$ for which the projection maps are finite to one. There will be a natural way to express R as an increasing union of finite sets $R_n \subseteq R_{n+1} \subseteq \omega$. From this point on $(\mathcal{A}, \mathcal{B})$ will be a gap modulo an F_σ P-ideal $\mathcal{I}$. The ideal $\mathcal{I}$ will moreover be generated by the collection of all finite changes of some compact set $\mathcal{K} \subseteq \mathcal{I} \subseteq \mathcal{P}(R)$ (it is easy to verify that in fact all F_σ P-ideals are of this form). I will also assume that all finite subsets of R are in $\mathcal{I}$ and that $\mathcal{K}$ is closed under subsets.

Let $T = \{(t,n) : t \subseteq R_n\}$ and define $(s,m) \lessdot (t,n)$ to be end extension, that is $m \leq n$ and $t \cap R_m = s$. Also, for $n \in \mathbb{N}$, set $\mathcal{K}_n = \{K \cap R_n : K \in \mathcal{K}\}$. Instead of considering an arbitrary member of $\mathcal{A} \times \mathcal{B}$, it will be necessary to restrict our attention to

$$\mathcal{A} \otimes \mathcal{B} = \{(A,B) \in \mathcal{A} \times \mathcal{B} : A \cap B \in \mathcal{K}\}.$$

Note that for every $A \in \mathcal{A}$, $B \in \mathcal{B}$ there is an n such that $(A \setminus R_n, B \setminus R_n)$ is in $\mathcal{A} \otimes \mathcal{B}$ and hence $\mathcal{A} \otimes \mathcal{B}$ is also a gap modulo $\mathcal{I}$.

In addition to $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$, and $\mathcal{K}$, the parameters will also include a subset Γ of $\mathcal{A} \otimes \mathcal{B}$ which also forms a gap modulo $\mathcal{I}$. Additional restrictions will be placed on Γ in section three. If $(A,B) \in \Gamma$ and $(t,n) \in T$ then define

(1) The type (a) generator $T_{(A,B)} = \{(s,m) \in T : ((A \cap R_m) \setminus s \in \mathcal{K}_m)$ and $(B \cap R_m \cap s \in \mathcal{K}_m)\}$.
(2) The type (b) generator $T_{(t,n)} = \{(s,m) \in T : ((s,m) \lessdot (t,n))$ or $((t,n) \lessdot (s,m))\}$.

If $C \subseteq R$ then let

(3) $b_C = \{(C \cap R_m, m) \in T : m \in \mathbb{N}\}$.

From this define the Boolean algebras

(4) $\mathcal{X} = \langle T_{(A,B)}, T_{(t,n)} : (A,B) \in \Gamma, (t,n) \in T\rangle/\mathrm{fin}$.
(5) $\mathcal{Y} = \langle T_{(t,n)} : (t,n) \in T\rangle/\mathrm{fin}$.

It is now useful to make some observations. First note that $(T, \lessdot)$ is a finitely branching tree. The following two facts are useful in dealing with elements of $\mathcal{X}$.

Fact 2.4. *A is in $\mathcal{K}$ if and only if $A \cap R_n$ is in $\mathcal{K}_n$ for all n.*

Proof. This follows from the compactness of $\mathcal{K}$. □

Fact 2.5. *If F is a positive element of $\mathcal{X}$ then there is a finite collection of generators whose meet is positive and contained in F.*

Proof. Observe that if B is the complement of a generator and (t,n) is in B, then

$$T_{(t,n)} \subseteq_* \{(s,m) \in T : (t,n) \lessdot (s,m)\} \subseteq B$$

(for type (a) generators this is a consequence of the previous fact). By considering the disjunctive normal form of F and applying this observation the fact follows immediately. □

An ultrafilter v in $\mathcal{Y}$ records a unique subset C of R which splits a portion of the gap (for every n there is a unique $t_n \subseteq n$ such that $T_{(t_n,n)} \in v$ — let $C = \bigcup_{n=1}^{\infty} t_n$). If this ultrafilter is extended to a filter in $\mathcal{X}$ which contains type (a) generators, then the pairs (A, B) corresponding to these generators must be split modulo $\mathcal{K}$ by the set C (note that even though $\mathcal{K}$ is not an ideal, this notion still makes sense). The Boolean algebra we will be interested in is $\mathcal{X}$ and the map $f : \mathrm{st}(\mathcal{X}) \to \mathrm{st}(\mathcal{Y})$ defined by $u \mapsto u \cap \mathcal{Y}$ will be the map which witnesses that $\mathrm{st}(\mathcal{X})$ is linearly fibered.

We are now ready to make an important observation about $\mathcal{X}$. It is time to place our first restriction on $(\mathcal{A}, \mathcal{B})$:

I. Both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed when ordered by $\subseteq_*$.

This guarantees that the algebra will not be σ-centered. To see this suppose $\mathcal{X}$ is the union of countably many ultrafilters $\{v_n\}_{n=1}^{\infty}$. Then it is possible to find a countable sequence C_n of subsets of R which correspond to the unique infinite branch each v_n determines in $(T, \lessdot)$. For each n pick a pair $(A_n, B_n) \in \mathcal{A} \otimes \mathcal{B}$ which is not split by C_n modulo $\mathcal{I}$ (i.e. either $A_n \not\subseteq_{\mathcal{I}} C_n$ or $B_n \not\perp_{\mathcal{I}} C_n$). Since both sides of the gap are σ-directed, find a pair (A, B) such that $A_n \subseteq_* A \in \mathcal{A}$ and $B_n \subseteq_* B \in \mathcal{B}$ for all n. Notice that we may assume (A, B) is in $\mathcal{A} \otimes \mathcal{B}$. Pick a m such that $T_{(A,B)}$ is in v_m. The set C_m splits (A, B) modulo $\mathcal{K}$ and hence splits (A_m, B_m) modulo $\mathcal{I}$, a contradiction.

3. How to ensure $\mathcal{X}$ will satisfy the c.c.c..

Element of the Boolean algebra $\mathcal{X}$ can be thought of as a collection of splitters for some portion of the gap Γ modulo the compact set $\mathcal{K}$. If we are given that

II. in $\mathcal{A}$, every uncountable $\mathcal{C} \subseteq \mathcal{A}$ contains an uncountable $\mathcal{C}_0$ which is $\subseteq_*$ bounded

then there is a standard approach to showing that $\mathcal{X}$ satisfies the c.c.c.. As we will see later, this is usually the case if $\mathrm{MA} + \neg\mathrm{CH}$ holds (see the remark on the role of Martin's Axiom in the next section). The general idea is as follows. If $\mathcal{F} \subseteq \mathcal{X}$ is uncountable, then consider the members of $\mathcal{A}$ used in the definitions of the members of $\mathcal{F}$. Using II, refine $\mathcal{F}$ to an uncountable subfamily $\mathcal{F}_0$ such that there is a C_0 in $\mathcal{A}$ which bounds any A' such that $T_{(A',B')}$ is mentioned in $\mathcal{F}_0$. By making a finite modification to C_0, it is possible to produce a $C \subseteq R$ which splits many members of $\mathcal{F}_0$.

The real trick turns out to be how to make this finite modification. Loosely speaking, we are given a collection $\mathcal{F}$ of finite pieces of the gap, where each $F \in \mathcal{F}$ is split by some "local" splitter C_F modulo $\mathcal{K}$. We are also given some "global" splitter C which works for all of the pieces in $\mathcal{F}$, but only modulo the larger object $\mathcal{I}$. The goal is to repair C by altering some finite portion of it so that it also splits many members of $\mathcal{F}$ modulo $\mathcal{K}$.

The sufficient condition which I will use is that $(\mathcal{A}, \mathcal{B})$ is actually orthogonal modulo a smaller ideal $\mathcal{J}$ which satisfies the following "exchange" property:

III. For every J in $\mathcal{J}$ there are infinitely many n such that for every $K \in \mathcal{K}$ the set $(J \setminus R_n) \cup (K \cap R_n)$ is in $\mathcal{K}$.

We are now ready to prove the following lemma about $\mathcal{X}$ assuming that $(\mathcal{A}, \mathcal{B})$, Γ, $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ satisfies conditions II and III.

Claim 3.1. *$\mathcal{X}$ is has precaliber $\aleph_1$ and in particular satisfies the c.c.c..*

Proof. Pick an uncountable family $\mathcal{F}$ of positive elements of $\mathcal{X}$. Applying Fact 2.5 it may be assumed without loss of generality that the members of $\mathcal{F}$ are the meet of finitely many generators.

For each $F \in \mathcal{F}$ pick a finite set $S_F \subseteq \Gamma$ and a (t_F, n_F) in T such that

$$F = T_{(t_F,n_F)} \cap \bigcap_{(A,B)\in S_F} T_{(A,B)}.$$

Applying II it is possible to find an uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ and a $C_0 \in \mathcal{A}$ such that $A \subseteq_* C_0$ whenever $A \in \pi_{\mathcal{A}}(S_F)$ and $F \in \mathcal{F}_0$. If $F \in \mathcal{F}_0$, let

$$J_F = \bigcup_{(A,B)\in S_F} [(A \setminus C_0) \cup (B \cap C_0)]$$

and choose a C_F such that $b_{C_F} \subseteq F$.

Applying property III find a $N_F > n_F$ such that $(J_F \setminus R_{N_F}) \cup (K \cap R_{N_F})$ is in $\mathcal{K}$ whenever K is in $\mathcal{K}$. Now pick an uncountable subset $\mathcal{F}_1$ of $\mathcal{F}_0$ such that $N_F = N$ and $C_F \cap R_N = t$ for some fixed $N \in \mathbb{N}$ and $t \subseteq R_N$ whenever $F \in \mathcal{F}_1$. Let $C = (C_0 \setminus R_N) \cup t$.

I will now show that $b_C \subseteq F$ for all $F \in \mathcal{F}_1$. Let $F \in \mathcal{F}_1$ and $(A, B) \in S_F$. Since $F \subseteq T_{(t,N)} \cap T_{(A,B)}$ is nonempty, $K_A = A \cap R_N \setminus t$ and $K_B = B \cap R_N \cap t$ are both in $\mathcal{K}_N \subseteq \mathcal{K}$. It follows from the choice of N that

$$K_A \cup (J_F \setminus R_N), K_B \cup (J_F \setminus R_N) \in \mathcal{K}.$$

Thus

$$\begin{aligned} A \setminus C &\subseteq K_A \cup (J_F \setminus R_N), \\ B \cap C &\subseteq K_B \cup (J_F \setminus R_N) \end{aligned}$$

are both in $\mathcal{K}$ and therefore $b_C \subseteq T_{(A,B)}$. □

4. How to ensure $\mathcal{X}$ does not contain an uncountable independent family.

I will start this section with a condition which ensures a compact space will not map onto $[0,1]^{\aleph_1}$.

Definition 4.1. A compact space X is said to be *linearly fibered* if there is a continuous map $f : X \to [0,1]$ such that the inverse images of points are linearly orderable compacta.

Remark. Clearly this property is inherited to all closed subspaces. Consequently every closed subset E of a linearly fibered compact space X contains a linearly orderable subspace which is a G_δ set. Since every linear compactum contains a point of countable π-character, E must contain a point of countable π-character and X can not map onto $[0,1]^{\aleph_1}$ by a well known result of Shapirovskiĭ[7].

A c.c.c. nonseperable linearly fibered compactum can be thought of as a generalization of a Souslin line (which is linearly fibered by the constant map) and also of the c.c.c. nonseparable metrizably fibered compactum which exists if $\mathrm{MA}_{\aleph_1}$ fails (see [8]). The latter example shows that I am justified in assuming that $\mathfrak{c}$ is greater than $\aleph_1$ in this construction.

For $\mathrm{st}(\mathcal{X})$ to linearly fibered it is sufficient that

IV. Γ is well ordered by $\subseteq_* \times \subseteq_*$.

To ensure that it is possible to find such a linear sequence it suffices (in the presence of MA) to know that both $\mathcal{A}$ and $\mathcal{B}$ are analytic subsets of $\mathcal{P}(R)$. Before doing this, however, I will make a few remarks concerning analytic P-ideals.

It has recently been shown by S. Todorčević (see [9]) that if $\mathcal{A}$ is an analytic P-ideal then $(\ell_1, \leq)$ can be mapped monotonicly and cofinally into $(\mathcal{A}, \subseteq)$. Here ℓ_1 is the collection $\{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x(n)| < \infty\}$ of absolutely convergent series and the order $\leq$ is the coordinatewise order. Thus ideals which are associated with Lebesgue measure are the most complex, at least as far as analytic P-ideals go. If $\mathrm{add}_*(\mathcal{A})$ and $\mathrm{add}_*(\ell_1)$ are defined to be the sizes of the smallest unbounded families in $(\mathcal{A}, \subseteq_*)$ and $(\ell_1, \leq_*)$ respectively, then it follows that $\mathrm{add}_*(\ell_1) \leq \mathrm{add}_*(\mathcal{A})$. Since it also known (see [5, 33C]) that Martin's Axiom implies that $\mathrm{add}_*(\ell_1) = \mathfrak{c}$ we can conclude that, assuming Martin's Axiom, $\mathrm{add}_*(\mathcal{A}) = \mathfrak{c}$ for every analytic P-ideal $\mathcal{A}$.

Returning to our construction, if both sides of our gap are analytic P-ideals and Martin's Axiom holds, then there are cofinal $\subseteq_*$-increasing sequences $\{A_\xi\}_{\xi<\mathfrak{c}}$ and $\{B_\xi\}_{\xi<\mathfrak{c}}$ in $(\mathcal{A}, \subseteq_*)$ and $(\mathcal{B}, \subseteq_*)$ respectively. Certainly if this is true then there is a linear subgap $\Gamma \subseteq \mathcal{A} \otimes \mathcal{B}$. Note that if $\mathfrak{c}$ is greater than $\aleph_1$ then condition II will be satisfied as well.

Remark. The role Martin's axiom is twofold in this paper. First, due to the construction in [8] mentioned above, $\mathrm{MA}_{\aleph_1}$ can be assumed throughout the construction. The purpose of doing this is to guaranteed that condition II will be satisfied. This is not really a set theoretical assumption as far a constructing the c.c.c. non-separable linearly fibered compactum is concerned. The second role of Martin's Axiom (and its use as quoted in the title) is to obtain a linear subgap of the definable gap $(\mathcal{A}, \mathcal{B})$. It is unknown whether any assumption is necessary to find such a subgap — this question may be of interest in it's own right.

In the next section I will construct a $F_{\sigma\delta}$ gap $(\mathcal{A}, \mathcal{B})$ modulo an F_σ P-ideal $\mathcal{I}$ satisfying properties I-III and then apply MA to obtain the linear subgap Γ thus satisfying IV and completing the construction. From now on I will write $\Gamma = \{(A_\xi, B_\xi) : \xi < \lambda\}$ for some λ where $\eta < \xi$ implies that $A_\eta \subseteq_* A_\xi$ and $B_\eta \subseteq_* B_\xi$. To simplify the notation I will write T_ξ instead of $T_{(A_\xi, B_\xi)}$. Notice that since $\mathcal{Y}$ is countable, $\mathrm{st}(\mathcal{Y})$ is a 0-dimensional compact metric space and thus homeomorphic to a subspace of 2^ω. The following theorem now finishes the proof of our claims about $\mathrm{st}(\mathcal{X})$.

Proposition 4.2. *The map $f : \mathrm{st}(\mathcal{X}) \to \mathrm{st}(\mathcal{Y})$ defined by $f(u) = u \restriction \mathcal{Y}$ has fibers which are linear compacta.*

Proof. Let v be an ultrafilter on $\mathcal{Y}$ and define Γ_v to be the collection of all $\xi < \lambda$ for which $\{T_\xi\} \cup v$ is a filter. It now suffices to show that if $\eta < \xi \in \Gamma_v$ then $T_\eta \restriction v \supseteq T_\xi \restriction v$.

Pick a $m \in \mathbb{N}$ such that

$$\begin{aligned} A_\eta \setminus R_m &\subseteq A_\xi \setminus R_m, \\ B_\eta \setminus R_m &\subseteq B_\xi \setminus R_m \end{aligned}$$

and let $s \subseteq R_m$ be the unique set such that $T_{(s,m)} \in v$. If $(s,m) \lessdot (t,n)$ is in T_ξ then

$$\begin{aligned} (A_\xi \cap (R_n \setminus R_m)) \setminus t &\in \{K \cap (R_n \setminus R_m) : K \in \mathcal{K}\}, \\ (B_\xi \cap (R_n \setminus R_m)) \cap t &\in \{K \cap (R_n \setminus R_m) : K \in \mathcal{K}\}. \end{aligned}$$

Since

$$\begin{array}{rcl} A_\eta \cap (R_n \setminus R_m) & \subseteq & A_\xi \cap (R_n \setminus R_m), \\ B_\eta \cap (R_n \setminus R_m) & \subseteq & B_\xi \cap (R_n \setminus R_m), \end{array}$$

we also have that

$$\begin{array}{rcl} (A_\eta \cap (R_n \setminus R_m)) \setminus t & \in & \{K \cap (R_n \setminus R_m) : K \in \mathcal{K}\}, \\ (B_\eta \cap (R_n \setminus R_m)) \cap t & \in & \{K \cap (R_n \setminus R_m) : K \in \mathcal{K}\}. \end{array}$$

Because $T_\eta \cap T_{(s,m)} \neq \emptyset$, $(A_\eta \cap R_m) \setminus t \in \mathcal{K}_m$. Therefore $(A_\eta \cap R_n) \setminus t \in \mathcal{K}_n$ and $B_\eta \cap R_n \cap t \in \mathcal{K}_n$. Thus $(t, n) \in T_\eta$ and $T_\eta \restriction v \supseteq T_\xi \restriction v$. $\square$

5. How to build the parameters which satisfy conditions I-IV.

I will now construct an analytic gap with the properties specified in the previous sections. First it is necessary to make some preliminary definitions. For $A \subseteq \mathbb{N}$ define $\mu(A) = \sum_{n \in A} 1/n$. Let $a_n = \ln 2 - \max\{\mu(A) : A \subseteq n \text{ and } \mu(A) < \ln 2\}$. [1] Then for all n it is true that $1 > a_n \geq a_{n+1} > 0$. Also it is clear that $\lim_n a_n = 0$. Define $h : \mathbb{N} \to \mathbb{N}$ by setting $h(k)$ to be the least integer n such that $1/n$ is less than $a_k/2^{k+1}$. Define $g : \mathbb{N} \to \mathbb{N}$ recursively so that $g(1) = h(1)$ and $g(n+1) = h(g(n))$. For convenience I will also define $h_k(n) = h(kn)$. Let $E : [\mathbb{N}]^\omega \leftrightarrow \mathbb{N}^{\uparrow\mathbb{N}}$ denote the canonical bijection which identifies subsets of $\mathbb{N}$ with their increasing enumeration. It will be useful to think of E as being defined on the on the finite sets as well: if a set A has no n^{th} element then set $E(A)(n) = \infty$. I will use $[m, n]$ denote the interval of integers between (and including) m and n. Define

(6) $u_n = [g(n) + 1, g(n+1)] = [g(n) + 1, h(g(n))]$,
(7) $R = \bigcup_{n=1}^{\infty} u_n \times u_n$, and
(8) $R_n = \bigcup_{i=1}^{n} u_i \times u_i$.

Note that $h(n)$ is at least $2n$ and therefore

$$\mu(u_n) \geq \sum_{i=g(n)+1}^{2g(n)} 1/n \geq g(n)\frac{1}{2g(n)} = \frac{1}{2}.$$

Define the following:

(9) $\mathcal{L}_0 = \{L \subseteq \mathbb{N} : \mu(\{n \in \mathbb{N} : u_n \cap L \neq \emptyset\}) < \infty\} \cap \{L \subseteq \mathbb{N} : \forall k (h_k <_* E(L))\}$
(10) $\mathcal{L}_1 = \{L \subseteq \mathbb{N} : \mu(L) < \infty\}$
(11) $\mathcal{A} = \{A \subseteq R : \pi_1(A) \in \mathcal{L}_0\}$
(12) $\mathcal{B} = \{B \subseteq R : \pi_2(B) \in \mathcal{L}_0\}$
(13) $\mathcal{I} = \{I \subseteq R : \pi_1(I) \cup \pi_2(I) \in \mathcal{L}_1\}$
(14) $\mathcal{J} = \mathcal{A} \cap \mathcal{B} = \{J \subseteq R : \pi_1(J) \in \mathcal{L}_0 \text{ and } \pi_2(J) \in \mathcal{L}_0\}$
(15) $\mathcal{K} = \{K \subseteq R : \mu(\pi_1(K)) \leq \ln 2 \text{ and } \mu(\pi_2(K)) \leq \ln 2\}$

Remark. Notice that $\mathcal{L}_0 \subseteq \mathcal{L}_1$ since whenever $2^n <_* E(L)$ it always follows that $\mu(L) < \infty$. From this it is immediate that $\mathcal{J} \subseteq \mathcal{I}$. Since $\mathcal{J} = \mathcal{A} \cap \mathcal{B}$, it follows automatically that $\mathcal{A} \perp_{\mathcal{J}} \mathcal{B}$ and $\mathcal{A} \perp_{\mathcal{I}} \mathcal{B}$.

The following lemma will handle conditions I and IV.

Lemma 5.1. *All of the collections mentioned in 9-15 are dense analytic P-ideals and $\mathcal{K}$ is compact.*

[1] There is nothing particularly special about $\ln 2$ — any irrational number between 0 and 1 will work equally well.

Proof. The compactness of $\mathcal{K}$ follows from the fact that for any set $L \subseteq \mathbb{N}$, $\mu(L) > \ln 2$ iff $\mu(F) > \ln 2$ for some finite subset F of L. It is a routine exercise in descriptive set theory to verify that all the remaining objects are $F_{\sigma\delta}$. It is easily seen that $\mathcal{L}_1$ is a dense P-ideal and since π_1, π_2 are finite-to-one maps, it suffices to show that

$$\mathcal{L} = \{L \subseteq \mathbb{N} : \forall k(h_k <_* E(L))\}$$

is a dense P-ideal. Let $\{L_k\}_{k=1}^{\infty}$ be a sequence of elements of $\mathcal{L}$. For each k, pick a $n_k > k$ such that $h_{k(k+1)}(n) < E(L_i)(n)$ whenever $i \leq k$ and $n > n_k$. Now let

$$L = \bigcup_{k=1}^{\infty} L_k \setminus [1, n_k].$$

To see that L is in $\mathcal{L}$, let $k \in \mathbb{N}$ be given and $q > n_k$ and $r < k$. Notice that

$$E(L)(qk + r) \geq E(L)(qk) \geq \min\{E(L_i)(q) : i \leq \max\{j : n_j \leq qk + r\}.$$

Furthermore the right hand side is at least

$$h_{k(k+1)}(q) = h_k(q(k + 1)) \geq h_k(qk + r)$$

by our choice of q (note that $r < k \leq n_k < q$).

The density of $\mathcal{L}$ follows from the fact that for any $f \in \mathbb{N}^{\mathbb{N}}$ and any infinite set $L \subseteq \mathbb{N}$, there is an infinite set L_0 such that $f <_* E(L_0)$. □

Lemma 5.2. *If J is in $\mathcal{J}$, there are infinitely many n such that $\mu(\pi_i(J \setminus R_n)) < a_{g(n+1)}$, for $i = 1, 2$ and hence $\mathcal{J}$ and $\mathcal{K}$ satisfy condition III.*

Proof. Pick a N_0 such that $h(n) < E(\pi_i(J))(n)$ for all $n > N_0$, $i = 1, 2$ and let $N = \max_i E(\pi_i(J)(N_0 + 1))$. Notice that for all n, $i = 1, 2$

$$E(E^{-1}(h) \setminus [1, g(N + 1)])(n) < E(\pi_i(J) \setminus [1, g(N + 1)])(n)$$

Now for infinitely many $n > N$, $u_n \times u_n \cap J = \emptyset$. Thus for such n

$$\begin{aligned} \mu(\pi_i(J \setminus R_n)) = \mu(\pi_i(J \setminus R_{n+1})) \quad & \leq \textstyle\sum_{k=g(n+1)}^{\infty} 1/h(k) \\ & \leq \textstyle\sum_{k=g(n+1)}^{\infty} a_k/2^{k+1} \\ & \leq a_{g(n+1)} \textstyle\sum_{k=g(n+1)}^{\infty} 1/2^{k+1} \\ & < a_{g(n+1)}. \end{aligned}$$

The proceeding inequality follows from this dominance and the fact that the least element in $\pi_i(J \setminus R_n)$ is at least $g(n + 2) = h(g(n + 1))$, for $i = 1, 2$. □

Finally I will use a Fubini style argument to show that $(\mathcal{A}, \mathcal{B})$ is indeed a gap as I have promised all along.

Lemma 5.3. *$(\mathcal{A}, \mathcal{B})$ is a gap in $\mathcal{P}(R)/\mathcal{I}$.*

Proof. Suppose that $C \subseteq R$. If $A \subseteq R$, define

$$\mu^2(A) = \sum_{(n,m) \in A} 1/mn$$

and note that this is just the product measure when restricted to the finite rectangles $u_n \times u_n$ (since μ is determined by its value on the singletons). Set

$$c_n = \mu^2(u_n \times u_n \cap C)/\mu^2(u_n \times u_n).$$

I now will consider two overlapping cases.

Case 1. $L = \{k \in \mathbb{N} : c_k \geq 1/2\}$ is infinite. Let L_1 be an infinite subset of L such that $\mu(L_1)$ is finite. Then for each k in L_1, let n_k be an element of u_k such that

$$\mu(\{m : (m, n_k) \in C\}) \geq (1/2)\mu(u_k).$$

This choice is possible by Fubini's theorem. Now let

$$B = C \cap \bigcup_{k \in L_1} u_k \times \{n_k\}.$$

By choice of n_k,

$$\mu(\pi_1(B)) \geq \sum_{k \in L_1} (1/2)\mu(u_k) = \infty.$$

On the other hand, $\pi_2(B) \cap u_k$ contains at most one element and thus $h_k <_* g \leq_* E(\pi_2(B))$ for all k. Furthermore $\{n \in \mathbb{N} : \pi_2(B) \cap u_n \neq \emptyset\} = L_1$ and therefore $B \in \mathcal{B} \setminus \mathcal{I}$.

Case 2. $\mathbb{N} \setminus L$ is infinite. It is now possible to apply a symmetric argument to find an $A \in \mathcal{A} \setminus \mathcal{I}$ such that $A \cap C = \emptyset$. □

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