CONTINUOUS COLORINGS ASSOCIATED WITH CERTAIN CHARACTERISTICS OF THE CONTINUUM

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ABSTRACT. A coloring $c : [X]^n \to Z$ is said to be irreducible if for every $Y \subseteq X$ of equal cardinality $c''[Y]^n = Z$. The focus of this note will be to show that there are continuous irreducible colorings on sets of reals associated with various cardinal invariants of the continuum. It is interesting that some of the colorings make crucial use of exponential lower bounds which have been proven for a certain class of finite Ramsey numbers.

1. Introduction

When presented with a coloring $c : [X]^n \to Z$ one would frequently like to know whether it is possible to find a set $Y \subseteq X$ of substantial size such that the image of $[Y]^n$ under c is as small as possible and in particular not equal to the whole set of colors Z (here $[X]^n$ is the collection of all n element subsets of X). It is often the case that "substantial size" means "having the same cardinality as X." If a coloring $c : [X]^n \to Z$ does not admit to such a Y that would have the same size as X then c is said to be *irreducible*. The general problem is to find irreducible colorings when X and n are small, Z is large, and c is of low complexity. Since Ramsey's theorem trivializes the case when X is countable, this notion will only be considered when X is an uncountable set.

If no restrictions are placed on the type of coloring which is allowed, then we have the following optimal result due to S. Todorčević [8]: there is a coloring $c : [\omega_1]^2 \to \omega_1$ which is irreducible. In light of this, it is natural to ask what happens when some restrictions are placed on the complexity of the coloring c. A natural such restriction is to require that c be continuous (another is the countable chain condition — see the appendix of this note). Here X and Z are typically subspaces of ω^ω and $[X]^n$ is given the natural topology by identifying it with an appropriate subspace of X^n. Thus requiring that c is continuous is the same as saying that to determine a finite initial segment of $c(x, y)$ one only needs to know finite initial segments of x and y. Again, in dimension 3 there is an optimal result due to S. Todorčević [12]: there is a continuous irreducible coloring $c : [X]^3 \to X$, where X is a subset of ω^ω of size $\aleph_1$.

The situation $n = 2$ is much different for continuous colorings. As we will see in a moment, it seems to be most natural to ask which cardinal invariants of the continuum have continuous irreducible colorings associated with them. Each of these invariants reflect a different aspect of the continuum and have values which can change in different models of set theory. Some examples of these cardinal invariants are the size of the smallest family of measure 0 sets whose union is not measure 0 (denoted $\mathrm{add}(\mathcal{N})$), the size of the smallest cover of the real line by nowhere dense sets (denoted $\mathrm{cov}(\mathcal{M})$), and the size of the smallest subset of

$(\omega^\omega, <^*)$ which is cofinal in the order (the dominating number $\mathfrak{d}$). These so called "small cardinals" all have values that lie somewhere between $\aleph_1$ and $\mathfrak{c}$ and can be used to gauge which effects of the Continuum Hypothesis remain when $\mathfrak{c}$ is greater than $\aleph_1$. Both the relationships of these cardinals to one another and their values in specific models of ZFC have been the subject of extensive research (see [2], [13], and [3] for an introduction to this subject).

It is known to be consistent with ZFC that for every uncountable set of reals X and every continuous coloring $c : [X]^2 \to 2$ there is an uncountable $Y \subseteq X$ such that $c \restriction [Y]^2$ is constant [1] (this is in fact a trivial consequence of the formulation of OCA given below). As the existence of continuous irreducible colorings in two dimensions seems to be most interesting (and approachable) when $Z = \omega$, our discussion will be restricted to this case for the duration of this note. What remains is the question of which cardinal invariants of the continuum have continuous irreducible colorings associated with them.

In the 1980's it was shown that the cardinality of the continuum has such a coloring associated with it (see [10] for a proof and also [6] for an earlier, noncontinuous version of this result). The focus of this note will be to add two new cardinals characteristics of the continuum to this list. One is the well known unbounding number $\mathfrak{b}$ of the structure ω^ω under the ordering of eventual dominance. The second is the additivity of a certain family $\mathcal{N}_*$ of measure 0 sets. In particular $\text{add}(\mathcal{N}_*)$ is at most $\text{non}(\mathcal{N})$, the cardinality of the smallest sets of reals which does not have measure 0.

I will close the introduction with a definition of OCA and a few remarks on how the results of this note can be interpreted. The currently quoted formulation of OCA [1] is the statement that every open graph G on a set of reals is either countably chromatic or else contains an uncountable complete subgraph (the simplest reason why a graph can't be countably chromatic). Here open means that the edge set E of G is an open subset of $[V]^2$, where V is the vertex set of G. For an introduction to OCA and the fact that is implies $\mathfrak{b} = \aleph_2$, the reader is referred to [9, §8].

One way to interpret the results which follow is that they are statements about what OCA can prove and how much of the full power of OCA is needed when proving a given statement. I prefer to think of the results as ZFC theorems instead and simply note that these implications are corollaries, as this approach sheds much more light on the situation. For instance, while the results of this note show that OCA implies $\text{non}(\mathcal{N}) \geq \text{add}(\mathcal{N}_*) > \aleph_1$, there seems to be no reason to believe that there is a continuous irreducible coloring associated with $\text{non}(\mathcal{N})$. We only know that there is a such a coloring on a possibly smaller set of reals of size $\text{add}(\mathcal{N}_*)$.

2. Notation

Before beginning I will first introduce some notation, most of which has become standard. The ordinal ω is the set $\{0, 1, \ldots\}$ of all nonnegative integers with the discrete topology. Each nonnegative integer n is viewed as the set $\{0, 1, \ldots, n-1\}$ of its predecessors ($0 = \emptyset$). If A and B are sets, A^B is the set of all functions from B to A. The collections $\omega^{<\omega}$ and $2^{<\omega}$ are the sets of all finite sequences of elements of ω and 2 respectively. If A is a set then $\#(A)$ refers to its cardinality and if t is

[1]Several versions of OCA first appeared in [1]. The above definition of OCA is due to S. Todorčević and is the one which is generally quoted in current literature.

a sequence, $|t|$ refers to its length. To avoid confusion and to improve the esthetics of the notation I will write $c''S$ to denote the image of a set S under a map c.

The spaces 2^ω and ω^ω are equipped with the product topology which is also compatible with the metric topology induced by $d(x,y) = 1/\Delta(x,y)$, where $\Delta(x,y) = \min\{n : x(n) \neq y(n)\}$. If t is in ω^n for some n then t defines the basic open set $[t] = \{x \in \omega^\omega : x \restriction n = t\}$. Similarly if $F \subseteq 2^{<\omega}$ is finite then it determines the basic open set $[F] = \{x \in 2^\omega : \exists n(x \restriction n \in F)\}$. Thus if $X \subseteq \omega^\omega$ and $c : [X]^2 \to \omega$ is a coloring, c is continuous iff the value of c at a pair $\{x, y\}$ can be determined by only knowing a finite amount of information about x and y. Furthermore if $F \subseteq 2^{<\omega}$ is finite and no two elements of F are comparable in the ordering of end extension, the measure $\mu([F])$ of this open set is equal to $\sum_{t\in F} 2^{-|t|}$.

If $X \subseteq \omega^\omega$ and $t \in \omega^n$ then t is said to be a node of X if $[t] \cap X$ is nonempty. A splitting node t of a set X is a node of X such that for infinitely many i the concatenation $t^\frown i$ is also a node of X. If $f, g \in \omega^\omega$ then I will write $f <^* g$ iff for all but finitely many n $f(n) < g(n)$. The cardinal $\mathfrak{b}$ is the size of the smallest unbounded family in the ordering $(\omega^\omega, <^*)$.

The null ideal $\mathcal{N}$ is the collection of all measure 0 subsets of 2^ω. Suppose that $\mathcal{I}, \mathcal{J} \subseteq \mathcal{N}$ are closed under subsets and finite unions. The cardinals $\mathrm{add}(\mathcal{I})$, $\mathrm{add}(\mathcal{I}, \mathcal{J})$, and $\mathrm{non}(\mathcal{I})$ are defined in the usual way:

$$\begin{aligned}\mathrm{add}(\mathcal{I}) &= \min\{\#(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{I} \wedge \cup\mathcal{A} \notin \mathcal{I}\}\\ \mathrm{add}(\mathcal{I}, \mathcal{J}) &= \min\{\#(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{I} \wedge \cup\mathcal{A} \notin \mathcal{J}\}\\ \mathrm{non}(\mathcal{I}) &= \min\{\#(S) : S \subseteq 2^\omega \wedge S \notin \mathcal{I}\}.\end{aligned}$$

The reader is referred to [2] for more information on real line combinatorics.

The main result of Section 3 relies heavily on some lower bounds which have been proven for a certain class of finite Ramsey numbers. These numbers make the notion of "substantially large" mentioned earlier precise in the context of finite colorings. If m, k, l and r are integers, then $(m)^r_{k/l}$ is the smallest integer n such that for any coloring $c : [n]^r \to k$, there is a set $S \subseteq n$ of size m such that $c''[S]^2$ has size at most l. I will use the notation $[m]^r_k$ to abbreviate $(m)^r_{k/k-1}$. For more information on partition calculus, the reader is referred to [5].

3. An irreducible coloring associated with measure

In this section I will introduce an ideal $\mathcal{N}_* \subseteq \mathcal{N}$ such that there is a continuous irreducible coloring associated with $\mathrm{add}(\mathcal{N}_*)$.

It is well known that a set $G \subseteq 2^\omega$ has measure 0 iff there is a sequence $F_G = \langle F_G(n) : n \in \omega\rangle$ such that

1. $F_G(n) \subseteq 2^n$,
2. $G \subseteq \bigcap_{M=1}^\infty \bigcup_{n=M}^\infty [F_G(n)]$, and
3. $\sum_{n=1}^\infty \#(F_G(n))/2^n < \infty$.

Such a sequence will be called a *cover* of G. If $r \in (0,1)$, cover F_G of G is said to be r *nice* if

$$\lim_n \#(F_n) n^{2-r}/2^n = 0.$$

Thus nice covers are those for which the sum mentioned above converges for a specific reason. Define $\mathcal{N}_r$ to be the collection of all subsets of 2^ω which have a r nice cover. Notice that if $r < s$ then $\mathcal{N}_r \subseteq \mathcal{N}_s \subseteq \mathcal{N}$. It is easy to see that $\mathrm{add}(\mathcal{N}_r, \mathcal{N}_s) \leq \mathrm{non}(\mathcal{N}_s) \leq \mathrm{non}(\mathcal{N})$. Also, if $r \leq a < b \leq s$ then $\mathrm{add}(\mathcal{N}_a, \mathcal{N}_b) \leq \mathrm{add}(\mathcal{N}_r, \mathcal{N}_s)$. Thus using a well foundedness argument on the ordinals, it is possible to find a pair

$r < s$ in $(0,1)$ such that for all $a < b$ in $[r,s]$, $\mathrm{add}(\mathcal{N}_r,\mathcal{N}_s) = \mathrm{add}(\mathcal{N}_a,\mathcal{N}_b)$. Define $\mathcal{N}_* = \bigcap_{b>r} \mathcal{N}_b$.

Lemma 3.1. *The invariants* $\mathrm{add}(\mathcal{N}_*)$ *and* $\mathrm{add}(\mathcal{N}_r,\mathcal{N}_s)$ *are equal. Moreover there are* $a < b$ *in* (r,s) *and* $\mathcal{A} \subseteq \mathcal{N}_a$ *of size* $\mathrm{add}(\mathcal{N}_*)$ *which is well ordered by* $\subseteq$ *and unbounded in* $\mathcal{N}_b$.

Proof. It is easy to see that $\mathcal{N}_r \subseteq \mathcal{N}_* \subseteq \mathcal{N}_s$ and therefore that $\mathrm{add}(\mathcal{N}_r,\mathcal{N}_s) \geq \mathrm{add}(\mathcal{N}_*)$. Now let $\mathcal{A}$ be an unbounded subset of $\mathcal{N}_*$ of size $\mathrm{add}(\mathcal{N}_*)$. It can be assumed without loss of generality that $\mathcal{A}$ is well ordered by $\subseteq$. Since $\mathcal{A}$ is unbounded, there is a $b > r$ such that $\cup\mathcal{A}$ is not in $\mathcal{N}_b$. Let a be any element of (r,b). Now clearly $\mathcal{A} \subseteq \mathcal{N}_a$ and is unbounded in $\mathcal{N}_b$. Thus $\mathcal{A}$ must have size $\mathrm{add}(\mathcal{N}_a,\mathcal{N}_b) = \mathrm{add}(\mathcal{N}_r,\mathcal{N}_s)$. □

From this point on a and b will remain fixed and d will be any number such that $a < d < b$.

Theorem 3.2. *There is a subset* X *of* ω^ω *of size* $\mathrm{add}(\mathcal{N}_*)$ *and a continuous irreducible coloring* $c : [X]^2 \to \omega$.

The existence the desired irreducible partition for the cardinal $\mathrm{add}(\mathcal{N}_*)$ is a consequence of the following sequence of results. It is rather interesting that this coloring makes crucial use of exponential lower bounds for a certain class of finite Ramsey numbers.

The following fact for $k = 2$ and $l = 1$ is essentially a well known result of P. Erdös [4] (see also [5, §26]) [2] and the methods presented in the proof can readily be adapted to give us the following result. While this theorem was almost certainly known to P. Erdös (and others), I have included the proof for completeness and because it is not to my knowledge in print elsewhere in this generality.

Theorem 3.3. *If* $\binom{k}{l} \leq m!$ *then* $(m)^2_{k/l} > (k/l)^{(m-1)/2}$. *In particular there is a constant* $\alpha > 0$ *such that* $[m]^2_k > 2^{\alpha m/k}$.

Proof. First note that for a fixed integer n, the number of colorings $c : [n]^2 \to k$ is $N = k^{\binom{n}{2}}$. If S is a fixed subset of n of size m and $L \subseteq k$ is a collection of l colors, then there are $l^{\binom{m}{2}} k^{\binom{n}{2}-\binom{m}{2}}$ many colorings $c : [n]^2 \to k$ which also satisfy $c''[S]^2 \subseteq L$. Since there are $\binom{n}{m}$ ways to choose S and $\binom{k}{l}$ ways to choose L, there are at most

$$N_{m,l} = \binom{n}{m}\binom{k}{l} l^{\binom{m}{2}} k^{\binom{n}{2}-\binom{m}{2}}$$

many colorings c of $[n]^2$ such that there is a subset of n of size m which realizes at most l colors. It now suffices to show that if $n \leq (k/l)^{(m-1)/2}$ then $N_{m,l} < N$. I will use the approximation $\binom{n}{m} < n^m/m!$.

$$\begin{aligned}
n &\leq (k/l)^{(m-1)/2} \\
n^m &\leq (k/l)^{m(m-1)/2} \\
\frac{n^m}{m!}\binom{k}{l} &< (k/l)^{\binom{m}{2}} \\
\binom{n}{m}\binom{k}{l} &< (k/l)^{\binom{m}{2}} \\
\binom{n}{m}\binom{k}{l} l^{\binom{m}{2}} k^{\binom{n}{2}-\binom{m}{2}} &< k^{\binom{n}{2}} \\
N_{m,l} &< N
\end{aligned}$$

[2] Erdös actually showed $(m)^2_{2/1} \geq 2^{m/2}$.

To see that the constant α exists, note that

$$1/k \leq \int_{k-1}^{k} \frac{1}{x} dx = \ln(\frac{k}{k-1}).$$

□

Suppose that T is a subset of ω^ω and $f \in \omega^\omega$ is a function. I will say that T is f *thin* if for all but finitely many n in ω and every t in ω^n the concatenation $t\hat{\ }i$ is a node of T for at most $f(n)$ many i (i.e. the splitting of the nodes of T is bounded by f). Similarly I will call T f *splitting* if T does not contain a f thin subset of the same size. Define $h(n) = 2^{\sqrt{n}}$ and let h_k be the k fold composition of h with itself.

Lemma 3.4. *Suppose T is any subset of ω^ω which is $h(f) = 2^{\sqrt{f}}$ thin for some f with $\lim_n f(n) = \infty$. It follows that there is a continuous coloring $c : [T]^2 \to \omega$ such that $c''[T_0]^2 = \omega$ for any $T_0 \subseteq T$ which is not f thin.*

Proof. Notice first that for any $h(f)$ thin subset T of ω^ω there is an isometry which embeds T into $\prod_{n=1}^{\infty} h(f(n))$ (this is in fact an equivalent formulation of "thinness"). Thus I will work in $\prod_{n=1}^{\infty} h(f(n))$ for convenience. Let $G(n)$ be the greatest integer k such that $2^{\alpha f(n)/k} \leq h(f(n))$. Note that G is both well defined and satisfies $\lim_n G(n) = \infty$. For each n pick a coloring $c_n : [h(f(n))]^2 \to G(n)$ such that for every subset S of $h(f(n))$ having size $f(n)$, $c_n''[S]^2 = G(n)$ (this is precisely what the definition of $G(n)$ and Theorem 3.3 guaranteed). Now define $c : [T]^2 \to \omega$ by $c(x, y) = c_n(x(n), y(n))$ where $n = \Delta(x, y)$. To see that c has the desired properties, let $k \in \omega$ be arbitrary and $T_0 \subseteq T$ be as in the statement of the theorem. Pick a m such that $G(m) > k$. Now find a $n \geq m$ and a t in ω^n such that the set $S = \{i \in h(f(n)) : \exists x \in S(t\hat{\ }i \subseteq x)\}$ has at least $f(n)$ elements in it. Then $c_n''[S]^2 = G(n)$ contains k and is contained in $c''[T_0]^2$ by definition. □

Notice that if $T \subseteq \omega^\omega$ is a $h_k(f)$ thin set which is f splitting then Lemma 3.4 tells us that there is a continuous irreducible coloring $c : [T_0]^2 \to \omega$ for some $T_0 \subseteq T$ of the same size. To see this, set $S_k = T$. If S_k is $h_{k-1}(f)$ splitting then this is a consequence of the lemma. If not let S_{k-1} be a $h_{k-1}(f)$ thin subset of S_k having the same cardinality as S_k. Now try to apply Lemma 3.4 to S_{k-1} and so on. For some $i \geq 1$, S_i has the same size as T and is $h_i(f)$ thin but $h_{i-1}(f)$ splitting (otherwise this would contradict the fact that T is $f = h_0(f)$ splitting).

Lemma 3.5. *Suppose that $T \subseteq \omega^\omega$ is 2^{2^n} thin but there is not a $T_0 \subseteq T$ of the same size such that $\#(T_0 \restriction n) \leq n^{d-a}$ for all n. Then there is a subset X of ω^ω with the same cardinality as T and a continuous irreducible coloring of $[X]^2$.*

Proof. For each x in T define $z_x(n)$ to be the finite sequence

$$z_x(n) = \langle x(n+1), \ldots, x((n+1)^2)\rangle.$$

Notice that since T is 2^{2^n} thin, $Z = \{z_x : x \in T\}$ is $(2^{2^{(n+1)^2}})^{(n+1)^2}$ thin. It is easy to verify that for some k, $h_k(n^{d-a}) \geq (2^{2^{(n+1)^2}})^{(n+1)^2}$ for all n. It therefore suffices to show that Z is n^{d-a} splitting.

Suppose that Z_0 is a n^{d-a} thin subset of Z of the same size. By refining Z_0 if necessary it may be assumes that n^{d-a} controls the splitting at all nodes of Z_0. Let $T_0 = \{x \in T : z_x \in Z_0\}$ and note that T_0 has the same size as T. I will now prove by induction that $\#(T_0 \restriction n) \leq n^{d-a}$ for all n. Notice that, if necessary, I can go to an appropriate neighborhood of T_0 to ensure that the base case of the induction is satisfied. Now suppose that $\#(T_0 \restriction m) \le m^{d-a}$ for all m less than n. Fix a m such that $m^2 \le n < (m+1)^2$. Then $\#(T_0 \restriction m) \le m^{d-a}$ and for each t in $T_0 \restriction m$ there are at most m^{d-a} many t' in $T_0 \restriction (m+1)^2$ which extend t. This latter fact follows from our assumption that Z_0 is n^{d-a} thin. Thus $T_0 \restriction n$ can have at most $m^{d-a} \cdot m^{d-a} = m^{2(d-a)} \le n^{d-a}$ many members, a contradiction. □

The following result is the last lemma which is needed to show that there is an irreducible coloring associated with $\mathrm{add}(\mathcal{N}_*)$.

Lemma 3.6. *There is a 2^{2^n} thin subset T of ω^ω of cardinality $\mathrm{add}(\mathcal{N}_*)$ which has no subset T_0 of the same size satisfying $\#(T_0 \restriction n) \le n^{d-a}$ for all n.*

Proof. Fix a family $\mathcal{A} \subseteq \mathcal{N}_a$ of size $\mathrm{add}(\mathcal{N}_*)$ which is well ordered by $\subseteq$ and unbounded in $\mathcal{N}_b$. For each A in $\mathcal{A}$ choose an a nice cover F_A of A which also satisfies $F_A(n)n^2/2^n \le n^a$ for all n. Let $T = \{F_A : A \in \mathcal{A}\}$ and notice that T is 2^{2^n} thin.

Suppose for contradiction that there is a $T_0 \subseteq T$ with the same size as T and which satisfies $\#(T_0 \restriction n) \le n^{d-a}$ for all n. Define $F_G(n) = \cup\{F_A(n) : F_A \in T_0\}$ and notice that

$$\#(F_G(n))n^2/2^n \le n^{d-a}n^a = n^d.$$

Thus $\#(F_G(n))n^{2-d}/2^n \le 1$ for all n. Since $\lim_n n^d/n^b = 0$, F_G is a b nice cover for $\cup\{A \in \mathcal{A} : F_A \in T_0\}$. If T_0 has the same size as T, then

$$\cup\mathcal{A} = \cup\{A \in \mathcal{A} : F_A \in T_0\} \in \mathcal{N}_b,$$

a contradiction. □

4. A new coloring associated with unboundedness in $(\omega^\omega, <^*)$

In this section I will modify the oscillation map of S. Todorčević (see [9, §1]) to produce a version which is continuous. Let $\mathbb{P}$ denote the collection of all strictly increasing functions from ω to ω. Suppose that x and y are two members of $\mathbb{P}$ such that either $x <^* y$ or $y <^* x$. Define the sequence $s(x,y) \in \omega^{<\omega}$ inductively as follows. Set $s(x,y)(0) = \Delta(x,y)$ and define $M(x,y,n) = \max\{x(n), y(n)\}$. If $s(x,y)(n)$ is defined and either

$$x(s(x,y)(n)) > y(s(x,y)(n)) \text{ and } x(M(x,y,s(x,y)(n))) < y(M(x,y,s(x,y)(n)))$$

or

$$x(s(x,y)(n)) < y(s(x,y)(n)) \text{ and } x(M(x,y,s(x,y)(n))) > y(M(x,y,s(x,y)(n)))$$

then set $s(x,y)(n+1) = M(x,y,s(x,y)(n))$. If neither of these conditions are satisfied then stop the procedure and set the length of $s(x,y)$ to be $n+1$. Define $c(x,y) = |s(x,y)| - 1$. Notice that the procedure must stop at or before the last oscillation between x and y. Also note that the procedure only uses a finite amount of information about x and y — if x' and y' have the same restriction to $M(x,y,s(x,y)(c(x,y))) + 1$ then they will yield the same finite sequence.

Theorem 4.1. *If $X \subseteq \mathbb{P}$ is unbounded and well ordered by $<^*$ then $c''[X]^2 = \omega$.*

Proof. Fix a countable dense set $D \subseteq X$ and pick an a in X such that $d <_* a$ for all d in D. Also fix an unbounded subset Y of X and a k_0 in ω such that every neighborhood of Y is unbounded and for every $m \ge k_0$ and every y in Y the inequality $a(m) < y(m)$ holds. I will now proceed by induction on n to show that for all n there are x and y in X such that

1. $c(x, y) = |s(x, y)| = n$,
2. $x(s(x, y)(n)) < y(s(x, y)(n))$,
3. $y \in Y$, and
4. $x \restriction i$ is a splitting node of Y for some i between $s(x, y)(n)$ and $M(s(x, y)(n))$ exclusively.

Suppose that x and y satisfy the induction hypothesis for n. Let i be fixed between $s(x, y)(n)$ and $M(x, y, s(x, y)(n))$ such that $x \restriction i$ is a splitting node of Y. Pick a j greater than $M(x, y, s(x, y)(n))$ such that $y \restriction j$ is a splitting node of Y. Choose a d in the dense set D such that $d \restriction j = y \restriction j$ and let $k_1 > k_0$ be fixed such that $d(m) < a(m)$ for all $m > k_1$. Now choose a z in Y such that $z(i) > \max\{a(k_1), j\}$.

I will now verify that d, z satisfy the role of x, y in the induction hypothesis for $c(x, y) = n + 1$. It is easy to see that by choice of z and d that $s(x, y) \restriction n + 1 = s(z, d) \restriction n + 1$. By monotonicity of z and definition of k_1 it follows that

$$z(M(z, d, s(z, d)(n))) > d(M(z, d, s(z, d)(n)))$$

and thus

$$s(z, d)(n + 1) = M(z, d, s(z, d)(n)).$$

Also, since $z(m) > a(m) > d(m)$ for all $m \geq s(z, d)(n + 1)$ it follows that $c(z, d) = n + 1$ satisfying part (1) of the induction hypothesis. Since

$$j < z(s(z, d)(n + 1)) = M(z, d, s(z, d)(n + 1)),$$

part (4) of the induction hypothesis is satisfied. Finally z is in Y, satisfying part (3). □

5. Questions

I will now leave the reader with a few open questions. It would be interesting if some application of finite Ramsey theory is necessary to obtain the main result in Section 3. A negative answer to the following question would seem to indicate that this is so.

Question 1. *Suppose that $h_k <_* f$ for all k for some $f \in \omega^\omega$ and that there is a set $X \subseteq \omega^\omega$ which is f thin but not n splitting. Is there a continuous irreducible coloring $c : [Y]^2 \to \omega$ for some Y with the same cardinality as X?*

The next question was raised by Zapletal in [14] and seems natural to include after the results of Section 3.

Question 2. *Does OCA imply that the additivity of Lebesgue measure is larger than $\aleph_1$?*

It is also unclear whether or not $\text{add}(\mathcal{N}_*)$ has some deeper relation to the other cardinal invariants of the continuum than just the inequality $\text{add}(\mathcal{N}_*) \leq \text{non}(\mathcal{N})$. I don't know, for instance, the relation between $\mathfrak{b}$ and $\text{add}(\mathcal{N}_*)$ or whether $\text{add}(\mathcal{N}_*)$ is in fact equal to $\text{add}(\mathcal{N})$.

6. Appendix

In the process of obtaining the above results, I noticed that the methods which had been developed could also be used to extend existing results in c.c.c. partition calculus. These two results have been left until the end since they don't really fit the feel of the rest of the paper. The reader is referred to [11] for the definitions of all new terms which follow (see also [7] and [9]). The following result, which is at the heart of the new theorems of this appendix, is due to S. Todorčević. It can be found in [11] in the proof that Σ^3 implies that there is an ω_1-scale.

Theorem. There is a c.c.c. partition of $[\omega^\omega]^3$ such that every homogeneous set is 2 thin.

Theorem 6.1. *If* $\omega_1 \xrightarrow{ccc} (\omega_1)^3$ *then* $\mathrm{add}(\mathcal{N})$ *is greater than* $\aleph_1$.

Proof. If G is any measure 0 set, it is possible to find a sequence $F^*_G(n) \subseteq 2^{<\omega}$ of codes for basic open sets such that $\mu([F^*_G(n)]) < 2^{-2n}$ and

$$G \subseteq \bigcap_{M=1}^{\infty} \bigcup_{n=M}^{\infty} [F^*_G(n)]$$

(see [2, p. 52]). Let T be the collection of all F^*_G such that G is in $\mathcal{N}$. Since there is a bijection between $\omega^{<\omega}$ and ω, T may be thought of as a subset of ω^ω. Combining the above result of S. Todorčević with the partition relation $\omega_1 \xrightarrow{ccc} (\omega_1)^3$ it is possible to find an uncountable $T_0 \subseteq T$ which is 2 thin. Define $F^*_H(n) = \cup\{F^*_G(n) : F^*_G \in T_0\}$ and notice that

$$\mu([F^*_H(n)]) < 2^{-2n} \cdot 2^n = 2^{-n}.$$

Thus $\cup\{G \in \mathcal{N} : F^*_G \in T_0\}$ is contained in the measure 0 set

$$H = \bigcap_{M=1}^{\infty} \bigcup_{n=M}^{\infty} [F^*_H(n)].$$

□

A similar argument can be used to show the following.

Theorem 6.2. Σ^3 *implies* $\mathrm{cof}(\mathcal{N}) = \aleph_1$.

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