Random forcing and (S) and (L) *

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Abstract

In this article I will analyze the impact of forcing with a measure algebra on various topological statements. In particular our interest will focus on the study of hereditary separability and the hereditary Lindelöf property in the classes of compact, extremally disconnected, and cometrizable spaces.

1 Introduction

For a given class of regular topological spaces, consider the following two questions:

(S) Is every hereditarily separable space hereditarily Lindelöf?

(L) Is every hereditarily Lindelöf space hereditarily separable?

These questions have long been at the center of active research in set theoretic topology in such classes as compact spaces, extremally disconnected spaces, cometrizable spaces, and arbitrary regular spaces. In the 1980's it was shown under the assumption of $\mathbf{MA}_{\aleph_1}$ that both questions have positive answers in all but the last class of spaces ([17], [18], [5] respectively).

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On another front, set theorists have been working towards understanding which consequences of $\mathbf{MA}_{\aleph_1}$ are preserved by forcing with a measure algebra. Probably the most influential result of this sort is the following theorem due to Laver.

Theorem 1.1. *[10] After forcing with a measure algebra over a model of* $\boldsymbol{MA}_{\aleph_1}$ *there are no Souslin continua.*

The focus of this note will be to examine the impact of forcing with a measure algebra on (S) and (L) in the class of compact, cometrizable, and extremally disconnected spaces.

2 Measure algebras and random reals

Before we begin, let us first review some basic notions and fix some terminology.

Definition 2.1. A *measure algebra* is a pair $(\mathcal{R}, \mu)$ such that

1. $\mathcal{R}$ is a complete Boolean algebra,
2. $\mu : \mathcal{R} \to [0, 1]$ is positive on positive elements of $\mathcal{R}$ and satisfies $\mu(\mathbf{1}) = 1$, and
3. $\mu(a \vee b) = \mu(a) + \mu(b)$ whenever $a \wedge b = \mathbf{0}$.

Sometimes I will abuse notion and write $\mathcal{R}$ when I really mean $(\mathcal{R}, \mu)$. While this definition seems quite general, the class of all measure algebras is actually rather small as the following remarkable theorem of Maharam shows.

Theorem 2.2. *[11] There is exactly one homogeneous measure algebra up to measure preserving isomorphism of each infinite character* κ*.*

Here the character of a measure algebra refers to the smallest number of elements required to completely generate it. We, moreover, know what these examples look like. If I is any infinite index set then we can define the product measure μ on the clopen subsets of 2^I. Extending this to the Baire subsets of 2^I and taking the quotient by the measure 0 sets gives us $\mathcal{R}_I$, the unique homogeneous measure algebra of character $\#(I)$.

As we will be interested in considering measure algebras as forcing notions, this tells us that forcing with a measure algebra is equivalent to adding some number of random reals. I have chosen to treat measure algebras in an abstract setting since this often highlights what is important in the proofs and, at times, makes the presentation more transparent. Occasionally, however, it will be more convenient to think of the generic object as a sequence of reals (or elements of 2^ω, ω^ω, etc.). I will not hesitate to switch back and forth between these two methods of presentation when it benefits the discussion.

The following theorem is at the heart of many of the arguments concerning the impact of $\mathbf{MA}_{\aleph_1}$ on forcing extensions of the form $\mathbf{V}^{\mathcal{R}}$ where $\mathcal{R}$ is a measure algebra.

Theorem 2.3. *($\mathbf{MA}_{\aleph_1}$) If $\mathcal{R}$ is a measure algebra and $\dot{G} : [\omega_1]^2 \to \mathcal{R}$ is a $\mathcal{R}$-name for a graph on ω_1 then either*

1. *There is a sequence $\mathcal{R}$-names $\dot{X}_n : \omega_1 \to \mathcal{R}$ indexed by ω such that for all $\alpha < \omega_1$*
$$\bigvee_{n<\omega} \dot{X}(\alpha) = \mathbf{1}$$
and for all n and $\alpha, \beta < \omega_1$
$$\dot{X}_n(\alpha) \wedge \dot{X}_n(\beta) \wedge \dot{G}(\alpha, \beta) = \mathbf{0}$$
(i.e. $\dot{G}$ is forced to be countably chromatic) or else

2. *there is a sequence F_ξ ($\xi < \omega_1$) of disjoint finite subsets of ω_1 and a $\delta > 0$ such that for all $\xi \neq \eta$*
$$\bigvee_{\alpha \in F_\xi} \bigvee_{\beta \in F_\eta} \dot{G}(\alpha, \beta)$$
has measure at least δ.

The proof of this theorem is carried out explicitly in [14]. The techniques of the proof already appear in [10] and the argument can readily be extracted from section 2 of [22]. It should be emphasized that for the questions which we are considering this is often the only use of $\mathbf{MA}_{\aleph_1}$ and that the bulk of the work lies in analyzing the implications of the second alternative of this theorem in a specific context.

3 Cometrizable spaces

A topological space X is *cometrizable* if there is a weaker metric topology on X such that every point of X has a neighborhood base of sets which are closed in the metric topology. In this section we will consider the influence of forcing with a measure algebra on (S) and (L) in the class of cometrizable spaces. It turns out that whether these objects are introduced by this forcing depends on the character of the algebra.

Theorem 3.1. *($\boldsymbol{MA}_{\aleph_1}$) After forcing with a separable measure algebra (S) and (L) have a positive answer in the class of cometrizable spaces.*

Proof. I will only present the proof for (S) as the proof for (L) is symmetric. Let $\mathcal{R}$ be a separable measure algebra and $(\dot{X}, \dot{d})$ be a $\mathcal{R}$-name for a metric space which supports a non-Lindelöf cometrizable topology $\dot{\tau}$. Since $\dot{\tau}$ is a refinement of the metric topology on $\dot{X}$ we may assume that $\dot{X}$ is forced to be separable. Select a sequences $\dot{x}_\alpha, \dot{E}_\alpha$ $(\alpha < \omega_1)$ such that

1. $\dot{x}_\alpha$ is a $\mathcal{R}$-name for an element of $\dot{X}$
2. $\dot{E}_\alpha$ is a $\mathcal{R}$-name for a τ neighborhood of $\dot{x}_\alpha$ which is closed in the metric topology.
3. if $\alpha < \beta$ then it is forced that $\dot{x}_\beta$ is not in $\dot{E}_\alpha$.

Define $\dot{G} : [\omega_1]^2 \to \mathcal{R}$ by $\dot{G}(\alpha, \beta)$ is the event "$\dot{x}_\alpha$ is in $\dot{E}_\beta$" where $\alpha < \beta < \omega_1$.

Now apply Theorem 2.3 to $\dot{G}$. It is easy to see that the first alternative gives a decomposition of $\{\dot{x}_\alpha : \alpha < \omega_1\}$ into countably many discrete sets. It therefore suffices to show that the second alternative cannot hold. Suppose that such a sequence F_ξ $(\xi < \omega_1)$ and a $\delta > 0$ are given. For each ξ, pick a rational $\varepsilon_\xi > 0$ such that

$$A_\xi = [\![\min_{\alpha \in F_\xi} \min_{\beta \in F_{\xi+1}} \dot{d}(\dot{x}_\beta, \dot{E}_\alpha) > \varepsilon_\xi]\!]$$

has measure greater than $1 - \delta/2$. Fix an uncountable $\Gamma \subseteq \omega_1$ and $\varepsilon > 0$ such that $\varepsilon_\xi = \varepsilon$ for all ξ in Γ. By the separability of $\mathcal{R}$ it is possible to find $\xi < \xi + 1 < \eta$ such that for all $i = 0, \ldots, n-1$

$$B_{\xi,\eta} = [\![\dot{d}(\dot{x}_{F_{\xi+1}(i)}, \dot{x}_{F_{\eta+1}(i)}) < \varepsilon]\!]$$

has measure greater than $1 - \delta/2$. It follows that $A_\eta \wedge B_{\xi,\eta}$ has measure greater than $1 - \delta$ and forces that $d(\dot{x}_\alpha, \dot{E}_\beta) > 0$ whenever α is in $F_{\xi+1}$ and β is in F_η and hence is disjoint from

$$\bigvee_{\alpha \in F_\xi} \bigvee_{\beta \in F_\eta} \dot{G}(\alpha, \beta).$$

□

For nonseparable measure algebras, the picture is quite different.

Theorem 3.2. *After forcing with any homogeneous nonseparable measure algebra, there are counterexamples to (S) and (L) in the class of cometrizable spaces.*

Proof. By modifying our ground model if necessary, we may assume that the forcing extension is of the form $\mathbf{V}[r_\xi : \xi < \omega_1]$ where $\dot{r}_\xi(\xi < \omega_1)$ is a sequence of random reals in $[0, 1]$. The set $\{r_\xi : \xi < \omega_1\}$ is a Sierpinski set and therefore is a counterexample to (L) when viewed as a subspace of the density topology [19]. Since the density topology is cometrizable (see the proof of Theorem 22.9 of [15]), this gives a cometrizable counterexample to (L).

Also, the set $\{\dot{r}_\xi : \xi < \omega_1\}$ is, in fact, hereditarily Lindelöf in all of its finite powers. For each $\xi < \omega_1$ pick a compact set $\dot{E}_\xi$ of positive measure which misses $\dot{r}_\eta$ for all $\eta < \xi$ but which contains $\dot{r}_\xi$ as a point of density 1. Now the sets $\dot{U}_\xi = \{\dot{E}_\eta : \dot{r}_\xi \in \dot{E}_\eta\}$ are closed in the Vietoris topology on the closed subsets of $[0, 1]$ and, by duality, generate a hereditarily separable, non Lindelöf topology on $\{E_\alpha : \alpha < \omega_1\}$. This gives a counterexample to (S) in the class of cometrizable spaces. □

It is perhaps worth noting at this point that the cometrizable spaces just mentioned are built on somewhat generic sets of reals. One might ask whether an ω_1-sequence of random reals could be used to construct cometrizable counterexamples to (S) or (L) on any set of reals of size $\aleph_1$. This is a reasonable question since Todorčević has shown in Chapter 2 of [21] that $\mathfrak{b} = \omega_1$ implies that any set of reals of size $\aleph_1$ supports a cometrizable locally compact topology which is a counterexample to (S). A closer inspection of the proof of Theorem 3.1, however, reveals that the use of the separability of $\mathcal{R}$ only requires that the underlying set of reals be added by a separable subalgebra.

4 Extremally Disconnected Spaces

A topological space X is *extremally disconnected* if the closure of every open set is open. Counterexamples to (S) which are subspaces of extremally disconnected spaces were constructed by Ginsburg [4], Szymański [18], and Wage [24] using a variety of set theoretic assumptions. In the last of the three papers written on extremally disconnected and (S), Szymański shows that Wage's Lemma, a known consequence of $\mathbf{MA}_{\aleph_1}$ (see [23]), implies that (S) has a positive answer in the class of all subspaces of extremally disconnect spaces. In this section I will extended Wage's result, showing that if $\mathbf{MA}_{\aleph_1}$ holds then Wage's Lemma remains true after forcing with any measure algebra. Combined with Szymanski's result, this establishes the consistency of "(S) has a positive answer for all subspaces of extremally disconnected spaces" with statements such as "there is a Sierpiński set" (which yields an extremally disconnected L space when identified with a subspace of the Stone space of the Lebesgue measure algebra [1]) and "there is an Ostaszewski space" (see section 5). The latter is perhaps surprising since historically extremally disconnect S spaces and Ostaszewski spaces have been constructed from similar axioms (see, e.g., [16], [18]).

Consider the following combinatorial statement for a regular cardinal θ:

W(θ) If $\mathcal{A}$ is a family of countable subsets of θ such that $\mathcal{A}$ has size at most θ and every pair has a finite intersection, then there is a cofinal subset of θ which has finite intersection with every element of $\mathcal{A}$.

This statement is known as Wage's Lemma and was considered by Wage in [23] through the course of working on topological problems related to (S) and (L). It also has uses in combinatorics — the interested reader is referred to [23]. I will now prove that $\mathbf{W}(\theta)$ is a consequence of $\mathbf{MA}_\theta$ which can hold after forcing with an arbitrary measure algebra.

Theorem 4.1 ($\mathbf{MA}_\theta$). ***W***(θ) *holds after forcing with any measure algebra.*

Proof. Let $\{\dot{A}_\alpha : \alpha < \theta\}$ be a sequence of $\mathcal{R}$-names for almost disjoint countable subsets of θ. For each $\alpha < \theta$ let $\mathcal{R}_\alpha$ be the separable complete subalgebra of $\mathcal{R}$ generated by $\{[\![\xi \in \dot{A}_\alpha]\!] : \xi < \theta\}$ (note that $[\![\xi \in \dot{A}_\alpha]\!]$ is $\mathbf{0}$ for all but countably many ξ). Let $(\mathcal{Q}, \leq)$ be the forcing notion of all pairs $(X, D, \vec{P})$ which satisfy the following properties:

[1] It is unclear even today whether (L) can consistently be true in the class of extremally disconnected spaces (i.e. this is apparently an open problem).

1. X, D are finite subsets of θ.
2. $\vec{P} = \langle P^0, \dots, P^{k-1} \rangle$ is a finite sequence of maps from D into $\mathcal{R}^+$.
3. If α is in D then $P^i(\alpha)$ is in $\mathcal{R}_\alpha$ for each P^i in $\vec{P}$.
4. If α is in D then $P^i(\alpha)$ forces $X \cap \dot{A}_\alpha$ has size at most i.

The ordering on elements of $\mathcal{Q}$ is defined by $q \leq r$ if the following hold:

1. X_q contains X_r and D_q contains D_r.
2. For every α in D_r and every i less than the length of $\vec{P}_r$, $\delta(P^i_r(\alpha)) = \delta(P^i_q(\alpha))$. Here if B is in $\mathcal{R}^+$, $\delta(B)$ is the largest number of the form $1/n$ which is less than $\mu(B)$.

For each condition $(X, D, \vec{P})$, there is a rational $\varepsilon = \varepsilon(X, D, \vec{P}) > 0$ such that for all α in D and P^i in $\vec{P}$, $\mu(P^i(\alpha)) > \delta(P^i(\alpha)) + \varepsilon$.

Claim 4.2. $(\mathcal{Q}, \leq)$ *satisfies the countable chain condition.*

Proof. Let $\{(X_\xi, D_\xi, \vec{P}_\xi) : \xi < \omega_1\}$ be a sequence of conditions in $\mathcal{Q}$. Select an uncountable $\Gamma \subseteq \omega_1$ and an $\varepsilon > 0$ such that the following conditions hold:

1. $\varepsilon(X_\xi, D_\xi, \vec{P}_\xi) = \varepsilon$.
2. There is a k such that all the sequences $\vec{P}_\xi$ have length k.
3. The families $\{X_\xi : \xi < \omega_1\}$ and $\{D_\xi : \xi < \omega_1\}$ form Δ-systems with roots X and D respectively.
4. If α is in D, $\xi, \eta < \omega_1$ then for all $i < k$ $P^i_\xi(\alpha)$ and $P^i_\eta(\alpha)$ differ by a set of measure at most $\varepsilon/2$.
5. If $\xi < \eta$ are in Γ then $X_\xi \subseteq \eta$ and it is forced that $\dot{A}_\alpha$ is contained in η for all α in D_ξ.

Let m denote the cardinality of $X_\xi \setminus X$ and I be the first ω elements of Γ. For $\eta > \sup I$ and $i \leq m$, define $f^i_\eta : I \to \mathcal{R}$ so that $f^i_\eta(\xi)$ is the event that the i^{th} element of $X_\xi \setminus X$ is in $\dot{A}_\alpha$ for some α in the domain of P_η. Notice that if $\eta \neq \zeta$ are in $\Gamma \setminus I$ then $\mu(f^i_\eta(\xi) \cdot f^i_\zeta(\xi))$ vanishes for all i as $\xi \to \sup I$. So if $\mathcal{U}$ is a nonprinciple ultrafilter on I, for each i there are only countably many η

in $\Gamma \setminus I$ such that the limit of $\mu(f^i_\eta(\xi))$ as $\xi \to \mathcal{U}$ is nonzero. In particular, this means that for some η in $\Gamma \setminus I$ and some ξ in I $\mu(f^i_\eta(\xi)) < \varepsilon/m$ for all i. It is now easy to verify that $(X_\xi, D_\xi, \vec{P}_\xi)$ and $(X_\eta, D_\eta, \vec{P}_\eta)$ are compatible. $\square$

Now let $\mathcal{D}_{\alpha,\varepsilon}$ be the collection of all $(X, D, \vec{P})$ in $\mathcal{Q}$ such that

1. α is in D.
2. $X \setminus \alpha$ is nonempty.
3. The measure of $\bigcup_i P_i(\alpha)$ is at least $1 - \varepsilon/2$.
4. The sum of $\mu(P(\alpha)) - \delta(P(\alpha))$ as is less than $\varepsilon/2$

It is routine to verify that $\mathcal{D}_{\alpha,\varepsilon}$ is dense for all $\alpha < \theta$ and $\varepsilon > 0$. Also, if X is the union of the first coordinates of a filter meeting all of these dense sets then X is forced to have finite intersection with each $\dot{A}_\alpha$. $\square$

5 Compact S Spaces

One of the primary reasons for studying (S) and (L) after forcing with a measure algebra is that this may yield a solution to an old problem of Katětov. While Katětov's problem has recently been resolved using techniques different than those developed in this paper (see [9]), the status of Katětov's problem after forcing with a measure algebra still seems to be of interest. In particular a resolution of Katětov's problem in the models which we are studying may require or lead to a better understanding of perfectly normal compacta than we presently have.

In [7] Katětov proved that if X is a compact space and X^3 is hereditarily normal then X must be metrizable. He then asked whether dimension 3 could be lowered to dimension 2. Gruenhage and Nyikos have shown in [6] that under **CH** and also under $\mathbf{MA}_{\aleph_1}$ this question has a negative answer. Moreover they show that if X is a counterexample, one of the following must hold:

1. There is a Q set.
2. X is a compact counterexample to (L).
3. X^2 is a compact counterexample to (S).

4. X^2 contains counterexamples to both (S) and (L).

After forcing with a measure algebra of character at least $\aleph_1$, there are no Q sets. Also, Todorčević has shown the following. [2]

Theorem 5.1. *[22] ($\boldsymbol{MA}_{\aleph_1}$) After forcing with any measure algebra every compact space containing a counterexample to (L) also contains an uncountable free sequence.*

Thus any counterexample to Katětov's problem in this model would have to be a perfectly normal compactum X whose square is a counterexample to (S). For some time it was unclear whether Todorčević's theorem could be dualized by replacing (L) with (S). The following theorem gives a negative answer to this question.

Theorem 5.2. *In any forcing extension by a homogeneous nonseparable measure algebra there is a perfectly normal countably compact non compact topology on ω_1. In particular, after such a forcing there is a compact counterexample to (S).*

Remark 5.3. Eisworth and Roitman [2] have shown that such spaces can not be constructed from **CH** alone. Thus it is not reasonable to hope to weaken the parameter in the construction to the existence of a Sierpinski set. It turns out that there is indeed a guessing principle which the above construction factors through. Interested readers are referred to [12] where a general discussion is given of guessing principles of this sort.

Proof. By modifying our ground model if necessary, we may assume that $\mathbf{V}^{\mathcal{R}} = \mathbf{V}[\dot{r}_\alpha : \alpha < \omega_1]$ for some sequence of random reals $\dot{r}_\alpha$ $(\alpha < \omega_1)$. It will be convenient to view $\dot{r}_\alpha$ as a random real in ω^ω where ω is given the atomic measure determined by $\mu(\{n\}) = 2^{-n-1}$. In $\mathbf{V}$ fix, for each limit ordinal δ, a strictly increasing sequence δ_n cofinal in δ. Define a sequence of topologies $\dot{\tau}_\alpha$ on the limit ordinals α by recursion so that $\dot{\tau}_\alpha$ is locally compact, non compact topology in $\mathbf{V}[\dot{r}_\xi : \xi < \alpha]$ and $\dot{\tau}_\beta \restriction \alpha$ is $\dot{\tau}_\alpha$ for $\alpha < \beta$. $\dot{\tau}_\omega$ is the discrete topology. Suppose now that $\dot{\tau}_\alpha$ has been defined (limit stages are trivial). Define a topology $\dot{\tau}_{\alpha+\omega}$ on $\alpha + \omega$ as follows. In $\mathbf{V}[\dot{r}_\xi : \xi < \alpha]$, let $\{\dot{U}_\alpha(k) : k < \omega\}$ be a partition of $(\alpha, \dot{\tau}_\alpha)$ into disjoint compact open pieces.

[2] While this result is certainly of independent interest — there is no mention of Katětov's problem in [22] — it should be noted that it is open whether one can prove in **ZFC** that if X^2 is compact and hereditarily normal then X is separable.

Neighborhoods of $\alpha + n$ in $\dot{\tau}_{\alpha+\omega}$ are of the form $\{\alpha + n\} \cup \bigcup \dot{\mathcal{V}}$ where $\dot{\mathcal{V}}$ is a cofinite subset of

$$\{\dot{U}_\alpha(k) : \dot{r}_\alpha(k) = n\}.$$

The space we are interested in is, of course, $(\omega_1, \dot{\tau}_{\omega_1})$. The topology is clearly locally compact and locally countable. It should also be clear that the genericity of the $\dot{r}_\alpha$'s ensure that, for a fixed $\alpha < \omega_1$, the closure of $\{\alpha + n : n < \omega\}$ is the set of all $\beta \geq \alpha$. I will now show that $(\omega_1, \dot{\tau}_{\omega_1})$ has the property that the closure of any set is either compact or cocountable. Suppose that $\dot{E}$ is a name for an infinite subset of ω_1 and assume without loss of generality that $\dot{E}$ is forced to be either countable or uncountable. Let α be a limit ordinal such that $\dot{E} \cap \alpha$ is infinite and is in $\mathbf{V}[\dot{r}_\xi : \xi < \alpha]$. If $\dot{E}$ is forced to be uncountable, then also arrange that $\dot{E} \cap \alpha$ is forced to be cofinal in α. Now we will work in $\mathbf{V}[\dot{r}_\xi : \xi < \alpha]$. If $\dot{E} \cap \alpha$ has compact closure in $(\alpha, \dot{\tau}_\alpha)$ then we are done (note that this is impossible if $\dot{E} \cap \alpha$ is cofinal in α). If $\dot{E} \cap \alpha$ does not have compact closure in $\dot{\tau}_\alpha$ then there are infinitely many k such that $\dot{U}_\alpha(k) \cap \dot{E}$ is nonempty. Moreover, an easy genericity argument shows that, for each n there are infinitely many k such that $\dot{U}_\alpha(k) \cap \dot{E}$ is non empty and $\dot{r}_\alpha(k) = n$. It follows that $\dot{E} \cap \alpha$ accumulates to each of the $\alpha + n$'s which are in turn dense the set of all $\beta \geq \alpha$. □

It is unclear whether the above example above can be made hereditarily separable in all finite powers. In particular, the following question is open.

Question 1. *($\boldsymbol{MA}_{\aleph_1}$) After forcing with an arbitrary measure algebra, does X^2 always contain an uncountable discrete subspace whenever X is non-metrizable and compact?*

Also the following question remains open, serving as a reminder that this example is actually quite irrelevant to Katětov's problem.

Question 2. *($\boldsymbol{MA}_{\aleph_1}$) After forcing with an arbitrary measure algebra, does (S) hold for the class of first countable compact spaces?*

6 Small Compactifications of L Spaces

By a result of Fremlin [3, 44A] (see also section 6 of [20]), $\mathbf{MA}_{\aleph_1}$ implies that every compact space containing a counterexample to (L) must map onto $[0, 1]^{\omega_1}$. By a theorem of Todorčević, $\mathbf{MA}_{\aleph_1}$ implies that after forcing with an arbitrary measure algebra, compact countably tight spaces can't contain counterexamples to (L). It is reasonable to ask whether these two results can be combined to obtain Fremlin's conclusion even after forcing with an arbitrary measure algebra. The following result indicates that this is not so.

Theorem 6.1. *If there is a Sierpiński set then there is a compact space K which contains a counterexample to (L) and which also maps continuously into 2^ω with linear fibers (and hence does not map onto $[0,1]^{\omega_1}$).*

Proof. Let $\{x_\alpha : \alpha < \omega_1\}$ be a Sierpiński subset of 2^ω and let $e_\alpha : \alpha \to \omega$ ($\alpha < \omega_1$) be a coherent sequence of injections. Suppose further that for all $\alpha < \beta$ $\Delta(x_\alpha, x_\beta) \leq e_\beta(\alpha)$ (the sequence e_α can always be modified to have this property — see Chapter 4 and in particular page 96 of [1]). Define

$$E_\beta = 2^\omega \setminus \bigcup_{\alpha<\beta} [x_\alpha \restriction e_\beta(\alpha)].$$

Then E_β is a compact set of positive measure all of whose elements are of Lebesgue density 1 and which contains x_β. Hence if a topology on 2^ω is generated by the clopen sets and the E_β's, $X = \{x_\alpha : \alpha < \omega_1\}$ is an L space when given the subspace topology. Let $\mathcal{B}$ be the Boolean algebra generated by the clopen subsets of 2^ω and the E_β's.

Claim 6.2. *The map from the Stone space of $\mathcal{B}$ to 2^ω given by restricting ultrafilters to the algebra of clopen sets has linear fibers.*

Proof. Suppose that x is in 2^ω. Let Γ_x be the collection of all β such that $x \in E_\beta$. We must show that if $\alpha < \beta$ are in Γ_x then, for some neighborhood U of x, $E_\beta \cap U \subseteq E_\alpha \cap U$. To this end, it suffices to show that $E_\beta \setminus E_\alpha$ is closed. Let $F \subseteq \alpha$ be a finite set such that e_α and e_β agree on $\alpha \setminus F$. Then

$$E_\beta \setminus E_\alpha = \bigcup_{\gamma \in F} \{x \in E_\beta : \Delta(x, x_\gamma) \leq e_\alpha(\gamma)\}$$

is clearly closed. □

□

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