STRUCTURE OF THE LODAY-RONCO HOPF ALGEBRA OF TREES

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ABSTRACT. Loday and Ronco defined an interesting Hopf algebra structure on the linear span of the set of planar binary trees. They showed that the inclusion of the Hopf algebra of non-commutative symmetric functions in the Malvenuto-Reutenauer Hopf algebra of permutations factors through their Hopf algebra of trees, and these maps correspond to natural maps from the weak order on the symmetric group to the Tamari order on planar binary trees to the boolean algebra.

We further study the structure of this Hopf algebra of trees using a new basis for it. We describe the product, coproduct, and antipode in terms of this basis and use these results to elucidate its Hopf-algebraic structure. We also obtain a transparent proof of its isomorphism with the non-commutative Connes-Kreimer Hopf algebra of Foissy, and show that this algebra is related to non-commutative symmetric functions as the (commutative) Connes-Kreimer Hopf algebra is related to symmetric functions.

INTRODUCTION

In 1998, Loday and Ronco defined a Hopf algebra LR on the linear span of the set of rooted planar binary trees [23]. This Hopf algebra is the free dendriform algebra on one generator [21]. In [23, 24], Loday and Ronco showed how natural poset maps between the weak order on the symmetric groups, the Tamari order on rooted planar binary trees with n leaves, and the Boolean posets induce injections of Hopf algebras

$$\mathcal{N}Sym \hookrightarrow LR \hookrightarrow \mathfrak{S}Sym\,,$$

where $\mathcal{N}Sym$ is the Hopf algebra of non-commutative symmetric functions [12] and $\mathfrak{S}Sym$ is the Malvenuto-Reutenauer Hopf algebra of permutations [26].

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Simultaneously, Hopf algebras of trees were proposed by Connes and Kreimer [9, 20] and Brouder and Frabetti [7, 8] to encode renormalization in quantum field theories. The obvious importance of these algebras led to intense study, and by work of Foissy [10, 11], Hivert-Novelli-Thibon [16], Holtkamp [19], and Van der Laan [38], the Hopf algebras of Loday-Ronco, Brouder-Frabetti, and the non-commutative Connes-Kreimer Hopf algebra are known to be isomorphic, self-dual, free (and cofree).

We described the elementary structure of $\mathfrak{S}Sym$ with respect to a new basis and used those results to further elucidate its structure as a Hopf algebra [1]. Here, we use a similar approach to study $\mathcal{Y}Sym := (LR)^*$, the graded dual Hopf algebra to LR. We define a new basis for $\mathcal{Y}Sym$ related to the (dual of) the Loday-Ronco basis via Möbius inversion on the poset of trees. We next describe the elementary structure of $\mathcal{Y}Sym$ with respect to this new basis, use those results to show that it is cofree, and then study its relation to $\mathfrak{S}Sym$ and $\mathcal{Q}Sym$, the Hopf algebra of quasi-symmetric functions. This basis allows us to give an explicit isomorphism between the Loday-Ronco Hopf algebra LR and the noncommutative Connes-Kreimer Hopf algebra of Foissy (this coincides with the isomorphism constructed by Holtkamp [19] and Palacios [28]). We use it to show that a canonical involution of $\mathcal{Q}Sym$ can be lifted to $\mathcal{Y}Sym$ and to deduce a commutative diagram involving the Connes-Kreimer Hopf algebras (commutative and non-commutative) on one hand, and symmetric and non-commutative symmetric functions on the other.

Our approach provides a unified framework to understand the structures of $\mathcal{Y}Sym$ and explain them in the context of the well-understood Hopf algebras $\mathfrak{S}Sym$, $\mathcal{Q}Sym$, and $\mathcal{N}Sym$ of algebraic combinatorics. A similarly unified approach, through realizations of the algebras $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ via combinatorial monoids, has been recently obtained by Hivert, Novelli, and Thibon [15, 16, 17]. Another interesting approach, involving lattice congruences, has been proposed by Reading [29, 30].

1. Basic definitions

1.1. **Compositions, permutations, and trees.** Throughout, n is a non-negative integer and $[n]$ denotes the set $\{1, 2, \ldots, n\}$. A *composition* α of n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of positive integers whose sum is n. Associating the set $I(\alpha) := \{\alpha_1, \alpha_1{+}\alpha_2, \ldots, \alpha_1{+}\cdots{+}\alpha_{k-1}\}$ to a composition α of n gives a bijection between compositions of n and subsets of $[n{-}1]$. Compositions of n are partially ordered by *refinement*, which is defined by its cover relations

$$(\alpha_1, \ldots, \alpha_i + \alpha_{i+1}, \ldots, \alpha_k) \lessdot (\alpha_1, \ldots, \alpha_k) .$$

Under the association $\alpha \leftrightarrow I(\alpha)$, refinement corresponds to set inclusion, so we simply identify the poset of compositions of n with the Boolean poset $\mathcal{Q}_n$ of subsets of $[n{-}1]$.

Let $\mathfrak{S}_n$ be the group of permutations of $[n]$. We use one-line notation for permutations, writing $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ and sometimes omitting the parentheses and commas. The *standard permutation* $\mathrm{st}(a_1, \ldots, a_p) \in \mathfrak{S}_p$ of a sequence $(a_1, \ldots, a_p)$ of distinct integers is the unique permutation σ such that

$$\sigma(i) < \sigma(j) \iff a_i < a_j .$$

An *inversion* in a permutation $\sigma \in \mathfrak{S}_n$ is a pair of positions $1 \leq i < j \leq n$ with $\sigma(i) > \sigma(j)$. Let $\mathrm{Inv}(\sigma)$ denote the set of inversions of σ. Given $\sigma, \tau \in \mathfrak{S}_n$, we write $\sigma \leq \tau$ if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau)$. This defines the (left) *weak order* on $\mathfrak{S}_n$. The identity permutation id_n

is the minimum element in $\mathfrak{S}_n$ and $\omega_n = (n, \ldots, 2, 1)$ is the maximum. See [1, Figure 1] for a picture of the weak order on $\mathfrak{S}_4$.

The grafting of two permutations $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$ is the permutation $\sigma \vee \tau \in \mathfrak{S}_{p+q+1}$ with values

$$\sigma(1)+q,\ \sigma(2)+q,\ \ldots,\ \sigma(p)+q,\ \ p+q+1,\ \ \tau(1),\ \tau(2),\ \ldots,\ \tau(q)\,. \tag{1}$$

Similarly, let $\sigma \underline{\vee} \tau \in \mathfrak{S}_{p+q+1}$ be the permutation with values

$$\sigma(1),\ \sigma(2),\ \ldots,\ \sigma(p),\ \ p+q+1,\ \ \tau(1)+p,\ \tau(2)+p,\ \ldots,\ \tau(q)+p\,.$$

This is the operation considered in [24, Def. 1.6]. As in [24, Def. 1.9], let $\sigma\backslash\tau \in \mathfrak{S}_{p+q}$ be the permutation whose values are

$$\sigma(1)+q,\, \sigma(2)+q,\, \ldots,\, \sigma(p)+q,\ \ \tau(1),\, \tau(2),\, \ldots,\, \tau(q)\ .$$

Consider decompositions $\rho = \sigma\backslash\tau$ of a permutation ρ. Every permutation ρ may be written as $\rho = \rho\backslash id_0 = id_0\backslash\rho$, where $id_0 \in \mathfrak{S}_0$ is the empty permutation. A permutation $\rho \neq id_0$ has *no global descents* if these are its only such decompositions. The operation $\backslash$ is associative and a permutation $\rho \neq id_0$ has a unique decomposition into permutations with no global descents. We similarly have the associative operation $/$ to form the permutation σ/τ (often denoted $\sigma \times \tau$ in the literature) whose values are

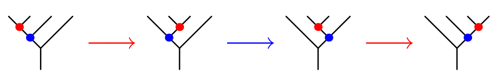
$$\sigma(1),\, \sigma(2),\, \ldots,\, \sigma(p),\ \ \tau(1)+p,\, \tau(2)+p,\, \ldots,\, \tau(q)+p\,.$$

The following properties are immediate from the definitions. For any permutations ρ, σ, τ,

$$(\rho \vee \sigma)\backslash\tau = \rho \vee (\sigma\backslash\tau)\,, \tag{2}$$

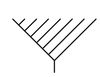
$$\rho/(\sigma\underline{\vee}\tau) = (\rho/\sigma)\underline{\vee}\tau\,. \tag{3}$$

Let $\mathcal{Y}_n$ be the set of rooted, planar binary trees with n interior nodes (and thus $n+1$ leaves). The *Tamari order* on $\mathcal{Y}_n$ is the partial order whose cover relations are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus

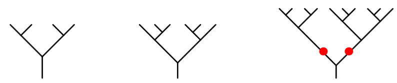


is an increasing chain in $\mathcal{Y}_3$ (the moving vertices are marked with dots). Only basic properties of the Tamari order are needed in this paper; their proofs will be provided. For more properties, see [4, Sec. 9]. Figure 1.1 shows the Tamari order on $\mathcal{Y}_3$ and $\mathcal{Y}_4$.

Let 1_n be the minimum tree in $\mathcal{Y}_n$. It is called a *right comb* as all of its leaves are right pointing:

$1_4 =$  $\quad 1_7 =$  .

Given trees $s \in \mathcal{Y}_p$ and $t \in \mathcal{Y}_q$, the tree $s \vee t \in \mathcal{Y}_{p+q+1}$ is obtained by grafting the root of s onto the left leaf of the tree $\curlyvee$ and the root of t onto its right leaf. Below we display trees s, t, and $s \vee t$, indicating the position of the grafts with dots.



For $n > 0$, every tree $t \in \mathcal{Y}_n$ has a unique decomposition $t = t_l \vee t_r$ with $t_l \in \mathcal{Y}_p$, $t_r \in \mathcal{Y}_q$, and $n = p+q+1$. Thus $\mathcal{Y}_n$ is in bijection with $\coprod_{p+q=n-1} \mathcal{Y}_p \times \mathcal{Y}_q$, and since $\mathcal{Y}_0 = \{|\}$ and $\mathcal{Y}_1 = \{\curlyvee\}$, we see that $\mathcal{Y}_n$ contains the Catalan number $\frac{(2n)!}{n!(n+1)!}$ of trees.

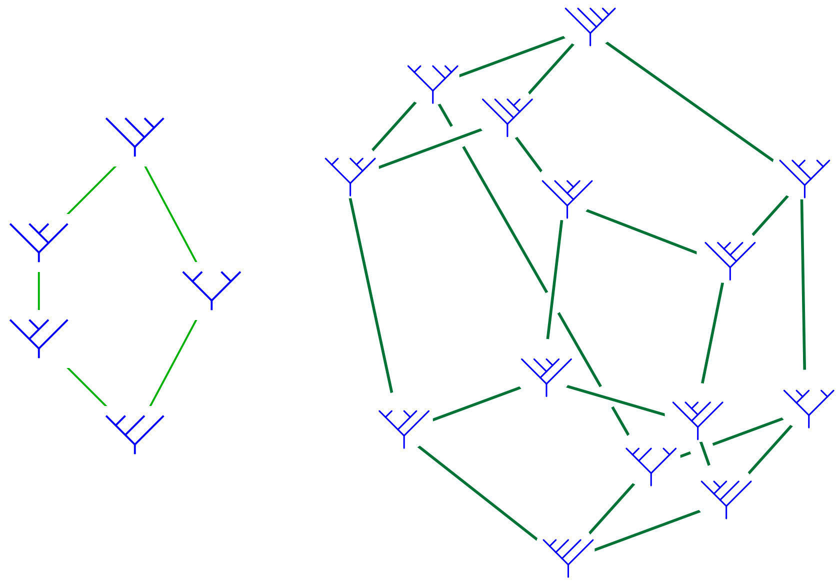
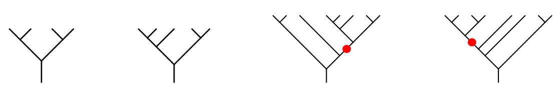


FIGURE 1. The Tamari order on $\mathcal{Y}_3$ and $\mathcal{Y}_4$.

For trees s and t, let $s\backslash t$ be the tree obtained by adjoining the root of t to the rightmost branch of s. Similarly, s/t is obtained by grafting the root of s to the leftmost branch of t. These operations are associative. Here, we display trees s, t, $s\backslash t$, and s/t, indicating the position of the graft with a dot.



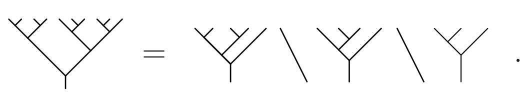
The following properties are immediate from the definitions. For any trees s and t,

$$s\backslash t \quad = \quad s_l \vee (s_r\backslash t)\,, \tag{4}$$

$$s/t \quad = \quad (s/t_l) \vee t_r\,. \tag{5}$$

A (right) decomposition of a tree t is a way of writing t as $r\backslash s$. Note that $t = t\backslash 1_0 = 1_0\backslash t$, so every tree has two trivial decompositions. We say that a tree $t \neq 1_0$ is *progressive* if these are its only right decompositions. For any tree $t \in \mathcal{Y}_n$ we have, by (4), $t = t_l \vee t_r = t_l \vee (1_0\backslash t_r) = (t_l \vee 1_0)\backslash t_r$. Also, for any trees s, r we have $s\backslash r = (s_l \vee s_r)\backslash r = s_l \vee (s_r\backslash r)$. Therefore, t is progressive if and only if $t_r = 1_0 = |$. Geometrically, progressive trees have no branching along the right branch from the root; equivalently, all internal nodes are to the *left* of the root.

Every tree $t \neq 1_0$ has a unique decomposition into progressive trees, $t = t_1\backslash t_2\backslash \cdots \backslash t_k$. For example,



1.2. **Some maps of posets.** Order-preserving maps (poset maps) between the posets $\mathcal{Q}_n$, $\mathcal{Y}_n$, and $\mathfrak{S}_n$ are central to the structures of the Hopf algebras $\mathcal{Q}Sym$, $\mathcal{Y}Sym$, and $\mathfrak{S}Sym$. A permutation $\sigma \in \mathfrak{S}_n$ has a *descent* at a position p if $(p, p{+}1) \in \mathrm{Inv}(\sigma)$, that is if $\sigma(p) >$

$\sigma(p{+}1)$. Let $\mathrm{Des}(\sigma) \in \mathcal{Q}_n$ denote the set of descents of a permutation σ. Then $\mathrm{Des}\colon \mathfrak{S}_n \to \mathcal{Q}_n$ is a surjection of posets. Given $\mathsf{S} = \{p_1, \ldots, p_k\} \in \mathcal{Q}_n$, let $Z(\mathsf{S}) \in \mathfrak{S}_n$ be

$$Z(\mathsf{S}) \ := \ id_{p_1} \backslash id_{p_2-p_1} \backslash \cdots \backslash id_{n-p_k}\,.$$

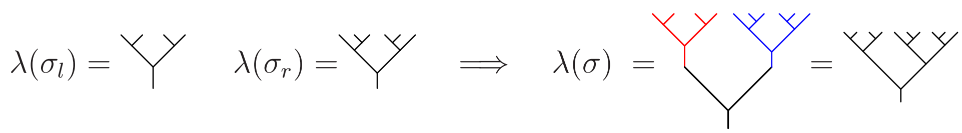
This is the maximum permutation in $\mathfrak{S}_n$ whose descent set is S. The map $Z\colon \mathcal{Q}_n \hookrightarrow \mathfrak{S}_n$ is an embedding of posets, in the sense that $\mathsf{S} \subseteq \mathsf{T} \iff Z(\mathsf{S}) \leq Z(\mathsf{T})$.

The image of Z is described as follows. A permutation $\sigma \in \mathfrak{S}_n$ is 132-*avoiding* if whenever $i < j < k \leq n$, then we *do not* have $\sigma(i) < \sigma(k) < \sigma(j)$. For example, 43512 is 132-avoiding. Similarly, σ is 213-avoiding if whenever $i < j < k \leq n$, then we do not have $\sigma(j) < \sigma(i) < \sigma(k)$. The definition of Z implies that $Z(\mathsf{S})$ is both 132 and 213-avoiding. Since the number of $(132, 213)$-avoiding permutations is 2^{n-1} [6, Ch. 14, Ex. 4], the map Z embeds $\mathcal{Q}_n$ as the subposet of $\mathfrak{S}_n$ consisting of $(132, 213)$-avoiding permutations.

There is a well-known map that sends a permutation to a tree [35, pp. 23-24], [4, Def. 9.9]. We are interested in the following variant $\lambda\colon \mathfrak{S}_n \to \mathcal{Y}_n$, as considered in [23, Section 2.4]. We define $\lambda(id_0) = 1_0$. For $n \geq 1$, let $\sigma \in \mathfrak{S}_n$ and $j := \sigma^{-1}(n)$. We set $\sigma_l := \mathrm{st}(\sigma(1), \ldots, \sigma(j{-}1))$, $\sigma_r := \mathrm{st}(\sigma(j{+}1), \ldots, \sigma(n))$, and define

$$\lambda(\sigma) \ := \ \lambda(\sigma_l) \vee \lambda(\sigma_r)\,. \tag{6}$$

In other words, we construct $\lambda(\sigma)$ recursively by grafting $\lambda(\sigma_l)$ and $\lambda(\sigma_r)$ onto the left and right branches of $\curlyvee$. For example, if $\sigma = 564973812$ then $j = 4$, $\sigma_l = 231$, $\sigma_r = 43512$, and



Note that if $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$ then $\lambda(\sigma \vee \tau) = \lambda(\sigma) \vee \lambda(\tau)$.

It is known that λ is a surjective morphism of posets [4, Prop. 9.10], [24, Cor. 2.8].

Consider the maps $\gamma, \underline{\gamma}\colon \mathcal{Y}_n \to \mathfrak{S}_n$ defined recursively by $\gamma(1_0) = \underline{\gamma}(1_0) := id_0$ and

$$\gamma(t) \ := \ \gamma(t_l) \vee \gamma(t_r) \qquad \text{and} \qquad \underline{\gamma}(t) \ := \ \underline{\gamma}(t_l) \underline{\vee} \underline{\gamma}(t_r)\,. \tag{7}$$

These are the maps denoted Max and Min by Loday and Ronco [24, Def. 2.4]. They show that [24, Thm. 2.5]

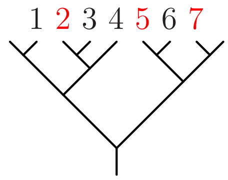
$$\gamma(t) \ := \ \max\{\sigma \in \mathfrak{S}_n \mid \lambda(\sigma) = t\} \qquad \text{and} \qquad \underline{\gamma}(t) \ := \ \min\{\sigma \in \mathfrak{S}_n \mid \lambda(\sigma) = t\}\,. \tag{8}$$

In particular, both γ and $\underline{\gamma}$ are sections of λ. In this paper, we are mostly concerned with the map γ. The recursive definition of γ implies that $\gamma(t)$ is 132-avoiding. Since $\mathcal{Y}_n$ and the set of 132-avoiding permutations in $\mathfrak{S}_n$ are equinumerous [36, p. 261], the map γ embeds $\mathcal{Y}_n$ as the subposet of $\mathfrak{S}_n$ consisting of 132-avoiding permutations.

Since $Z(\mathsf{S})$ is 132-avoiding, there is a unique map $C : \mathcal{Q}_n \hookrightarrow \mathcal{Y}_n$ such that $Z = \gamma \circ C$. It follows that C is an embedding of posets. Explicitly, if $\mathsf{S} = \{p_1, \ldots, p_k\} \in \mathcal{Q}_n$, then

$$C(\mathsf{S}) \ := \ 1_{p_1} \backslash 1_{p_2-p_1} \backslash \cdots \backslash 1_{n-p_k}\,. \tag{9}$$

A tree $t \in \mathcal{Y}_n$ has $n{+}1$ leaves, which we number from 1 to $n{-}1$ left-to-right, excluding the two outermost leaves. Let $L(t)$ be the set of labels of those leaves that point left. For the tree $t \in \mathcal{Y}_8$ below, $L(t) = \{2, 5, 7\}$.



Loday and Ronco [23, Sec. 4.4] note that $\mathrm{Des} = \lambda \circ L$. It follows that $L(t) = \mathrm{Des}(\gamma(t))$ and L is a surjective morphism of posets. In summary:

Proposition 1.1. *We have the following commutative diagrams of poset maps.*

$$\begin{tikzcd} & \mathcal{Y}_n \arrow[dl, two heads, "L"'] & \\ \mathcal{Q}_n & & \mathfrak{S}_n \arrow[ll, two heads, "\mathrm{Des}"'] \arrow[ul, two heads, "\lambda"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{Y}_n \arrow[dr, hook, "\gamma"] & \\ \mathcal{Q}_n \arrow[ur, hook, "C"] \arrow[rr, hook, "Z"] & & \mathfrak{S}_n \end{tikzcd}$$

In addition, $\mathrm{Des} \circ Z = id_{\mathcal{Q}_n}$, $\lambda \circ \gamma = id_{\mathcal{Y}_n}$, $L \circ C = id_{\mathcal{Q}_n}$.

The maps λ, γ, and $\underline{\gamma}$ are well-behaved with respect to the operations $\backslash$ and $/$.

Proposition 1.2. *Let $\sigma \in \mathfrak{S}_p$, $\tau \in \mathfrak{S}_q$, $s \in \mathcal{Y}_p$, and $t \in \mathcal{Y}_q$. Then*

$$\lambda(\sigma\backslash\tau) \;=\; \lambda(\sigma)\backslash\lambda(\tau) \quad \textit{and} \quad \lambda(\sigma/\tau) \;=\; \lambda(\sigma)/\lambda(\tau) \tag{10}$$

$$\gamma(s\backslash t) \;=\; \gamma(s)\backslash\gamma(t) \quad \textit{and} \quad \underline{\gamma}(s/t) \;=\; \underline{\gamma}(s)/\underline{\gamma}(t)\,. \tag{11}$$

Proof. The assertions about λ are given in [24, Thm. 2.9]. For γ, note that by (4), $s\backslash t = s_l \vee (s_r\backslash t)$. Since by definition (7) γ preserves the grafting operations $\vee$, we have $\gamma(s\backslash t) = \gamma(s_l) \vee \gamma(s_r\backslash t)$. Proceeding inductively we derive $\gamma(s\backslash t) = \gamma(s_l) \vee \big(\gamma(s_r)\backslash\gamma(t)\big)$. Finally, from (2) we conclude $\gamma(s\backslash t) = \big(\gamma(s_l) \vee \gamma(s_r)\big)\backslash\gamma(t) = \gamma(s)\backslash\gamma(t)$. The assertion about $\underline{\gamma}$ can be similarly obtained from (5) and (3).

We discuss these maps further in Section 2.

Remark 1.3. The weak order on $\mathfrak{S}_n$ was defined by the inclusion of inversion sets. If we define the inversion set of a tree t to be the inversion set of the permutation $\gamma(t)$, then we obtain that for trees $s, t \in \mathcal{Y}_n$, $s \leq t \Leftrightarrow \mathrm{Inv}(s) \subseteq \mathrm{Inv}(t)$. Hugh Thomas pointed out that this inversion set can de described directly in terms of the tree as follows. Suppose that planar binary trees are drawn with branches either pointing right (⁄) or left (∖). If we illuminate a tree t from the Northeast, then its inversion set $\mathrm{Inv}(t)$ is the shaded region among its branches. For example, Figure 2 shows the inversion set for the tree $\lambda(41253)$. We relate

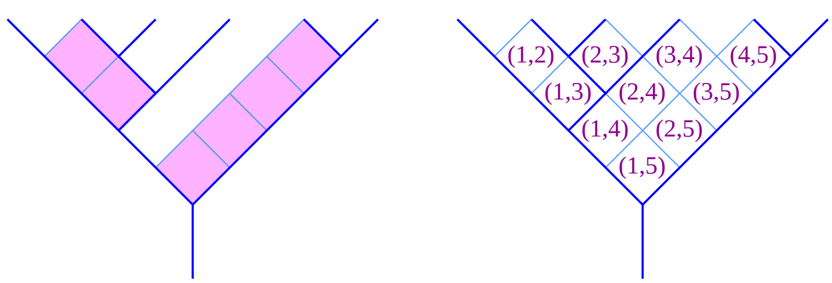


FIGURE 2. The inversion set of a tree.

this to the inversion set of the permutation $\gamma(t)$. If we draw trees in $\mathcal{Y}_n$ on a tilted grid rotated 45° counterclockwise (as we do), then the region between the leftmost and rightmost branches is a collection of boxes which may be labeled from $(1,2)$ in the leftmost box to $(n-1,n)$ in the rightmost box, with the first index increasing in the Northeast direction and the second in the Southeast direction. (See Figure 2.) Then, given a tree t, the labels of the boxes in the shade is the inversion set of $\gamma(t)$. For example, if $t = \lambda(41253)$, then $\gamma(t) = 42351$, and we see that

$$\mathrm{Inv}(t) \;=\; \{(1,2),\,(1,3),\,(1,5),\,(2,5),\,(3,5),\,(4,5)\} \;=\; \mathrm{Inv}(42351)\,.$$

1.3. **The Hopf algebra of quasi-symmetric functions.** We use elementary properties of Hopf algebras, as given in the book [27]. Our Hopf algebras H will be graded connected Hopf algebras over $\mathbb{Q}$. Thus the $\mathbb{Q}$-algebra H is the direct sum $\bigoplus\{H_n \mid n = 0,1,\ldots\}$ of its homogeneous components H_n, with $H_0 = \mathbb{Q}$, the product and coproduct respect the grading, and the counit is the projection onto H_0.

The algebra $\mathcal{Q}Sym$ of quasi-symmetric functions was introduced by Gessel [13] in connection to work of Stanley [34]. Malvenuto described its Hopf algebra structure [25, Section 4.1]. See also [32, 9.4] or [36, Section 7.19].

$\mathcal{Q}Sym$ is a graded connected Hopf algebra. The component of degree n has a linear basis of *monomial quasi-symmetric functions* M_α indexed by compositions $\alpha = (a_1,\ldots,a_k)$ of n. The coproduct is

$$\Delta(M_\alpha) = \sum_{i=0}^{k} M_{(a_1,\ldots,a_i)} \otimes M_{(a_{i+1},\ldots,a_k)}\,. \tag{12}$$

The product of two monomial functions M_α and M_β can be described in terms of quasi-shuffles of α and β. A geometric description for the structure constants in terms of faces of the cube was given in [1, Thm. 7.6].

Gessel's *fundamental* quasi-symmetric function F_β is defined by

$$F_\alpha \;=\; \sum_{\alpha\leq\beta} M_\beta\,.$$

By Möbius inversion, we have

$$M_\alpha \;=\; \sum_{\alpha\leq\beta} (-1)^{k(\beta)-k(\alpha)} F_\beta\,,$$

where $k(\alpha)$ is the number of parts of α. Thus the set $\{F_\alpha\}$ forms another basis of $\mathcal{Q}Sym$.

We often index these monomial and fundamental quasi-symmetric functions by subsets of $[n-1]$. Accordingly, given a composition α of n with $\mathsf{S} = I(\alpha)$, we define

$$F_{\mathsf{S}} \;:=\; F_\alpha \qquad\text{and}\qquad M_{\mathsf{S}} \;:=\; M_\alpha\,.$$

This notation suppresses the dependence on n, which is understood from the context.

1.4. **The Hopf algebra of permutations.** Set $\mathfrak{S}_\infty := \coprod_{n\geq 0}\mathfrak{S}_n$. Let $\mathfrak{S}Sym$ be the graded vector space over $\mathbb{Q}$ with *fundamental basis* $\{F_\sigma \mid \sigma \in \mathfrak{S}_\infty\}$, whose degree n component is spanned by $\{F_\sigma \mid \sigma \in \mathfrak{S}_n\}$. Write 1 for the basis element of degree 0. Malvenuto and Reutenauer [25, 26] described a Hopf algebra structure on this space that was further elucidated in [1]. Here, as in [1], we study the self-dual Hopf algebra $\mathfrak{S}Sym$ with respect to bases dual to those in [25, 26].

The product of two basis elements is obtained by shuffling the corresponding permutations. For $p, q > 0$, set

$$\mathfrak{S}^{(p,q)} \;:=\; \{\zeta \in \mathfrak{S}_{p+q} \mid \zeta \text{ has at most one descent, at position } p\}\,.$$

This is the collection of minimal representatives of left cosets of $\mathfrak{S}_p \times \mathfrak{S}_q$ in $\mathfrak{S}_{p+q}$. These are sometimes called (p, q)-shuffles. For $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$, set

$$F_\sigma \cdot F_\tau \;=\; \sum_{\zeta \in \mathfrak{S}^{(p,q)}} F_{(\sigma/\tau)\cdot\zeta^{-1}}\,.$$

This endows $\mathfrak{S}Sym$ with the structure of a graded algebra with unit 1.

The algebra $\mathfrak{S}Sym$ is also a graded coalgebra with coproduct given by all ways of splitting a permutation. More precisely, define $\Delta\colon \mathfrak{S}Sym \to \mathfrak{S}Sym \otimes \mathfrak{S}Sym$ by

$$\Delta(F_\sigma) \;=\; \sum_{p=0}^{n} F_{\mathrm{st}(\sigma(1),\,\dots,\,\sigma(p))} \otimes F_{\mathrm{st}(\sigma(p+1),\,\dots,\,\sigma(n))}\,,$$

when $\sigma \in \mathfrak{S}_n$. With these definitions, $\mathfrak{S}Sym$ is a graded connected Hopf algebra.

The descent map induces a morphism of Hopf algebras.

$$\begin{array}{rcl} \mathcal{D} \;:\; \mathfrak{S}Sym & \longrightarrow & \mathcal{Q}Sym \\ F_\sigma & \longmapsto & F_{\mathrm{Des}(\sigma)} \end{array}$$

There is another basis $\{M_\sigma \mid \sigma \in \mathfrak{S}_\infty\}$ for $\mathfrak{S}Sym$. For each $n \geq 0$ and $\sigma \in \mathfrak{S}_n$, define

$$M_\sigma \;:=\; \sum_{\sigma \leq \tau} \mu_{\mathfrak{S}_n}(\sigma, \tau) \cdot F_v\,, \tag{13}$$

where $\mu_{\mathfrak{S}_n}(\cdot,\cdot)$ is the Möbius function of the weak order on $\mathfrak{S}_n$. By Möbius inversion,

$$F_\sigma \;:=\; \sum_{\sigma \leq \tau} M_\tau\,,$$

so these elements M_σ indeed form a basis of $\mathfrak{S}Sym$. The algebraic structure of $\mathfrak{S}Sym$ with respect to this M-basis was determined in [1].

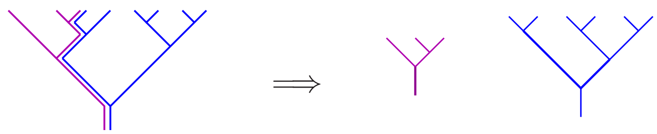
Proposition 1.4. *Let $w \in \mathfrak{S}_n$. Then*

$$\Delta(M_\rho) = \sum_{\rho = \sigma \backslash \tau} M_\sigma \otimes M_\tau\,, \tag{14}$$

$$\mathcal{D}(M_\sigma) = \begin{cases} M_{\mathsf{S}} & \text{if } \sigma = Z(\mathsf{S}), \quad \text{for some } \mathsf{S} \in \mathcal{Q}_n\,, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The multiplicative structure constants are non-negative integers with the following description. The 1-skeleton of the permutahedron Π_n is the Hasse diagram of the weak order on $\mathfrak{S}_n$. Its facets are canonically isomorphic to products of lower dimensional permutahedra. Say that a facet isomorphic to $\Pi_p \times \Pi_q$ has *type* (p, q). Given $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$, such a facet has a distinguished vertex corresponding to (σ, τ) under the canonical isomorphism. Then, for $\rho \in \mathfrak{S}_{p+q}$, the coefficient of M_ρ in $M_\sigma \cdot M_\tau$ is the number of facets of Π_{p+q} of type (p, q) with the property that the distinguished vertex is below ρ (in the weak order) and closer to ρ than to any other vertex in the facet.

1.5. **The Hopf algebra of planar binary trees.** Set $\mathcal{Y}_\infty := \coprod_{n\geq 0} \mathcal{Y}_n$. The Hopf algebra $\mathcal{Y}Sym$ of planar binary trees is the graded vector space over $\mathbb{Q}$ with *fundamental basis* $\{F_t \mid t \in \mathcal{Y}_\infty\}$ whose degree n component is spanned by $\{F_t \mid t \in \mathcal{Y}_n\}$. We describe its multiplication and comultiplication in terms of a geometric construction on trees. For any leaf of a tree t, we may divide t into two pieces—the piece left of the leaf and the piece right of the leaf—by dividing t along the path from the leaf to the root. We illustrate this on the tree $\lambda(67458231)$.

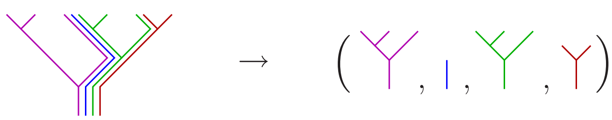


If r is the piece to the left of the leaf of t and s the piece to the right, write $t \to (r, s)$.

Suppose that $t \in \mathcal{Y}_p$ and $s \in \mathcal{Y}_q$. Divide t into $q+1$ pieces at a multisubset of its $p+1$ leaves of cardinality q:

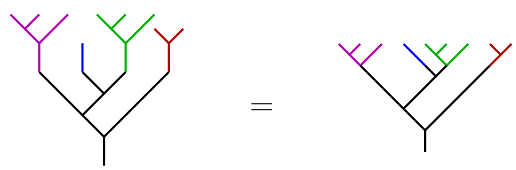
$$t \;\to\; (t_0, t_1, \ldots, t_q)\,.$$

This may be done in $\binom{p+q}{p}$ ways. Label the leaves of s from 0 to q left-to-right. For each such division $t \to (t_0, t_1, \ldots, t_q)$, attach t_i to the ith leaf of s to obtain the tree $(t_0, t_1, \ldots, t_q)/s$. For example, we divide $\lambda(45231)$ at three leaves to obtain



$$= \;(t_0, t_1, t_2, t_3)\,.$$

Then if $s = \lambda(213)$, the tree $(t_0, t_1, t_2, t_3)/s$ is



We define a coproduct and a product on $\mathcal{Y}Sym$. For $t \in \mathcal{Y}_p$ and $s \in \mathcal{Y}_q$, set

$$\Delta(F_t) \;=\; \sum_{t\to(t_0,t_1)} F_{t_0} \otimes F_{t_1} \qquad \text{and} \qquad F_t \cdot F_s \;=\; \sum_{t\to(t_0,t_1,\ldots,t_q)} F_{(t_0,t_1,\ldots,t_q)/s}\,.$$

These are compatible with the operations on $\mathfrak{S}Sym$ and $\mathcal{Q}Sym$. The maps $\lambda\colon \mathfrak{S}_n \to \mathcal{Y}_n$ and $L\colon \mathcal{Y}_n \to \mathcal{Q}_n$ induce linear maps

$$\begin{array}{cccccccccc} \Lambda & : & \mathfrak{S}Sym & \longrightarrow & \mathcal{Y}Sym & \qquad \mathcal{L} & : & \mathcal{Y}Sym & \longrightarrow & \mathcal{Q}Sym \\ & & F_\sigma & \longmapsto & F_{\lambda(\sigma)} & & & F_t & \longmapsto & F_{L(t)} \end{array} \tag{16}$$

Proposition 1.5. *The maps $\Lambda\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$ and $\mathcal{L}\colon \mathcal{Y}Sym \to \mathcal{Q}Sym$ are surjective morphisms of Hopf algebras.*

As with both $\mathcal{Q}Sym$ and $\mathfrak{S}Sym$, we define another basis $\{M_t \mid t \in \mathcal{Y}_\infty\}$, related to the fundamental basis via Möbius inversion on $\mathcal{Y}_n$. For each $n \geq 0$ and $t \in \mathcal{Y}_n$, define

$$M_t \;:=\; \sum_{t\leq s} \mu_{\mathcal{Y}_n}(t, s) \cdot F_s\,, \tag{17}$$

where $\mu_{\mathcal{Y}_n}(\cdot,\cdot)$ is the Möbius function of $\mathcal{Y}_n$. By Möbius inversion,

$$F_t \;:=\; \sum_{t\leq s} M_s\,,$$

so these elements M_t indeed form a basis of $\mathcal{Y}Sym$. For instance,

In this paper we determine the algebraic structure of $\mathcal{Y}Sym$ with respect to this basis. For example, we will show (Theorems 3.1 and 5.1) that, given $t\in\mathcal{Y}_n$, then

$$\Delta(M_t) = \sum_{t=r\backslash s} M_r \otimes M_s\,;$$

$$\Lambda(M_\sigma) = \begin{cases} M_t & \text{if } \sigma=\gamma(t),\ t\in\mathcal{Y}_n,\\ 0 & \text{otherwise;}\end{cases}$$

$$\mathcal{L}(M_t) = \begin{cases} M_{\mathsf{T}} & \text{if } t=C(\mathsf{T}),\ \mathsf{T}\in\mathcal{Q}_n,\\ 0 & \text{otherwise.}\end{cases}$$

We will also obtain a geometric description for the structure constants of the multiplication of $\mathcal{Y}Sym$ on this basis in terms of the associahedron (Corollary 4.4), and an explicit description for the structure constants of the antipode (Theorem 6.1).

2. Some Galois connections between posets

In Section 1.2 we described order-preserving maps

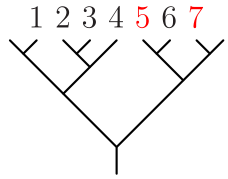
$$\begin{tikzcd} & \mathcal{Y}_n \arrow[dl, two heads, "L"'] & \\ \mathcal{Q}_n & & \mathfrak{S}_n \arrow[ll, two heads, "\mathrm{Des}"'] \arrow[ul, two heads, "\lambda"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{Y}_n \arrow[dr, hook, "\gamma"] & \\ \mathcal{Q}_n \arrow[ur, hook, "C"] \arrow[rr, hook, "Z"] & & \mathfrak{S}_n \end{tikzcd}$$

between the posets $\mathcal{Q}_n$, $\mathcal{Y}_n$, and $\mathfrak{S}_n$. Recall from Sections 1.4 and 1.5 that when the maps in the leftmost diagram are applied to the fundamental bases, they induce morphisms of Hopf algebras $\mathfrak{S}Sym \twoheadrightarrow \mathcal{Y}Sym \twoheadrightarrow \mathcal{Q}Sym$. The values of these morphisms on the monomial bases can be described through another set of poset maps given below.

$$\begin{tikzcd} & \mathcal{Y}_n \arrow[dl, two heads, "R"'] & \\ \mathcal{Q}_n & & \mathfrak{S}_n \arrow[ll, two heads, "\mathrm{GDes}"'] \arrow[ul, two heads, "\rho"'] \end{tikzcd}$$

A permutation σ has a *global descent* at a position $p\in[n-1]$ if $\sigma=\rho\backslash\tau$ with $\rho\in\mathfrak{S}_p$. The map GDes sends a permutation to its set of global descents. Global descents were studied in [1] in connection to the structure of the Hopf algebra $\mathfrak{S}Sym$.

To define the map R, take a tree $t\in\mathcal{Y}_n$ and number its leaves from 1 to $n-1$ left-to-right, excluding the two outermost leaves as before. Let $R(t)$ be the set of labels of those leaves that belong to a branch that emanates from the rightmost branch of the tree. In other words, $R(t)$ is set of $j\in[n-1]$ for which the tree admits a decomposition $r\backslash t$ with $r\in\mathcal{Y}_j$. For the tree $t \in \mathcal{Y}_8$ below, $R(t) = \{5, 7\}$.



The map ρ is defined below. It appears to be new, but by Theorem 2.1 and Lemma 2.2 below it is quite natural.

These maps have very interesting order-theoretic properties. A *Galois connection* between posets P and Q is a pair (f, g) of order-preserving maps $f\colon P \to Q$ and $g\colon Q \to P$ such that for any $x \in P$ and $y \in Q$,

$$f(x) \;\leq\; y \quad\Longleftrightarrow\quad x \;\leq\; g(y)\,.$$

We also say that f is left adjoint to g, and g is right adjoint to f.

Theorem 2.1. *We have the following commutative diagrams of order-preserving maps.*

$$\begin{tikzcd} & \mathcal{Y}_n \arrow[dl, two heads, "L"'] & \\ \mathcal{Q}_n & & \mathfrak{S}_n \arrow[ll, two heads, "\mathrm{Des}"'] \arrow[ul, two heads, "\lambda"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{Y}_n \arrow[dr, hook, "\gamma"] & \\ \mathcal{Q}_n \arrow[ur, hook, "C"] \arrow[rr, hook, "Z"] & & \mathfrak{S}_n \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{Y}_n \arrow[dl, two heads, "R"'] & \\ \mathcal{Q}_n & & \mathfrak{S}_n \arrow[ll, two heads, "\mathrm{GDes}"'] \arrow[ul, two heads, "\rho"'] \end{tikzcd}$$

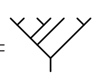
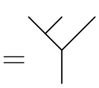
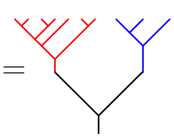
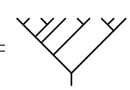
Moreover, the corresponding maps in adjacent diagrams form Galois connections between the appropriate posets. That is, the maps in the left diagram are left adjoint to the corresponding maps in the central diagram, and the maps in the right diagram are right adjoint to the corresponding maps in the central diagram.

Recall the recursive definition (6) of the map λ, where we split a permutation at its greatest value. The map ρ is similarly described in terms of splitting the permutation, except now we split it at its first global descent.

We define $\rho(id_0) = 1_0$. For $n \geq 1$, let $\sigma \in \mathfrak{S}_n$ and suppose that j is the position of its first global descent. Let $\sigma_l = \mathrm{st}(\sigma(1), \ldots, \sigma(j-1))$ and $\sigma_r = \mathrm{st}(\sigma(j+1), \ldots, \sigma(n))$. If there are no global descents, we set $j = n$ (and $\sigma_r = id_0$). Note that σ_l and σ_r generally differ from the permutations in the definition (6) of λ. Define

$$\rho(\sigma) \;:=\; \rho(\sigma_l) \vee \rho(\sigma_r)\,. \tag{18}$$

For example, if $\sigma = 564973812$, then the first global descent occurs at the position of the 8, and thus $\sigma_l = \mathrm{st}(564973) = 342761$, $\sigma_r = \mathrm{st}(12) = 12$, and

$\rho(\sigma_l) =$  $\rho(\sigma_r) =$  $\Longrightarrow$ $\rho(\sigma) =$  $=$ 

If σ is 132-avoiding, then the first global descent occurs at the maximum value of σ, and both σ_l and σ_r are 132-avoiding. Thus for 132-avoiding permutations σ, $\rho(\sigma) = \lambda(\sigma)$, and we conclude that $\rho(\gamma(t)) = t$.

Lemma 2.2. *Let $n \geq 0$. The map $\rho\colon \mathfrak{S}_n \to \mathcal{Y}_n$ is order-preserving and for each tree $t \in \mathcal{Y}_n$, we have*

$$\gamma(t) \;=\; \min\{\sigma \in \mathfrak{S}_n \mid \rho(\sigma) = t\}\,. \tag{19}$$

Proof. Suppose that $\sigma \in \mathfrak{S}_n$ is not 132-avoiding. Then we construct a permutation σ' with $\sigma' \lessdot \sigma$ such that $\rho(\sigma') = \rho(\sigma)$. Since this process terminates when σ is 132-avoiding, this, together with $\rho(\gamma(t)) = t$, proves (19).

Suppose that σ has a 132-pattern. Among all 132-patterns of σ choose a pattern ($i < j < k$ with $\sigma(i) < \sigma(k) < \sigma(j)$) with $\sigma(k)$ maximum, and among those, with $\sigma(j)$ minimum. Let m be the position such that $\sigma(m) = \sigma(k)+1$. We must have $m \leq j$ for $m > j$ contradicts the maximality of $\sigma(k)$. The choice of j implies that either $m = j$ or else $m < i$. Transposing the values $\sigma(m) = \sigma(k)+1$ and $\sigma(k)$ gives a new permutation $\sigma' \lessdot \sigma$. We then iterate this procedure, eventually obtaining a 132-avoiding permutation. For example, we iterate this procedure on a permutation in $\mathfrak{S}_7$:

$$\underline{47}5\underline{6}132 \;\to\; \underline{465}7132 \;\to\; 4567\underline{132} \;\to\; 4567213\,.$$

We use induction on n to prove that $\rho(\sigma') = \rho(\sigma)$. First note that σ and σ' have the same global descents. This is clear for global descents outside of the interval $[m, k]$. By the 132-pattern at $i < j < k$, the only other possibility is if σ or σ' has a global descent at k, but σ' has a global descent at k if and only if σ does. In particular, σ and σ' have the same first global descent. If this is at k, then $\sigma'_r = \sigma_r$ and $\sigma'_l = \sigma_l$, as only one of the transposed values $\sigma(k)$ and $\sigma(k)-1 = \sigma(m)$ occurs to the left of k and neither occurs to the right. If the first global descent is outside of the interval $[i, k]$, then one of σ_l or σ_r contains the pattern used to construct σ'. If σ_l contains that pattern, then $\sigma'_r = \sigma_r$, so $\rho(\sigma'_r) = \rho(\sigma_r)$, and our inductive hypothesis implies that $\rho(\sigma'_l) = \rho(\sigma_l)$. We reach the same conclusion if $\sigma'_l = \sigma_l$, and so we conclude that $\rho(\sigma') = \rho(\sigma)$.

To see that ρ is order-preserving, let $\tau \leq \sigma$. Since $\gamma(\rho(\tau)) \leq \tau$ and $\gamma(\rho(\tau))$ is 132-avoiding, we need only show that if $\tau \leq \sigma$ and τ is 132-avoiding, then $\tau \leq \gamma(\rho(\sigma))$. Consider the construction of the permutation $\sigma' \lessdot \sigma$ in the preceding paragraph. Observe that σ has exactly one more inversion, (m, k), than does σ', where either $m = j$ or $m < i$. Since $i < j < k$ is a 132-pattern in σ and $\tau \leq \sigma$ is 132-avoiding, we must have that $\tau(i) < \tau(j) < \tau(k)$. If (m, k) were an inversion of τ, then $\tau(m) > \tau(j)$, and so (m, j) is an inversion of σ, as $\tau \leq \sigma$ implies that $\mathrm{Inv}(\tau) \subseteq \mathrm{Inv}(\sigma)$. But this contradicts the choice of j, which implies that $\sigma(j) \geq \sigma(k)+1 = \sigma(m)$. We conclude by induction that ρ is order-preserving.

Proof of Theorem 2.1. The result for the horizontal maps appears in [1] (Propositions 2.11 and 2.13).

We first treat the maps between $\mathfrak{S}_n$ and $\mathcal{Y}_n$. Suppose that $\sigma \in \mathfrak{S}_n$ and $t \in \mathcal{Y}_n$ satisfy $\lambda(\sigma) \leq t$. Then by (8), we have $\sigma \leq \gamma(\lambda(w))$. Since γ is order-preserving, we have $\gamma(\lambda(\sigma)) \leq \gamma(t)$, and so $\sigma \leq \gamma(t)$. Conversely, suppose that $\sigma \leq \gamma(t)$. Since λ is order-preserving, $\lambda(\sigma) \leq \lambda(\gamma(t))$. But $\lambda \circ \gamma$ is the identity, so we conclude that $\lambda(\sigma) \leq t$. Thus λ is left adjoint to γ.

Virtually the same argument using Lemma 2.2 shows that for $t \in \mathcal{Y}_n$ and $\sigma \in \mathfrak{S}_n$,

$$\gamma(t) \leq \sigma \iff t \leq \rho(\sigma)\,.$$

Lastly, the remaining two equivalences, that for $t \in \mathcal{Y}_n$ and $\mathsf{S} \in \mathcal{Q}_n$, we have

$$\begin{aligned} L(t) \leq \mathsf{S} &\iff t \leq C(\mathsf{S}) \\ C(\mathsf{S}) \leq t &\iff \mathsf{S} \leq R(t) \end{aligned}$$

follow from the corresponding facts for the horizontal maps via $\gamma\colon \mathcal{Y}_n \hookrightarrow \mathfrak{S}_n$.

We remark that the Galois connection (λ, γ) can be traced back to [4, Sec. 9]. Generalizations appear in [29]. All the ingredients for the Galois connection (L, C) also appear in [24, Prop. 3.5].

3. Some Hopf morphisms involving $\mathcal{Y}Sym$

Consider the diagram

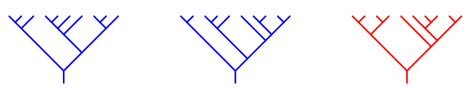
$$
\begin{tikzcd}
 & \mathcal{Y}Sym \arrow[dl, two heads, "\mathcal{L}"'] & \\
\mathcal{Q}Sym & & \mathfrak{S}Sym \arrow[ll, two heads, "\mathcal{D}"] \arrow[ul, two heads, "\Lambda"']
\end{tikzcd}
\tag{20}
$$

These are surjective morphisms of Hopf algebras (Proposition 1.5) and the diagram commutes (on the fundamental basis, this is the commutativity of the first diagram in Theorem 2.1). We use the Galois connections of Section 2 to determine the effect of these maps on the bases of monomial functions.

Recall that a permutation $\sigma \in \mathfrak{S}_n$ has the form $Z(\mathsf{S})$ for some $\mathsf{S} \in \mathcal{Q}_n$ if and only if it is $(132, 213)$-avoiding (Section 1.2). Thus (15) states that

$$
\mathcal{D}(M_\sigma) = \begin{cases} M_{\mathrm{Des}(\sigma)} & \text{if } \sigma \text{ is } (132,213)\text{-avoiding,} \\ 0 & \text{otherwise.} \end{cases}
$$

We derive similar descriptions for the maps Λ and $\mathcal{L}$ in terms of pattern avoidance. We say that a tree $t \in \mathcal{Y}_n$ is $\curlyvee$-avoiding if it has the form $t = C(\mathsf{S})$ for some $\mathsf{S} \in \mathcal{Q}_n$. Since $\gamma \circ C = Z$, the tree t is $\curlyvee$-avoiding if and only if the permutation $\gamma(t)$ is 213-avoiding. Geometrically, t is $\curlyvee$-avoiding if every leftward pointing branch emanates from the rightmost branch. Equivalently, if each indecomposable component is a right comb. For example, the two trees on the left below are $\curlyvee$-avoiding, while the tree on the right is not.



Theorem 3.1. *Let $\sigma \in \mathfrak{S}_n$ and $t \in \mathcal{Y}_n$. Then*

$$
\Lambda(M_\sigma) = \begin{cases} M_{\lambda(\sigma)} & \textit{if } \sigma \textit{ is } 132\textit{-avoiding,} \\ 0 & \textit{otherwise;} \end{cases}
\tag{21}
$$

$$
\mathcal{L}(M_t) = \begin{cases} M_{L(t)} & \textit{if } t \textit{ is } \curlyvee\textit{-avoiding,} \\ 0 & \textit{otherwise.} \end{cases}
\tag{22}
$$

Proof. If we have poset maps $f\colon P \to Q$ and $g\colon Q \to P$ with f left adjoint to g, then Rota [33, Theorem 1] showed that the Möbius functions of P and Q are related by

$$
\forall\, x \in P \text{ and } \ w \in Q, \qquad \sum_{\substack{y \in P \\ x \le y,\, f(y)=w}} \mu_P(x,y) \;=\; \sum_{\substack{v \in Q \\ v \le w,\, g(v)=x}} \mu_Q(v,w)\,.
$$

Thus, for $\sigma \in \mathfrak{S}_n$ and $t \in \mathcal{Y}_n$, we have

$$
\sum_{\substack{\sigma \le \tau \in \mathfrak{S}_n \\ \lambda(\tau)=t}} \mu_{\mathfrak{S}_n}(\sigma,\tau) \;=\; \sum_{\substack{s \le t \in \mathcal{Y}_n \\ \gamma(s)=\sigma}} \mu_{\mathcal{Y}_n}(s,t)\,.
$$

If σ is not 132-avoiding, then the index set on the right hand side is empty. If σ is 132-avoiding, then by (8), the index set consists only of the tree $s = \lambda(\sigma)$, so we have

$$\sum_{\substack{\sigma\leq\tau\in\mathfrak{S}_n\\ \lambda(\tau)=t}} \mu_{\mathfrak{S}_n}(\sigma,\tau) \;=\; \begin{cases} \mu_{\mathcal{Y}_n}(\lambda(\sigma),t) & \text{if } \sigma \text{ is 132-avoiding,}\\ 0 & \text{if not.}\end{cases} \tag{23}$$

Now, according to (13) and (16),

$$\begin{aligned}\Lambda(M_\sigma) \;&=\; \sum_{\sigma\leq\tau} \mu_{\mathfrak{S}_n}(\sigma,\tau)\cdot F_{\lambda(\tau)} \quad=\quad \sum_t \Bigl(\sum_{\substack{\sigma\leq\tau\\ \lambda(\tau)=t}} \mu_{\mathfrak{S}_n}(\sigma,\tau)\Bigr)\cdot F_t\\ &=\; \begin{cases} \sum_t \mu_{\mathcal{Y}_n}(\lambda(\sigma),t)\cdot F_t & \text{if } \sigma \text{ is 132-avoiding}\\ 0 & \text{if not.}\end{cases}\end{aligned}$$

This proves the first part of the theorem in view of (17).

For the second part, let $t \in \mathcal{Y}_n$. We just showed that $\Lambda(M_{\gamma(t)}) = M_t$. From the commutativity of (20) we deduce

$$\mathcal{L}(M_t) \;=\; \mathcal{L}\bigl(\Lambda(M_{\gamma(t)})\bigr) \;=\; \mathcal{D}(M_{\gamma(t)})\,.$$

The remaining assertion follows from this and (15).

4. Geometric interpretation of the product of $\mathcal{Y}Sym$

For $\mathfrak{S}Sym$, the multiplicative structure constants with respect to the basis $\{M_\sigma \mid \sigma \in \mathfrak{S}_\infty\}$ were given a combinatorial description in Theorem 4.1 of [1]. In particular, they are non-negative. An immediate consequence of this and (21) is that the Hopf algebra $\mathcal{Y}Sym$ has non-negative multiplicative structure constants with respect to the basis $\{M_t \mid t \in \mathcal{Y}_\infty\}$. We give a direct combinatorial interpretation of these structure constants and complement it with a geometric interpretation in terms of the facial structure of the associahedron.

For each permutation $\zeta \in \mathfrak{S}^{(p,q)}$ consider the maps

$$\varphi_\zeta\colon \mathfrak{S}_p\times\mathfrak{S}_q \to \mathfrak{S}_{p+q} \qquad\text{and}\qquad f_\zeta\colon \mathcal{Y}_p\times\mathcal{Y}_q\to\mathcal{Y}_{p+q}$$

defined by

$$\varphi_\zeta(\sigma,\tau) \;:=\; (\sigma/\tau)\cdot\zeta^{-1} \qquad\text{and}\qquad f_\zeta(s,t) \;:=\; \lambda\bigl(\gamma(s)/\gamma(t)\cdot\zeta^{-1}\bigr)\,. \tag{24}$$

We supress the dependence of φ_ζ and f_ζ on p and q; only when ζ is the identity permutation does this matter. We view $\mathfrak{S}_p\times\mathfrak{S}_q$ as the Cartesian product of the posets $\mathfrak{S}_p$ and $\mathfrak{S}_q$, and similarly for $\mathcal{Y}_p\times\mathcal{Y}_q$. The map φ_ζ is order-preserving [1, Prop. 2.7]. Since

$$f_\zeta(s,t) \;=\; (\lambda\circ\varphi_\zeta)(\gamma(s),\gamma(t))\,, \tag{25}$$

f_ζ is also order-preserving.

We describe the structure constants of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ in terms of these maps.

Proposition 4.1 (Theorem 4.1 of [1])**.** *Let $\sigma \in \mathfrak{S}_p$, $\tau\in\mathfrak{S}_q$, and $\rho\in\mathfrak{S}_{p+q}$. Then the coefficient of M_ρ in the product $M_\sigma\cdot M_\tau$ is*

$$\alpha^\rho_{\sigma,\tau} \;:=\; \#\{\zeta\in\mathfrak{S}^{(p,q)} \mid (\sigma,\tau) = \max\varphi_\zeta^{-1}[id_{p+q},\rho]\}\,,$$

where $[1_{p+q},\rho] = \{\rho'\in\mathcal{Y}_{p+q} \mid \rho'\leq\rho\}$, the interval in $\mathfrak{S}_{p+q}$ below ρ.

Theorem 4.2. *Let $s \in \mathcal{Y}_p$, $t \in \mathcal{Y}_q$, and $r \in \mathcal{Y}_{p+q}$. The coefficient of M_r in $M_s \cdot M_t$ is*

$$\#\{\zeta \in \mathfrak{S}^{(p,q)} \mid (s,t) = \max f_\zeta^{-1}[1_{p+q}, r]\}\,. \tag{26}$$

Proof. By (21), $M_s \cdot M_t = \Lambda\big(M_{\gamma(s)} \cdot M_{\gamma(t)}\big)$. We evaluate this using Proposition 4.1 and (21) to obtain

$$M_s \cdot M_t \;=\; \Lambda\Big(\sum_{\rho \in \mathfrak{S}_{p+q}} \alpha^{\rho}_{\gamma(s),\gamma(t)}\, M_\rho\Big) \;=\; \sum_{r \in \mathcal{Y}_{p+q}} \alpha^{\gamma(r)}_{\gamma(s),\gamma(t)} M_r\,.$$

The constant $\alpha^{\gamma(r)}_{\gamma(s),\gamma(t)}$ is equal to

$$\#\{\zeta \in \mathfrak{S}^{(p,q)} \mid (\gamma(s),\gamma(t)) = \max \varphi_\zeta^{-1}[id_{p+q}, \gamma(r)]\}\,.$$

The Galois connections between $\mathfrak{S}_{p+q}$ and $\mathcal{Y}_{p+q}$ of Theorem 2.1 imply that

$$\lambda\big(\gamma(s)/\gamma(t)\cdot\zeta^{-1}\big) \;\leq\; r \iff \gamma(s)/\gamma(t)\cdot\zeta^{-1} \;\leq\; \gamma(r)\,.$$

By the definitions of f_ζ and φ_ζ, it follows that

$$(s,t) \;=\; \max f_\zeta^{-1}[1_{p+q}, r] \iff (\gamma(s),\gamma(t)) \;=\; \max \varphi_\zeta^{-1}[id_{p+q}, \gamma(r)]\,,$$

and hence $\alpha^{\gamma(r)}_{\gamma(s),\gamma(t)}$ equals (26), which completes the proof. ∎

The Hasse diagram of $\mathcal{Y}_n$ is isomorphic to the 1-skeleton of the *associahedron* $\mathcal{A}_n$, an $(n-1)$-dimensional polytope. (See [39, pp. 304–310] and [36, p. 271].) The faces of $\mathcal{A}_n$ are in one-to one correspondence with collections of non-intersecting diagonals of a polygon with $n+2$ sides (an $(n+2)$-gon). Equivalently, the faces of $\mathcal{A}_n$ correspond to polygonal subdivisions of an $n+2$-gon with facets corresponding to diagonals and vertices to triangulations. The dual graph of a polygonal subdivision is a planar tree and the dual graph of a triangulation is a planar binary tree. If we distinguish one edge to be the root edge, the trees are rooted, and this furnishes a bijection between the vertices of $\mathcal{A}_n$ and $\mathcal{Y}_n$. Figure 3 shows two views of the associahedron $\mathcal{A}_3$, the first as polygonal subdivisions of the pentagon, and the second as the corresponding dual graphs (planar trees). The root is at the bottom.

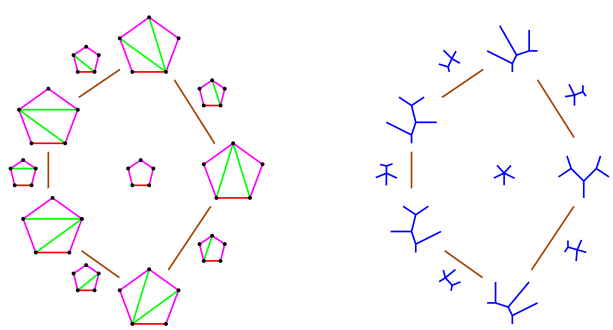


FIGURE 3. Two views of the associahedron $\mathcal{A}_3$

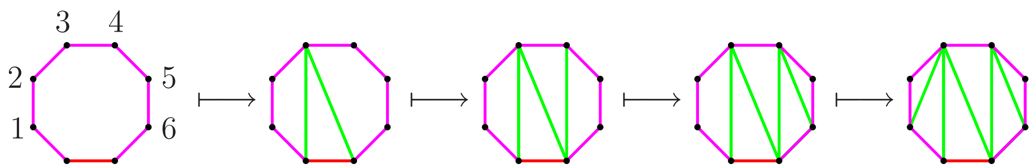
We describe the map $\lambda\colon \mathfrak{S}_n \to \mathcal{Y}_n$ in terms of triangulations of the $(n+2)$-gon where we label the vertices with $0, 1, \ldots, n, n+1$ beginning with the left vertex of the root edge and proceeding clockwise. Let $\sigma \in \mathfrak{S}_n$ and set $w_i := \sigma^{-1}(n+1-i)$, for $i = 1, \ldots, n$. This records the positions of the values of σ taken in decreasing order. We inductively construct the triangulation, beginning with the empty triangulation consisting of the root edge, and after i steps we have a triangulation T_i of the polygon

$$P_i \ := \ Conv\{0, n{+}1, w_1, \ldots, w_i\}\,.$$

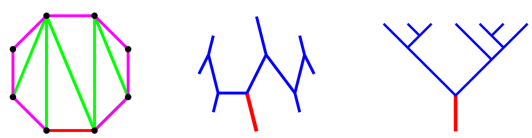
Some edges of P_i will be edges of the original $(n{+}2)$-gon and others will be diagonals. Each diagonal cuts the $(n{+}2)$-gon into two pieces, one containing P_i and the other a polygon which is not yet triangulated and whose root edge we take to be that diagonal. Subsequent steps add to the triangulation T_i and its support P_i.

First set $T_1 := Conv\{0, n{+}1, w_1\}$, the triangle with base the root edge and apex the vertex $w_1 = \sigma^{-1}(n)$. Set $P_1 := T_1$ and continue. After i steps we have constructed T_i and P_i in such a way that the vertex w_{i+1} is not in P_i. Hence it must lie in some untriangulated polygon consisting of some consecutive edges of the $(n{+}2)$-gon and a diagonal that is an edge of P_i. Add the join of the vertex w_{i+1} and the diagonal to the triangulation to obtain a triangulation T_{i+1} of the polygon P_{i+1}. The process terminates when $i = n$.

For example, we display this process for the permutation $\sigma = 316524$, where we label the vertices of the first octagon:

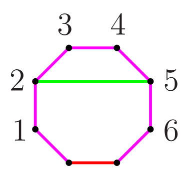


The last two steps are supressed as they add no new diagonals. The dual graph to the triangulation T_n is the planar binary tree $\lambda(\sigma)$. Here is the triangulation, its dual graph, and a 'straightened' version, which we recognize as the tree $\lambda(316524)$.

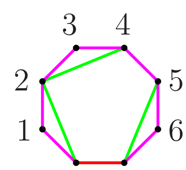


A subset S of $[n]$ determines a face Φ_{S} of the associahedron $\mathcal{A}_n$ as follows. Suppose that we label the vertices of the $(n{+}2)$-gon as above. Then the vertices labeled $0, n{+}1$ and those labeled by S form a $(\#\mathsf{S} + 2)$-gon whose edges include a set E of non-crossing diagonals of the original $(n{+}2)$-gon. These diagonals determine the face Φ_{S} of $\mathcal{A}_n$ corresponding to S. We give two examples of this association when $n = 6$ below.

$\{1, 2, 5, 6\} \longleftrightarrow$



$\{2, 4, 5\} \longleftrightarrow$



We determine the image of f_ζ using the above description of the map $\lambda\colon \mathfrak{S}_n \to \mathcal{Y}_n$.

Proposition 4.3. *Given $\zeta \in \mathfrak{S}^{(p,q)}$, the image of $\lambda \circ \varphi_\zeta$ coincides with the image of f_ζ and equals the face $\Phi_{\zeta\{p+1,\ldots,p+q\}}$ of $\mathcal{A}_{p+q}$. This is a facet if and only if ζ is 132-avoiding.*

Proof. Let $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$, and set $\rho := \varphi_\zeta(\sigma, \tau) = (\sigma/\tau) \cdot \zeta^{-1}$. Since the q largest values of ρ lie in the positions $\mathsf{S} := \zeta\{p{+}1, \ldots, p{+}q\}$, the triangulation $\lambda(\rho)$ is obtained by triangulating $Q := Conv\{0, p{+}q{+}1, \mathsf{S}\}$ with $\lambda(\tau)$, and then placing triangulations given by parts of σ in the polygons that lie outside Q. More precisely, suppose that $\{1, \ldots, p{+}q\} - \mathsf{S}$

consists of strings of $a_1, a_2, \ldots, a_r$ consecutive numbers. Then the ith polygon P_i outside of Q is triangulated according to $\lambda(\mathrm{st}(\sigma_{A_i+1}, \ldots, \sigma_{A_i+a_i}))$, where $A_i = a_1 + \cdots + a_{i-1}$. Thus all triangulations of the $(n{+}2)$-gon that include the edges of Q are obtained from permutations of the form $(\gamma(s_1)/\gamma(s_2)/\cdots/\gamma(s_r),\ \gamma(t))$, where $s_i \in \mathcal{Y}_{a_i}$ and $t \in \mathcal{Y}_q$. But this describes the face Φ_{S} and shows that $f_\zeta(\mathcal{Y}_p \times \mathcal{Y}_q) = (\lambda \circ \varphi_\zeta)(\mathfrak{S}_p \times \mathfrak{S}_q) = \Phi_{\mathsf{S}}$.

This face will be a facet only if $r = 1$, that is if $\{1, \ldots, p{+}q\} - \mathsf{S}$ consists of consecutive numbers, which is equivalent to ζ avoiding the pattern 132.

We say that a face of $\mathcal{A}_{p+q}$ of the form Φ_{S} with $\#\mathsf{S} = q$ has *type* (p,q). If a face has a type, this type is unique. A permutation $\zeta \in \mathfrak{S}^{(p,q)}$ is uniquely determined by the set $\zeta\{p{+}1, \ldots, p{+}q\}$. Therefore, a face of type (p,q) is the image of f_ζ for a unique permutation $\zeta \in \mathfrak{S}^{(p,q)}$. This allows us to speak of the vertex of the face corresponding to a pair $(s,t) \in \mathcal{Y}_p \times \mathcal{Y}_q$ (under f_ζ).

We deduce a geometric interpretation of the multiplicative structure constants from Proposition 4.3 and Theorem 4.2.

Corollary 4.4. *Let $s \in \mathcal{Y}_p$ and $t \in \mathcal{Y}_q$. The coefficient of M_r in the product $M_s \cdot M_t$ equals the number of faces of the associahedron $\mathcal{A}_{p+q}$ of type (p,q) such that the vertex corresponding to (s,t) is below r, and it is the maximum vertex on its face below r.*

Proposition 4.3 describes the image of a facet $\varphi_\zeta(\mathfrak{S}_p \times \mathfrak{S}_q)$ of the permutahedron for $\mathfrak{S}_{p+q}$ under the map λ. More generally, it is known that the image of any face of the permutahedron is a face of the associahedron Tonks [37]. The map from the permutahedron to the associahedron can also be understood by means of the theory of *fiber polytopes* [3, Sec. 5], [31, Sec. 4.3]. For more on the permutahedron and associahedron, see [22].

5. Cofreeness and the coalgebra structure of $\mathcal{Y}Sym$

We compute the coproduct on the basis $\{M_t \mid y \in \mathcal{Y}_\infty\}$ and deduce the existence of a new grading for which $\mathcal{Y}Sym$ is cofree. We show that $\{M_{t\vee|} \mid t \in \mathcal{Y}_\infty\}$ is a basis for the space of primitive elements and describe the coradical filtration of $\mathcal{Y}Sym$. Since $\mathcal{Y}Sym$ is the graded dual of the Loday-Ronco Hopf algebra LR, this work strengthens the result [23, Theorem 3.8] of Loday and Ronco that LR is a free associative algebra.

Theorem 5.1. *Let $r \in \mathcal{Y}_n$. Then*

$$\Delta(M_r) \ = \ \sum_{r=s\backslash t} M_s \otimes M_t \,. \tag{27}$$

Proof. Suppose that $\rho \in \mathfrak{S}_n$ is 132-avoiding and we decompose ρ as $\rho = \sigma\backslash\tau$. Then both σ and τ are 132-avoiding: a 132 pattern in either would give a 132 pattern in ρ. We use that Λ is a morphism of coalgebras and (14) to obtain

$$\begin{aligned}
\Delta(M_r) \ &= \ \Delta(\Lambda(M_{\gamma(r)})) \ = \ \Lambda(\Delta(M_{\gamma(r)})) \\
&= \ \Lambda\Big(\sum_{\gamma(r)=\sigma\backslash\tau} M_\sigma \otimes M_\tau \Big) \\
&= \ \sum_{\gamma(r)=\sigma\backslash\tau} M_{\lambda(\sigma)} \otimes M_{\lambda(\tau)} \quad = \quad \sum_{r=s\backslash t} M_s \otimes M_t \,,
\end{aligned}$$

the last equality by Proposition 1.2.

We recall the notion of cofree graded coalgebras. Let V be a vector space and set

$$Q(V) \;:=\; \bigoplus_{k\geq 0} V^{\otimes k}\,,$$

which is naturally graded by k. Given $v_1,\ldots,v_k \in V$, let $v_1\backslash v_2\backslash\cdots\backslash v_k$ denote the corresponding tensor in $V^{\otimes k}$. Under the *deconcatenation* coproduct

$$\Delta(v_1\backslash\cdots\backslash v_k) \;=\; \sum_{i=0}^{k}(v_1\backslash\cdots\backslash v_i)\otimes(v_{i+1}\backslash\cdots\backslash v_k)\,,$$

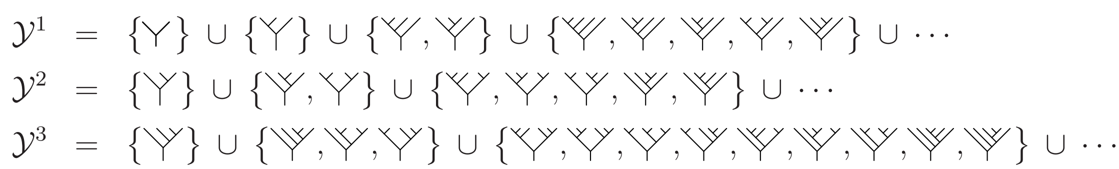
and counit $\epsilon(v_1\backslash\cdots\backslash v_k)=0$ for $k\geq 1$, $Q(V)$ is a graded connected coalgebra, the *cofree graded coalgebra on* V.

We show that $\mathcal{Y}Sym$ is cofree by first defining a second coalgebra grading, where the degree of M_t is the number of branches of t emanating from the rightmost branch (including the leftmost branch), that is, $1+\#R(t)$. This is also the number of components in the (right) decomposition of t into progressive trees $t=t_1\backslash t_2\backslash\cdots\backslash t_k$, as in Section 1.

First, set $\mathcal{Y}^0 := \mathcal{Y}_0$, and for $k\geq 1$, let

$$\begin{aligned}\mathcal{Y}^k_n \;&:=\; \{t\in\mathcal{Y}_n \mid t \text{ has exactly } k \text{ progressive components}\}, \quad\text{and}\\ \mathcal{Y}^k \;&:=\; \coprod_{n\geq k}\mathcal{Y}^k_n\,.\end{aligned}$$

In particular $\mathcal{Y}^1$ consists of the progressive trees, those of the form $t\vee|$. For instance,



Let $(\mathcal{Y}Sym)^k$ be the vector subspace of $\mathcal{Y}Sym$ spanned by $\{M_t \mid t\in\mathcal{Y}^k\}$.

Theorem 5.2. *The decomposition $\mathcal{Y}Sym=\oplus_{k\geq 0}(\mathcal{Y}Sym)^k$ is a coalgebra grading. With this grading $\mathcal{Y}Sym$ is a cofree graded coalgebra.*

Proof. Let $V := (\mathcal{Y}Sym)^1$, the span of $\{M_t \mid t \text{ is progressive}\}$. Then the map

$$M_{t_1}\otimes M_{t_2}\otimes\cdots\otimes M_{t_k} \;\longmapsto\; M_{t_1\backslash t_2\backslash\cdots\backslash t_k}\,,$$

identifies $V^{\otimes k}$ with $(\mathcal{Y}Sym)^k$. Together with the coproduct formula (27), this identifies $\mathcal{Y}Sym$ with the deconcatenation coalgebra $Q(V)$.

The coradical C^0 of a graded connected coalgebra C is the 1-dimensional component in degree 0. The primitive elements of C are

$$\mathrm{P}(C) \;:=\; \{x\in C \mid \Delta(x)=x\otimes 1+1\otimes x\}\,.$$

Set $C^1 := C^0\oplus\mathrm{P}(C)$, the first level of the coradical filtration. More generally, the k-th level of the coradical filtration is

$$C^k \;:=\; \left(\Delta^k\right)^{-1}\Bigl(\sum_{i+j=k} C^{\otimes i}\otimes C^0\otimes C^{\otimes j}\Bigr)\,.$$

We have $C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots \subseteq C = \bigcup_{k\geq 0} C^k$, and

$$\Delta(C^k) \ \subseteq \ \sum_{i+j=k} C^i {\otimes} C^j\, .$$

Thus, the coradical filtration measures the complexity of iterated coproducts.

When C is a cofree graded coalgebra $Q(V)$, its space of primitive elements is just V, and the k-th level of its coradical filtration is $\oplus_{i=0}^k V^{\otimes i}$. We record these facts for $\mathcal{Y}Sym$.

Corollary 5.3. *A linear basis for the k-th level of the coradical filtration of $\mathcal{Y}Sym$ is*

$$\{M_t \mid t \in \mathcal{Y}^k\}\, .$$

In particular, a linear basis for the space of primitive elements is

$$\{M_t \mid t \ \textit{is progressive}\}.$$

Remark 5.4. Recall that a tree $t = t_l \vee t_r$ is progressive if and only if $t_r = 1_0 = |$ (Section 1.1). It follows that the number of progressive trees in $\mathcal{Y}_n$ is $\dim(\mathcal{Y}Sym)^1_n = c_{n-1}$. Theorem 5.2 implies that the Hilbert series of $\mathcal{Y}Sym$ and $(\mathcal{Y}Sym)^1$ are related by

$$\sum_{n\geq 0} c_n t^n = \frac{1}{1-\sum_{n\geq 1} c_{n-1} t^n}\, . \tag{28}$$

This is equivalent to the usual recursion for the Catalan numbers

$$c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k} \quad \forall\, n \geq 1\,, \quad c_0 = 1\, .$$

Remark 5.5. Let gr$\mathcal{Y}Sym$ be the associated graded Hopf algebra to $\mathcal{Y}Sym$ under its coradical filtration. This is bigraded, as it also retains the original grading of $\mathcal{Y}Sym$. Greg Warrington showed that this is commutative, and it is in fact the shuffle Hopf algebra generated by the M_t for t a progressive tree. Grossman and Larson [14] defined a graded cocommutative Hopf algebra of planar trees, whose dual is isomorphic to gr$\mathcal{Y}Sym$ [2].

6. Antipode of $\mathcal{Y}Sym$

We give an explicit formula for the antipode of $\mathcal{Y}Sym$. This is a simple consequence of the formula in [1, Thm. 5.5] for the antipode of $\mathfrak{S}Sym$. (See Remark 9.5 in [30].) Let $\tau \in \mathfrak{S}_n$. Subsets $\mathsf{R} \subseteq \mathrm{GDes}(\tau)$ correspond to decompositions of τ

$$\mathsf{R} \ \leftrightarrow \ \tau \ = \ \tau_1\backslash\tau_2\backslash\cdots\backslash\tau_r\, .$$

For such a partial decomposition R, set

$$\tau_{\mathsf{R}} \ := \ \tau_1/\tau_2/\cdots/\tau_r\, .$$

For example, for the decomposition R of the permutation

$$\tau \ = \ 798563421 \ = \ 132\backslash 3412\backslash 21$$

we have

$$\tau_{\mathsf{R}} \ = \ 132/3412/21 \ = \ 132674598\, .$$

Lastly, given n and a subset S of $[n-1]$, $\mathfrak{S}^{\mathsf{S}}$ denotes the set of permutations $\sigma \in \mathfrak{S}_n$ such that $\mathrm{Des}(\sigma) \subseteq \mathsf{S}$; equivalently, $\mathfrak{S}^{\mathsf{S}} = [id_n, Z(\mathsf{S})]$.

Theorem 6.1. *For $t \in \mathcal{Y}_n$,*

$$S(M_t) \;=\; -(-1)^{\#R(t)} \sum_{s \in \mathcal{Y}_n} \kappa(t,s) M_s \;,$$

where $\kappa(t,s)$ records the number of permutations $\zeta \in \mathfrak{S}^{R(t)}$ that satisfy

(i) $\lambda(\gamma(t)_{R(t)} \cdot \zeta^{-1}) \leq s$,
(ii) $t \leq t'$ *and* $\lambda(\gamma(t')_{R(t)} \cdot \zeta^{-1}) \leq s$ *implies that* $t = t'$, *and*
(iii) *If* $\mathrm{Des}(\zeta) \subseteq \mathsf{R} \subseteq R(t)$ *and* $\lambda(\gamma(t)_{\mathsf{R}} \cdot \zeta^{-1}) \leq s$, *then* $\mathsf{R} = R(t)$.

Proof. Theorem 5.5 of [1] gives the following formula for the antipode on $\mathfrak{S}Sym$:

$$S(M_\tau) \;=\; -(-1)^{\#\mathrm{GDes}(\tau)} \sum_{\sigma \in \mathfrak{S}_n} k(\tau,\sigma) M_\sigma \,,$$

where $k(\tau,\sigma)$ records the number of permutations $\zeta \in \mathfrak{S}^{\mathrm{GDes}(\tau)}$ that satisfy

(a) $\tau_{\mathrm{GDes}(\tau)} \cdot \zeta^{-1} \leq \sigma$,
(b) $\tau \leq \tau'$ and $\tau'_{\mathrm{GDes}(\tau)} \cdot \zeta^{-1} \leq \sigma$ implies that $\tau = \tau'$, and
(c) If $\mathrm{Des}(\zeta) \subseteq \mathsf{R} \subseteq \mathrm{GDes}(\tau)$ and $\tau_{\mathsf{R}} \cdot \zeta^{-1} \leq \sigma$, then $\mathsf{R} = \mathrm{GDes}(\tau)$.

Since $\Lambda\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$ is a morphism of Hopf algebras, $\Lambda(S(M_\tau)) = S(\Lambda(M_\tau))$. Also, (21) says that $\Lambda(M_\sigma) = 0$ unless $\sigma = \gamma(s)$ for some $s \in \mathcal{Y}_n$. Since we also have $R(s) = \mathrm{GDes}(\gamma(s))$ for $s \in \mathcal{Y}_n$, the theorem will follow from this result for $\mathfrak{S}Sym$ if the set of permutations $\zeta \in \mathfrak{S}^{R(t)}$ satisfying conditions (i), (ii), and (iii) for trees $s, t \in \mathcal{Y}_n$ in the statement of the theorem equals the set satisfying (a), (b), and (c) for $\tau = \gamma(t)$ and $\sigma = \gamma(s)$.

Suppose that $\mathrm{Des}(\zeta) \subseteq \mathsf{R} \subseteq R(t)$ ($= \mathrm{GDes}(\tau)$). Then $\gamma(t)_{R(t)} \cdot \zeta^{-1} = \tau_{\mathrm{GDes}(\tau)} \cdot \zeta^{-1}$, and so

$$\lambda(\gamma(t)_{R(t)} \cdot \zeta^{-1}) \leq s \iff \tau_{\mathrm{GDes}(\tau)} \cdot \zeta^{-1} \leq \sigma \,,$$

by the Galois connection (Theorem 2.1). This shows that (i) and (a) are equivalent, as well as (iii) and (c).

We show that (b) implies (ii). Suppose that $t' \in \mathcal{Y}_n$ satisfies

$$t \leq t' \qquad \text{and} \qquad \lambda(\gamma(t')_{R(t)} \cdot \zeta^{-1}) \leq s \,.$$

Let $\tau' := \gamma(t')$. Applying γ to the first inequality and treating the second as in the preceding paragraph (replacing t' for t) we obtain

$$\tau \leq \tau' \qquad \text{and} \qquad \tau'_{\mathrm{GDes}(\tau)} \cdot \zeta^{-1} \leq \sigma \,.$$

Hypothesis (b) implies $\tau = \tau'$ and applying γ we conclude $t = t'$, so (ii) holds.

To see that (ii) implies (b), suppose that $\tau' \in \mathfrak{S}_n$ satisfies

$$\gamma(t) \;\leq\; \tau' \qquad \text{and} \qquad \tau'_{R(t)} \cdot \zeta^{-1} \;\leq\; \gamma(s) \,. \tag{29}$$

By Remark 6.3 below, $\lambda(\tau'_{R(t)} \cdot \zeta^{-1}) = \lambda\Big(\gamma\big(\lambda(\tau')\big)_{R(t)} \cdot \zeta^{-1}\Big)$. If we apply λ to (29), we obtain

$$t \;\leq\; \lambda(\tau') \qquad \text{and} \qquad \lambda\Big(\gamma\big(\lambda(\tau')\big)_{R(t)} \cdot \zeta^{-1}\Big) \;\leq\; s \,.$$

Assuming (ii), we conclude that $t = \lambda(\tau')$. This implies $\gamma(t) \geq \tau'$ by (8). Hence $\tau = \gamma(t) = \tau'$, and (b) holds. ⅋

We remark that by similar techniques one may derive an explicit formula for the antipode of $\mathcal{Y}Sym$ on the fundamental basis F_t, working from the corresponding result for $\mathfrak{S}Sym$ [1, Thm. 5.4].

For $\tau \in \mathfrak{S}_n$, let $\overline{\tau} := \gamma(\lambda(\tau))$, the unique 132-avoiding permutation such that $\lambda(\tau) = \lambda(\overline{\tau})$. Suppose that $1 \leq a < b \leq n$. We define a premutation $\overline{\tau}^{[a,b]}$ which has no 132-patterns having values in the interval $[a, b]$ (no occurrences of $i < j < k$ with $a \leq \tau(i) < \tau(k) < \tau(j) \leq b$), and which satisfies $\lambda(\tau) = \lambda(\overline{\tau}^{[a,b]})$. Set $\mathsf{S} = \{s_1 < \cdots < s_m\} := \tau^{-1}([a, b])$ and let σ be the permutation $\mathrm{st}(\tau(s_1), \ldots, \tau(s_m))$, the standard permutation formed by the values of τ in the interval $[a, b]$. Define $\overline{\tau}^{[a,b]} \in \mathfrak{S}_n$ to be the permutation

$$\overline{\tau}^{[a,b]}(i) \;=\; \begin{cases} \tau(i) & \text{if } i \notin \mathsf{S} \\ a - 1 + \overline{\sigma}(j) & \text{if } i = s_j \in \mathsf{S} \end{cases}$$

Lemma 6.2. *With the above definitions, $\lambda(\tau) = \lambda(\overline{\tau}^{[a,b]})$.*

Remark 6.3. Suppose that τ is a permutation and $\mathsf{R} = \{r_1 < \cdots < r_{m-1}\}$ is a subset of $\mathrm{GDes}(\tau)$. Thus $\tau = \tau_1 \backslash \tau_2 \backslash \cdots \backslash \tau_m$ with $\tau_i \in \mathfrak{S}_{r_i - r_{i-1}}$, where $0 = r_0$ and $r_m = n$. Then, by Proposition 1.2, $\overline{\tau} = \overline{\tau_1} \backslash \cdots \backslash \overline{\tau_m}$.

Let $\zeta \in \mathfrak{S}^{\mathsf{R}}$ and consider the permutations

$$\tau_{\mathsf{R}} \cdot \zeta^{-1} \;=\; (\tau_1/\tau_2/\cdots/\tau_m) \cdot \zeta^{-1} \quad \text{and} \quad \overline{\tau}_{\mathsf{R}} \cdot \zeta^{-1} \;=\; (\overline{\tau_1}/\overline{\tau_2}/\cdots/\overline{\tau_m}) \cdot \zeta^{-1}\,.$$

Observe that by the definitions preceeding the statement of the lemma,

$$(\tau_1/\cdots/\overline{\tau_i}/\cdots/\tau_m) \cdot \zeta^{-1} \;=\; \overline{\tau_{\mathsf{R}} \cdot \zeta^{-1}}^{[1+r_{i-1}, r_i]}\,.$$

Thus

$$\overline{\tau}_{\mathsf{R}} \cdot \zeta^{-1} \;=\; \overline{(\cdots(\overline{\tau_{\mathsf{R}} \cdot \zeta^{-1}}^{[1,r_1]})\cdots)}^{[1+r_{m-1},n]}\,.$$

By Lemma 6.2 we conclude that

$$\lambda(\tau_{\mathsf{R}} \cdot \zeta^{-1}) \;=\; \lambda(\overline{\tau}_{\mathsf{R}} \cdot \zeta^{-1}) \;=\; \lambda\Big(\gamma\big(\lambda(\tau)\big)\Big)_{\mathsf{R}} \cdot \zeta^{-1})\,,$$

which was needed in the proof of Theorem 6.1.

Proof of Lemma 6.2. We prove this by increasing induction on n and decreasing induction on the length of the permutation τ. The initial cases are trivial and immediate. Consider 132-patterns in τ with values in $[a, b]$. If τ has no 132-pattern with values in $[a, b]$, then $\overline{\tau}^{[a,b]} = \tau$, and there is nothing to show.

Otherwise, consider the 132-patterns in τ with values in $[a, b]$ where $\tau(j)$ is minimal, and among those, consider patterns where $\tau(k)$ is also minimal. Finally, among those, consider the one with $\tau(i)$ maximal. That is $(\tau(j), \tau(k), -\tau(i))$ is minimal in the lexicographic order. We claim that $\tau(i) = \tau(k) - 1$. Indeed, define m by $\tau(m) = \tau(k) - 1$. We cannot have $j < m$, for then $i < j < m$ would give a 132-pattern with values in $[a, b]$ where $(\tau(j), \tau(m))$ preceeds $(\tau(j), \tau(k))$ in the lexicographic order. Since $m < j$, the choice of i forces $m = i$.

Transposing the values $\tau(i)$ and $\tau(k)$ gives a permutation τ' with $\tau \lessdot \tau'$. We claim that $\lambda(\tau') = \lambda(\tau)$. This will complete the proof, as we are proceeding by downwward induction on the length of τ.

We prove this by induction on n. Consider forming the trees $\lambda(\tau)$ and $\lambda(\tau')$. Let $m := \tau^{-1}(n)$, then $\tau'(m) = n$, also. As in the definition of λ (6), form τ_l and τ_r, and the same for τ'. If $i < m < k$, then $\tau'_l = \tau_l$ and $\tau'_r = \tau_r$, and so $\lambda(\tau) = \lambda(\tau')$. If $k < m$, then $\tau'_r = \tau_r$, but $\tau'_l \neq \tau_l$. However, τ'_l is obtained from τ_l by interchanging the values $\tau_l(i)$ and $\tau_l(k)$, and $i < j < k$ is a 132-pattern in τ_l with values in an interval where $(\tau_l(j), \tau_l(k), -\tau_l(i))$ is minimal in the lexicographic order. By induction on n, $\lambda(\tau_l) = \lambda(\tau'_l)$, and so $\lambda(\tau) =$

$\lambda(\tau_l) \vee \lambda(\tau_r) = \lambda(\tau_l') \vee \lambda(\tau_r') = \lambda(\tau')$. Similar arguments suffice when $m < k$. This completes the proof.

7. Crossed product decompositions for $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$

We observe that the surjective morphisms of Hopf algebras of Section 3

$$\begin{tikzcd} & \mathcal{Y}Sym \arrow[dl, two heads, "\mathcal{L}"'] & \\ \mathcal{Q}Sym & & \mathfrak{S}Sym \arrow[ul, two heads, "\Lambda"'] \arrow[ll, two heads, "\mathcal{D}"'] \end{tikzcd}$$

admit splittings as coalgebras, and thus $\mathfrak{S}Sym$ is a crossed product over $\mathcal{Y}Sym$ and $\mathcal{Y}Sym$ is a crossed product over $\mathcal{Q}Sym$. We elucidate these structures.

Recall the poset embeddings of Section 1.2:

$$\begin{tikzcd} & \mathcal{Y}_n \arrow[dr, hook, "\gamma"] & \\ \mathcal{Q}_n \arrow[ur, hook, "C"] \arrow[rr, hook, "Z"] & & \mathfrak{S}_n \end{tikzcd}$$

We use them to define linear maps as follows:

$$\begin{aligned} \mathcal{C}\colon \mathcal{Q}Sym \to \mathcal{Y}Sym\,, \quad M_\alpha &\longmapsto M_{C(\alpha)}\,; \\ \Gamma\colon \mathcal{Y}Sym \to \mathfrak{S}Sym\,, \quad M_t &\longmapsto M_{\gamma(t)}\,; \\ \mathcal{Z}\colon \mathcal{Q}Sym \to \mathfrak{S}Sym\,, \quad M_\alpha &\longmapsto M_{Z(\alpha)}\,. \end{aligned}$$

The following theorem is immediate from the expression for the coproduct on the M-bases of $\mathcal{Q}Sym$ (12), $\mathfrak{S}Sym$ (14), and $\mathcal{Y}Sym$ (27), and the formulas for the maps $\mathcal{D}$ (15), Λ (21) and $\mathcal{L}$ (22) on these bases.

Theorem 7.1. *The following is a commutative diagram of injective morphisms of coalgebras which split the corresponding surjections of* (20)*.*

$$\begin{tikzcd} & \mathcal{Y}Sym \arrow[dr, hook, "\Gamma"] & \\ \mathcal{Q}Sym \arrow[ur, hook, "\mathcal{C}"] \arrow[rr, hook, "\mathcal{Z}"] & & \mathfrak{S}Sym \end{tikzcd}$$

We use a theorem of Blattner, Cohen, and Montgomery [5], [27, Ch. 7]. Suppose that $\pi\colon H \to K$ is a morphism of Hopf algebras admitting a coalgebra splitting $\gamma\colon K \to H$. Then there is a *crossed product* decomposition

$$H \;\cong\; A\#_c K$$

where A, a subalgebra of H, is the *left Hopf kernel* of π:

$$A \;:=\; \{h \in H \mid \sum h_1 \otimes \pi(h_2) = h \otimes 1\}$$

and the *Hopf cocycle* $c\colon K \otimes K \to A$ is

$$c(k,k') = \sum \gamma(k_1)\gamma(k_1')S\gamma(k_2k_2')\,. \tag{30}$$

Note that if π and γ preserve gradings, then so does the rest of this structure.

The crossed product decomposition of $\mathfrak{S}Sym$ over $\mathcal{Q}Sym$ corresponding to $(\mathcal{D}, \mathcal{Z})$ was described in [1, Sec. 8]. We describe the left Hopf kernels A of $\Gamma\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$ and B of $\mathcal{C}\colon \mathcal{Y}Sym \to \mathcal{Q}Sym$, which are graded with components A_n and B_n. Let $n > 0$. Recall (Section 1.1) that a permutation $\tau \in \mathfrak{S}_n$ admits a unique decomposition into permutations with no global descents and a tree $t \in \mathcal{Y}_n$ admits a unique decomposition into progressive trees:

$$\tau \;=\; \tau_1 \backslash \cdots \backslash \tau_k \qquad t \;=\; t_1 \backslash \cdots \backslash t_l\,.$$

We call τ_k and t_l the *last components* of τ and t. Recall that the minimum tree $1_n = \lambda(id_n)$ is called a right comb.

Theorem 7.2. *A basis for A_n is the set $\{M_\tau\}$ where τ runs over all permutations of n whose last component is not 132-avoiding. A basis for B_n is the set $\{M_t\}$ where t runs over all trees whose last component is not a right comb. In particular,*

$$\dim A_n \;=\; n! - \sum_{k=0}^{n-1} k!c_{n-k-1} \qquad \textit{and} \qquad \dim B_n \;=\; c_n - \sum_{k=0}^{n-1} c_k\,,$$

where $c_k = \#\mathcal{Y}_k$ is the kth Catalan number.

Proof. By the theorem of Blattner, Cohen, and Montgomery, we have

$$\mathfrak{S}Sym \cong A\#_c\mathcal{Y}Sym \quad \text{and} \quad \mathcal{Y}Sym \cong B\#_c\mathcal{Q}Sym\,.$$

In particular $\mathfrak{S}Sym \cong A\otimes\mathcal{Y}Sym$ and $\mathcal{Y}Sym \cong B\otimes\mathcal{Q}Sym$ as vector spaces. The Hilbert series for these graded algebras are therefore related by

$$\sum_{n\geq 0} n!t^n = \Bigl(\sum_{n\geq 0} a_n t^n\Bigr)\Bigl(\sum_{n\geq 0} c_n t^n\Bigr) \quad \text{and} \quad \sum_{n\geq 0} c_n t^n = \Bigl(\sum_{n\geq 0} b_n t^n\Bigr)\Bigl(1 + \sum_{n\geq 1} 2^{n-1}t^n\Bigr)\,,$$

where $a_n := \dim A_n$ and $b_n := \dim B_n$. Using (28) we deduce $a_n = n! - \sum_{k=0}^{n-1} k!c_{n-k-1}$, and using

$$1 + \sum_{n\geq 1} 2^{n-1}t^n = \frac{1}{1 - \sum_{n\geq 1} t^n}$$

we deduce $b_n = c_n - \sum_{k=0}^{n-1} c_k$ as claimed.

The number of permutations in $\mathfrak{S}_n$ which are 132-avoiding and have no global descents equals the number of progressive trees in $\mathcal{Y}_n$, which is c_{n-1}. Therefore, a_n counts the number of permutations in $\mathfrak{S}_n$ whose last component is not 132-avoiding. Suppose that τ is such a permutation and $\tau = \sigma\backslash\rho$ is an arbitrary decomposition. As long as $\rho \neq id_0$, the last component of ρ is the last component of τ and hence ρ is not 132-avoiding. Thus formulas (14) and (21) imply that $(id\otimes\Lambda)\Delta(M_\tau) = M_\tau\otimes 1$ and so M_τ lies in the Hopf kernel of Λ. Since these elements are linearly independent, they form a basis of A_n as claimed.

Similarly, b_n counts the number of trees in $\mathcal{Y}_n$ whose last component is not a comb, and analogous arguments using (22) and (27) show that if t is such a tree, then M_t lies in the Hopf kernel of $\mathcal{L}$.

8. The dual of $\mathcal{Y}Sym$ and the non-commutative Connes-Kreimer Hopf algebra

We turn now to the structure of the Loday-Ronco Hopf algebra LR, which we define as the graded dual of the Hopf algebra $\mathcal{Y}Sym$. It is known that these graded Hopf algebras are isomorphic [10, 11, 16, 19, 38]. An explicit isomorphism is obtained as the composite [16, Thm. 4], [17, Thm. 34]

$$LR \xrightarrow{\Lambda^*} (\mathfrak{S}Sym)^* \cong \mathfrak{S}Sym \xrightarrow{\Lambda} \mathcal{Y}Sym$$

where the isomorphism $\mathfrak{S}Sym^* \cong \mathfrak{S}Sym$ sends an element F^*_σ of the dual of the fundamental basis to $F_{\sigma^{-1}}$. Thus the results of this section also apply to $\mathcal{Y}Sym$ itself.

8.1. **The dual of the monomial basis.** Let $\{M^*_t \mid t \in \mathcal{Y}_\infty\}$ be the basis of LR dual to the monomial basis of $\mathcal{Y}Sym$.

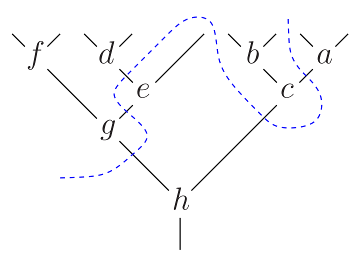
By Theorem 5.2, $\mathcal{Y}Sym$ is cofree as a coalgebra and $\{M_t \mid t \in \mathcal{Y}^1\}$ is a basis for its primitive elements. This and the form of the coproduct have the following consequence, which also appears in [17, Thm. 29].

Theorem 8.1. *LR is the free associative algebra generated by $\{M^*_t \mid t \in \mathcal{Y}^1\}$ with*

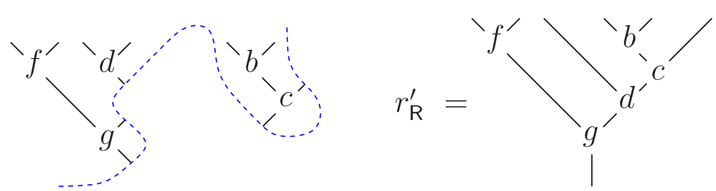
$$M^*_s \cdot M^*_t \;=\; M^*_{s\backslash t} \qquad \text{for } s,t \in \mathcal{Y}_\infty\,.$$

The description of the coproduct requires a definition.

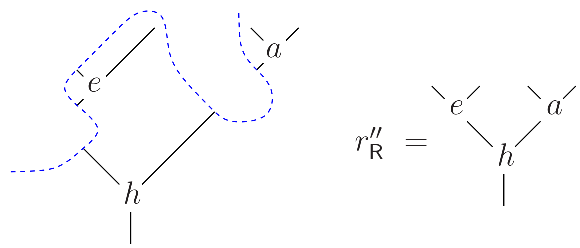
Definition 8.2. A subset R of internal nodes of a planar binary tree is *admissible* if for any node $x \in \mathsf{R}$, the left child y of x and all the descendants of y are in R. Thus any internal node in the left subtree above x also lies in R. An admissible set R of internal nodes in a planar binary tree gives rise to a *pruning:* cut each branch connecting a node from R to a node in its complement R^c. For example, here is a planar binary tree whose internal nodes are labeled $a, b, \ldots, h$ with an admissible set of nodes $\mathsf{R} = \{f, g, d, b, c\}$. The corresponding pruning is indicated by the dotted line.

(31) 

The branches removed in such a pruning of a planar binary tree r form a forest of planar binary trees $r_1, \ldots, r_p$, ordered from left to right by the positions of their leaves in r. Assemble these into a planar binary tree $r'_{\mathsf{R}} := r_1 \backslash r_2 \backslash \cdots \backslash r_p$. In (31), here is the pruned forest and the resulting tree r'_{R}:



The rest of the tree r also forms a forest, which is assembled into a tree in a different fashion. If a tree s in that forest is above another tree t (in the original tree r) and there are no intervening components, then there is a unique leaf of t that is below the root of s. Attach the root of s to that leaf of t. As R is admissible, there will be a unique tree in this forest below all the others whose root is the root of the planar binary tree r''_{R} obtained by this assembly. In (31), here is the forest that remains and the tree r''_{R}.



We record how this construction behaves under the grafting operation on trees.

Lemma 8.3. *Let $s, t \in \mathcal{Y}_\infty$ be planar binary trees and $r = s \vee t$. Let R be an admissible subset of the internal nodes of r, and S (respectively T) those nodes of R lying in s (respectively t).*
If the root node of r lies in R, then all the nodes of s lie in R, and

$$r'_{\mathsf{R}} \;=\; s \vee t'_{\mathsf{T}} \qquad \text{and} \qquad r''_{\mathsf{R}} \;=\; t''_{\mathsf{T}}\ .$$

If If the root node of r does not lie in R, then

$$r'_{\mathsf{R}} \;=\; s'_{\mathsf{S}} \backslash t'_{\mathsf{T}} \qquad \text{and} \qquad r''_{\mathsf{R}} \;=\; s''_{\mathsf{S}} \vee t''_{\mathsf{T}}\ .$$

We describe the coproduct of LR in terms of the basis $\{M^*_t \mid t \in \mathcal{Y}_\infty\}$.

Theorem 8.4. *For any tree $t \in \mathcal{Y}_\infty$,*

$$\Delta(M^*_t) \;=\; \sum M^*_{t'_{\mathsf{S}}} \otimes M^*_{t''_{\mathsf{S}}}\,, \tag{32}$$

the sum over all admissible subsets S of internal nodes of t.

We begin our proof of Theorem 8.4. Recall the product formula of Theorem 4.2. For $s \in \mathcal{Y}_p$ and $t \in \mathcal{Y}_q$,

$$M_s \cdot M_t \;=\; \sum_{r \in \mathcal{Y}_{p+q}} \alpha^r_{s,t}\, M_r\,,$$

where $\alpha^r_{s,t}$ enumerates the set

$$\{\zeta \in \mathfrak{S}^{(p,q)} \;\mid\; (s,t) \in \mathcal{Y}_p \times \mathcal{Y}_q \text{ is maximum such that } f_\zeta(s,t) \leq r\}$$

and $f_\zeta \colon \mathcal{Y}_p \times \mathcal{Y}_q \to \mathcal{Y}_{p+q}$ is the map

$$f_\zeta(s,t) \;=\; \lambda\big(\gamma(s)/\gamma(t) \cdot \zeta^{-1}\big)\,.$$

Dualizing this formula gives a formula for the coproduct of LR. Let $r \in \mathcal{Y}_n$. Then

$$\Delta(M^*_r) \;=\; \sum_{p=0}^{n} \sum_{\zeta} M^*_s \otimes M^*_t\,, \tag{33}$$

where (s,t) is the maximum element of $\mathcal{Y}_p \times \mathcal{Y}_q$ such that $f_\zeta(s,t) \leq r$, and the inner sum is over all $\zeta \in \mathfrak{S}^{(p,q)}$ such that $\{(s',t') \in \mathcal{Y}_p \times \mathcal{Y}_q \mid f_\zeta(s',t') \leq r\} \neq \emptyset$.

We will deduce (32) from (33). Key to this is another reformulation of the coproduct intermediate between these two.

For a subset $\mathsf{R} \subseteq [n]$ with p elements, let $\mathsf{R}^c := [n] - \mathsf{R}$ be its complement and set $q := n-p$. Write the elements of R and R^c in order

$$\mathsf{R} \;=\; \{R_1 < R_2 < \cdots < R_p\} \qquad \text{and} \qquad \mathsf{R}^c \;=\; \{R^c_1 < R^c_2 < \cdots < R^c_q\}\,.$$

Define the permutation $\pi_{\mathsf{R}} \in \mathfrak{S}_n$ by

$$\pi_{\mathsf{R}}^{-1} \ := \ (R_1, R_2, \ldots, R_p,\ R^c_1, R^c_2, \ldots, R^c_q) \ \in \ \mathfrak{S}^{(p,q)}\,.$$

Then $\pi_{\mathsf{R}}(R_i) = i$ and $\pi_{\mathsf{R}}(R^c_i) = p+i$. Any $\zeta \in \mathfrak{S}^{(p,q)}$ is of the form π_{R}^{-1} for a unique $\mathsf{R} \subseteq [n]$.

Let R be a subset of $[n]$ as above. For a permutation $\rho \in \mathfrak{S}_n$ and a tree $r \in \mathcal{Y}_n$ define

$$\rho|_{\mathsf{R}} \ := \ \mathrm{st}\big(\rho(R_1), \rho(R_2), \ldots, \rho(R_p)\big) \qquad \text{and} \qquad r|_{\mathsf{R}} \ := \ \lambda\big(\gamma(r)|_{\mathsf{R}}\big)\,.$$

Lemma 8.5. *For any* $\mathsf{R} \subseteq [n]$ *and* $r \in \mathcal{Y}_n$*,*

$$\gamma(r)|_{\mathsf{R}} = \gamma(r|_{\mathsf{R}})\,.$$

Proof. Let $\sigma := \gamma(r)|_{\mathsf{R}}$. Since $\gamma(r)$ is 132-avoiding (Section 1.2), so is σ. Hence $\sigma = \gamma\big(\lambda(\sigma)\big)$, and $\gamma(r)|_{\mathsf{R}} = \gamma\Big(\lambda\big(\gamma(r)|_{\mathsf{R}}\big)\Big) = \gamma(r|_{\mathsf{R}})$. ∎

Theorem 8.6. *Let* $r \in \mathcal{Y}_n$*. Then*

$$\Delta(M^*_r) \ = \ \sum_{\substack{\mathsf{R}\subseteq[n] \\ \lambda(\pi_{\mathsf{R}})\leq r}} M^*_{r|_{\mathsf{R}}} \otimes M^*_{r|_{\mathsf{R}^c}}\ . \tag{34}$$

Proof. Let $\mathsf{R} \subseteq [n]$ with $\#\mathsf{R} = p$ and $\zeta := \pi_{\mathsf{R}}^{-1} \in \mathfrak{S}^{(p,q)}$. Since the map f_ζ is order-preserving, the minimum element in its image is $f_\zeta(1_p, 1_q) = \lambda(\zeta^{-1})$, and so the sums in (33) and (34) are over the same sets. We only need show that if $\lambda(\zeta^{-1}) \leq r$ then $(r|_{\mathsf{R}}, r|_{\mathsf{R}^c})$ is maximum among those pairs $(s,t) \in \mathcal{Y}_p \times \mathcal{Y}_q$ such that $f_\zeta(s,t) \leq r$. We first establish the corresponding fact about permutations; namely that if $\zeta^{-1} \leq \rho$ then $(\rho|_{\mathsf{R}}, \rho|_{\mathsf{R}^c})$ is maximum among those pairs $(\sigma,\tau) \in \mathfrak{S}_p \times \mathfrak{S}_q$ such that $\varphi_\zeta(\sigma,\tau) \leq \rho$.

The permutation $\upsilon := \varphi_\zeta(\sigma,\tau) = (\sigma/\tau)\cdot\zeta^{-1}$ satisfies

$$\upsilon(R_i) \ = \ \sigma(i) \qquad \text{and} \qquad \upsilon(R^c_j) \ = \ p + \tau(j)\,,$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Thus $\upsilon|_{\mathsf{R}} = \sigma$ and $\upsilon|_{\mathsf{R}^c} = \tau$. We describe the inversion set of υ:

$$\begin{aligned}
(R_i, R_j) \in \mathrm{Inv}(\upsilon) &\iff (i,j) \in \mathrm{Inv}(\sigma)\\
(R^c_i, R^c_j) \in \mathrm{Inv}(\upsilon) &\iff (i,j) \in \mathrm{Inv}(\tau)\\
(R^c_i, R_j) \in \mathrm{Inv}(\upsilon) &\iff R^c_i < R_j
\end{aligned}$$

There are no inversions of υ of the form (R_i, R^c_j).

The above includes a description of $\mathrm{Inv}(\zeta^{-1})$ (choosing $\sigma = id_p$, $\tau = id_q$). Since the weak order on $\mathfrak{S}_n$ is given by inclusion of inversion sets, we see that for a permutation $\rho \in \mathfrak{S}_n$,

$$\zeta^{-1} \leq \rho \iff \big\{(R^c_i, R_j) \mid R^c_i < R_j\big\} \ \subseteq \ \mathrm{Inv}(\rho)\,.$$

Since (i,j) is an inversion of $\rho|_{\mathsf{R}}$ is and only if (R_i, R_j) is an inversion of ρ, we see that if $\zeta^{-1} \leq \rho$, then $(\rho|_{\mathsf{R}}, \rho|_{\mathsf{R}^c})$ is maximum among all pairs $(\sigma,\tau) \in \mathfrak{S}_p \times \mathfrak{S}_q$ such that $(\sigma/\tau)\cdot\zeta^{-1} \leq \rho$.

We finish the proof by deducing the fact about trees. Suppose $\lambda(\zeta^{-1}) \leq r$. Let $\rho := \gamma(r)$. Then $\zeta^{-1} \leq \rho$, by Theorem 2.1. Suppose $f_\zeta(s,t) \leq r$. Let $\sigma := \gamma(s)$ and $\tau = \gamma(t)$. Then $\lambda\big((\sigma/\tau)\cdot\zeta^{-1}\big) = f_\zeta(s,t) \leq r$, so $\varphi_\zeta(\sigma,\tau) \leq \rho$. By the fact about permutations, $\sigma \leq \rho|_{\mathsf{R}}$ and $\tau \leq \rho|_{\mathsf{R}^c}$. Applying λ we obtain $s \leq r|_{\mathsf{R}}$ and $t \leq r|_{\mathsf{R}^c}$. It remains to verify that $f_\zeta(r|_{\mathsf{R}}, r|_{\mathsf{R}^c}) \leq r$. This is equivalent to $\varphi_\zeta\big(\gamma(r|_{\mathsf{R}}), \gamma(r|_{\mathsf{R}^c})\big) \leq \rho$. This holds by the fact about permutations, since $\gamma(r|_{\mathsf{R}}) = \rho|_{\mathsf{R}}$ and $\gamma(r|_{\mathsf{R}^c}) = \rho|_{\mathsf{R}^c}$ by Lemma 8.5. ∎

Lemma 8.7. *Let $1 \le j \le n$, $\sigma \in \mathfrak{S}_{j-1}$, $\tau \in \mathfrak{S}_{n-j}$, $s \in \mathcal{Y}_{j-1}$, and $t \in \mathcal{Y}_{n-j}$. Let $\mathsf{R} \subseteq [n]$, $\mathsf{S} := \mathsf{R} \cap [1, j-1]$, and $\mathsf{T} := \mathsf{R} \cap [j+1, n]$. Then*

$$(\sigma \vee \tau)|_{\mathsf{R}} \;=\; \begin{cases} (\sigma|_{\mathsf{S}}) \vee (\tau|_{\mathsf{T}-j}) & \text{if } j \in \mathsf{R}, \\ (\sigma|_{\mathsf{S}}) \backslash (\tau|_{\mathsf{T}-j}) & \text{if } j \notin \mathsf{R}; \end{cases} \qquad (s \vee t)|_{\mathsf{R}} \;=\; \begin{cases} (s|_{\mathsf{S}}) \vee (t|_{\mathsf{T}-j}) & \text{if } j \in \mathsf{R}, \\ (s|_{\mathsf{S}}) \backslash (t|_{\mathsf{T}-j}) & \text{if } j \notin \mathsf{R}. \end{cases}$$

Proof. The statement for permutations is immediate from the definitions. Applying γ to both sides of the remaining equality, and using (7), (11), and Lemma 8.5 we deduce the statement for trees.

We complete the proof of Theorem 8.4 by showing that under a natural labeling of the internal nodes of a tree $r \in \mathcal{Y}_n$, admissible subsets of nodes are exactly those subsets $\mathsf{R} \subseteq [n]$ such that $\lambda(\pi_{\mathsf{R}}) \le r$, and that given such a subset R,

$$r|_{\mathsf{R}} \;=\; r'_{\mathsf{R}} \qquad \text{and} \qquad r|_{\mathsf{R}^c} \;=\; r''_{\mathsf{R}}\,.$$

Label the n internal nodes of a tree $r \in \mathcal{Y}_n$ with the integers $1, 2, \ldots, n$ in the following recursive manner. Write $r = s \vee t$ with $s \in \mathcal{Y}_{j-1}$ and $t \in \mathcal{Y}_{n-j}$, $1 \le j \le n$. Assume the nodes of s and t have been labeled. The root node of r is labeled with j, if a node comes from s, it retains its label, and if a node in comes from t, we increase its label by j. Note that the label of any internal node of r is bigger than all the labels of nodes in its left subtree and smaller than all the labels of nodes in its right subtree.

Lemma 8.8. *Let $\mathsf{R} \subseteq [n]$ and $r \in \mathcal{Y}_n$. We consider R to be a subset of internal nodes of r, under the above labeling. Then*

$$\lambda(\pi_{\mathsf{R}}) \;\le\; r \quad \iff \quad \mathsf{R} \textit{ is admissible.}$$

Proof. Let $\mathsf{R} \subseteq [n]$ and $r \in \mathcal{Y}_n$, and set $\rho := \gamma(r)$. By Theorem 2.1, $\lambda(\pi_{\mathsf{R}}) \le r \iff \pi_{\mathsf{R}} \le \rho$. In the proof of Theorem 8.6, we showed that

$$\pi_{\mathsf{R}} \;\le\; \rho \iff \text{whenever } i < j \text{ with } i \notin \mathsf{R}, \text{ and } j \in \mathsf{R}, \text{ then } \rho(i) > \rho(j)\,.$$

Equivalently, if $i < j$ with $\rho(i) < \rho(j)$ and $j \in \mathsf{R}$, then $i \in \mathsf{R}$. To show that this is equivalent to R being admissible, we only need to verify that if $i < j$ with $\rho(i) < \rho(j)$, then in r the node labeled i is in the left subtree above the node labeled j.

Let h be the label of the root node of r, $1 \le h \le n$. Thus $r = s \vee t$ with $s \in \mathcal{Y}_{h-1}$ and $t \in \mathcal{Y}_{n-h}$. By definition of γ (7), $\rho = \gamma(r) = \gamma(s) \vee \gamma(t)$. By definition of grafting of permutations (1), $\rho(h) = n$. Thus ρ achieves its maximum on the label h of the root node. Suppose $j = h$. By construction of the labeling, all labels $i < j$ belong to the left subtree above the root, which shows that the claim holds in this case. If $j \neq h$, since ρ is 132-avoiding, we must have either $i < j < h$ or $h < i < j$. In the former case, $\rho(i) < \rho(j) \iff \gamma(s)(i) < \gamma(s)(j)$; in the latter, $\rho(i) < \rho(j) \iff \gamma(t)(i-h) < \gamma(t)(j-h)$. The claim now follows by induction on n.

The following lemma completes the proof of Theorem 8.4.

Lemma 8.9. *Let $\mathsf{R} \subseteq [n]$ be an admissible subset of nodes of a tree $r \in \mathcal{Y}_n$, labeled as above. Then*

$$r|_{\mathsf{R}} \;=\; r'_{\mathsf{R}} \qquad \textit{and} \qquad r|_{\mathsf{R}^c} \;=\; r''_{\mathsf{R}}\,.$$

Proof. Write $r = s \vee t$ with $s \in \mathcal{Y}_{j-1}$ and $t \in \mathcal{Y}_{n-j}$. Thus j is the label of the root node of r, the set of labels of the nodes of s and t are respectively $[1, j-1]$ and $[j+1, n]$.

Suppose that $j \in \mathsf{R}$. As R is admissible, $[1, j-1] \subseteq \mathsf{R}$, and by Lemma 8.7,

$$r|_{\mathsf{R}} \;=\; s \vee (t|_{\mathsf{R}\cap[j+1,n]-j}) \qquad \text{and} \qquad r|_{\mathsf{R}^c} \;=\; t|_{\mathsf{R}^c\cap[j+1,n]-j}\,.$$

Proceeding by induction we may assume that $t|_{\mathsf{R}\cap[j+1,n]-j} = t'_{\mathsf{R}\cap[j+1,n]-j}$ and $t|_{\mathsf{R}^c\cap[j+1,n]-j} = t''_{\mathsf{R}\cap[j+1,n]-j}$. Together with Lemma 8.3 this gives $r|_{\mathsf{R}} = r'_{\mathsf{R}}$ and $r|_{\mathsf{R}^c} = r''_{\mathsf{R}}$.

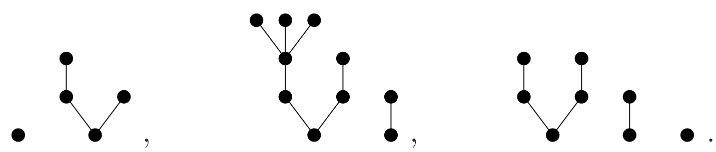
Similarly, if $j \notin \mathsf{R}$, then by Lemma 8.7,

$$r|_{\mathsf{R}} \;=\; (s|_{\mathsf{R}\cap[1,j-1]})\backslash(t|_{\mathsf{R}\cap[j+1,n]-j}) \qquad \text{and} \qquad r|_{\mathsf{R}^c} \;=\; (s|_{\mathsf{R}^c\cap[1,j-1]}) \vee (t|_{\mathsf{R}^c\cap[j+1,n]-j})\,.$$

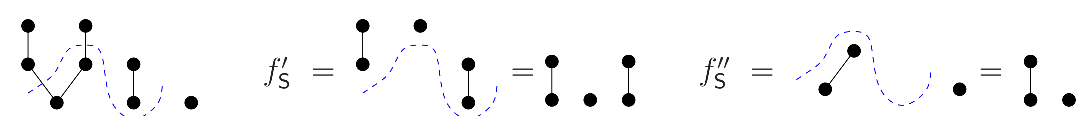
Induction and an application of Lemma 8.3 complete the proof.

8.2. *LR* **and the non-commutative Connes-Kreimer Hopf algebra.** We use Theorems 8.1 and 8.4 to give an explicit isomorphism between *LR* and the non-commutative Connes-Kreimer Hopf algebra, *NCK* of Foissy [10, Sec. 5]. Holtkamp constructed a less explicit isomorphism [19, Thm. 2.10]. Palacios [28, Sec. 4.4.1] obtained an explicit description of this isomorphism which is equivalent to ours. Foissy [11] showed that the two Hopf algebras are isomorphic by exhibiting a dendriform structure on *NCK*.

As an algebra, *NCK* is freely generated by the set of all finite rooted planar trees. Monomials of rooted planar trees are naturally identified with *ordered forests* (sequences of rooted planar trees), so *NCK* has a linear basis of such forests. The identity element corresponds to the empty forest $\emptyset$. The algebra *NCK* is graded by the total number of nodes in a forest. Here are some forests.



An subset R of nodes of a forest is *admissible* if for any node $x \in \mathsf{R}$, every node above x also lies in R. Given an admissible subset of nodes in a forest f, we prune the forest by removing the edges connecting nodes of R to nodes of its complement. The pruned pieces give a planar forest f'_{R}, and the pieces that remain also form a forest, f''_{R}. For example, here is a pruning of the third forest above, and the resulting forests:

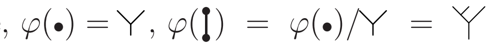
(35) 

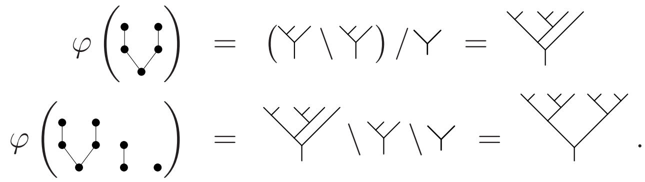
The coproduct in *NCK* is given by

$$\Delta(f) \;=\; \sum f'_{\mathsf{S}} \otimes f''_{\mathsf{S}}\,, \tag{36}$$

the sum over all admissible subsets S of nodes of the forest f. To prove $NCK \cong LR$, we furnish a bijection φ between planar forests f of rooted planar trees with n nodes and planar binary trees with n internal nodes that preserves these structures.

We construct φ recursively. Set $\varphi(\emptyset) := |$. Removing the root from a planar rooted tree t gives a planar forest f, and we set $\varphi(t) := \varphi(f)/\curlyvee$. Finally, given a forest $f = (t_1, t_2, \ldots, t_n)$, where each t_i is a planar rooted tree, set $\varphi(f) := \varphi(t_1)\backslash\varphi(t_2)\backslash \cdots \backslash\varphi(t_n)$.

For example, , and so



The last example above shows that the planar binary tree of (31) and the forest of (35) correspond to each other under φ. Compare the admissible subsets and prunings illustrated in (31) and (35). Under φ the images of the nodes above a node x consist of all the internal nodes in the left branch above the image of x. Thus admissible subsets of nodes of a forest f correspond to admissible subsets of internal nodes of the planar binary tree $\varphi(f)$. Similarly, the assembly of the pieces given by a cut corresponding to admissible sets also correspond, as may be seen from these examples and Lemma 8.3.

We deduce an isomorphism between the non-commutative Connes-Kreimer Hopf algebra and the Loday-Ronco Hopf algebra.

Define a linear map $\Phi : NCK \to LR$ by

$$\Phi(f) \;:=\; M^*_{\varphi(f)} \tag{37}$$

Theorem 8.10. *The map Φ is an isomorphism of graded Hopf algebras $NCK \cong LR$.*

Proof. Theorem 8.1 guarantees that Φ is a morphism of algebras and the preceding discussion shows that Φ is a morphism of coalgebras. It is easy to see that φ is a bijection between the set of ordered forests with n nodes and the set of planar binary trees with n internal nodes. Thus Φ is an isomorphism of graded Hopf algebras.

The non-commutative Connes-Kreimer Hopf algebra carries a canonical involution. Given a plane forest f, let f^{r} be its reflection across a vertical line on the plane. It is clear that

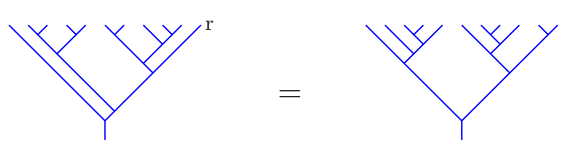
$$(f^{\mathrm{r}})^{\mathrm{r}} = f\,, \qquad (f\cdot g)^{\mathrm{r}} \;=\; g^{\mathrm{r}}\cdot f^{\mathrm{r}}\,, \qquad \text{and} \qquad \Delta(f)^{\mathrm{r}\otimes\mathrm{r}} \;=\; \Delta(f^{\mathrm{r}})\,;$$

in other words, the map $f \mapsto f^{\mathrm{r}}$ is an involution, an algebra anti-isomorphism, and a coalgebra isomorphism of the noncommutative Connes-Kreimer Hopf algebra with itself.

We deduce the existence of a canonical involution on $\mathcal{Y}Sym$, which we construct recursively. Define $|^{\mathrm{r}} := |$. For a progressive tree t, write $t = s \vee |$, and define $t^{\mathrm{r}} := s^{\mathrm{r}} \vee |$. Finally, for an arbitrary planar binary tree t, consider its decomposition into progressive trees $t = t_1\backslash t_2\backslash\cdots\backslash t_k$ (Section 1.1) and define

$$t^{\mathrm{r}} \;:=\; (t_k)^{\mathrm{r}}\backslash\cdots\backslash(t_2)^{\mathrm{r}}\backslash(t_1)^{\mathrm{r}}\,.$$

For instance,



Corollary 8.11. *The map $\mathcal{Y}Sym \to \mathcal{Y}Sym$, $M_t \mapsto M_{t^{\mathrm{r}}}$, is an involution, an algebra isomorphism, and a coalgebra anti-isomorphism.*

Proof. By construction, $\varphi(f^{\mathrm{r}}) = \varphi(f)^{\mathrm{r}}$. We may thus transport the result from NCK to LR via Φ (and to $\mathcal{Y}Sym$ via duality).

Since the map $t \mapsto t^{\mathrm{r}}$ does not preserve the Tamari order on $\mathcal{Y}_n$, the involution does not admit a simple expression on the $\{F_t\}$-basis of $\mathcal{Y}Sym$. We also remark that there is a commutative diagram

$$\begin{tikzcd} \mathcal{Y}Sym \arrow[r, "\mathrm{r}"] \arrow[d, "\mathcal{D}"'] & \mathcal{Y}Sym \arrow[d, "\mathcal{D}"] \\ \mathcal{Q}Sym \arrow[r, "\mathrm{r}"] & \mathcal{Q}Sym \end{tikzcd}$$

The bottom map sends $M_\alpha \mapsto M_{\alpha^{\mathrm{r}}}$, where $\alpha^{\mathrm{r}} = (a_k, \ldots, a_2, a_1)$ is the reversal of the composition $\alpha = (a_1, a_2, \ldots, a_k)$. This is an involution, an algebra isomorphism, and a coalgebra anti-isomorphism of $\mathcal{Q}Sym$ with itself. The map $\alpha \mapsto \alpha^{\mathrm{r}}$ *is* order-preserving, so the involution on $\mathcal{Q}Sym$ is also given by $F_\alpha \mapsto F_{\alpha^{\mathrm{r}}}$.

8.3. **Symmetric functions and the Connes-Kreimer Hopf algebra.** Let CK be the Connes-Kreimer Hopf algebra. It is the free commutative algebra generated by the set of all finite rooted (non-planar) trees. Commutative monomials of rooted trees are naturally identified with *unordered forests* (multisets of rooted trees), so CK has a linear basis consisting of such unordered forests. The coproduct of CK is defined in terms of admissible subsets of nodes in the same way as for NCK (36). CK is a commutative graded Hopf algebra.

Given an ordered forest f of planar trees, let $U(f)$ be the unordered forest obtained by forgetting the left-to-right order among the trees in f, and the left-to-right ordering among the branches emanating from each node in each tree in f. The map $U : NCK \to CK$ is a surjective morphism of Hopf algebras.

Consider rooted trees in which each node has at most one child. These are sometimes called *ladders*. Let ℓ_n be the ladder with n nodes. Clearly,

$$\Delta(\ell_n) = \sum_{i=0}^{n} \ell_i \otimes \ell_{n-i} \,.$$

It follows that the subalgebra of CK generated by $\{\ell_n\}_{n\geq 0}$, is a Hopf subalgebra, isomorphic to the Hopf algebra of symmetric functions via the map

$$Sym \hookrightarrow CK\,, \qquad h_n \mapsto \ell_n \,.$$

Here h_n denotes the *complete* symmetric function.

Recall that the graded dual of the Hopf algebra of quasi-symmetric functions is the Hopf algebra of non-commutative symmetric functions: $\mathcal{Q}Sym^* = \mathcal{N}Sym$. Dualizing the map $\mathcal{L} : \mathcal{Y}Sym \twoheadrightarrow \mathcal{Q}Sym$ (Proposition 1.5) we obtain an injective morphism of Hopf algebras, which by (22) is given by

$$\mathcal{N}Sym \hookrightarrow \mathcal{Y}Sym\,, \qquad M^*_\alpha \mapsto M^*_{C(\alpha)} \,.$$

By definition of the map C (9) and Theorem 8.1, if $\alpha = (a_1, \ldots, a_k)$ then

$$M^*_{C(\alpha)} = M^*_{1_{a_1}} \cdots M^*_{1_{a_k}} \,.$$

The bijection φ of Section 8.2 sends the ladder ℓ_n (viewed as a planar rooted tree) to the comb 1_n. Therefore, composing with the isomorphism of Theorem 8.10 we obtain an injective morphism of Hopf algebras

$$\mathcal{N}Sym \hookrightarrow NCK\,, \qquad M^*_\alpha \mapsto \ell_{a_1} \cdots \ell_{a_k} \,.$$

The canonical map $\mathcal{N}Sym \twoheadrightarrow Sym$ sends M^*_α to the complete symmetric function $h_{a_1} \cdots h_{a_k}$. We have shown:

Theorem 8.12. *There is a commutative diagram of graded Hopf algebras*

$$\begin{tikzcd} \mathcal{N}Sym \arrow[r, hook] \arrow[d] & NCK \arrow[d] \\ Sym \arrow[r, hook] & CK \end{tikzcd} \tag{38}$$

Let $GL := CK^*$ denote the graded dual of the Connes-Kreimer Hopf algebra. As shown by Hoffman [18], this is the (cocommutative) Hopf algebra of rooted trees constructed by Grossman and Larson in [14]. Dualizing (38) we obtain the following commutative diagram of graded Hopf algebras:

$$\begin{tikzcd} GL \arrow[r, hook] \arrow[d] & \mathcal{Y}Sym \arrow[d] \\ Sym \arrow[r, hook] & \mathcal{Q}Sym \end{tikzcd} \tag{39}$$

References

[1] Marcelo Aguiar and Frank Sottile, *Structure of the Malvenuto-Reutenauer Hopf algebra of permutations*, 2002, `math.CO/0203282`, to appear in Adv. Math.
[2] ______, *Cocommutative Hopf algebras of permutations and trees*, 2004, `math.QA/0403101`.
[3] Louis J. Billera and Bernd Sturmfels, *Iterated fiber polytopes*, Mathematika **41** (1994), no. 2, 348–363.
[4] Anders Björner and Michelle Wachs, *Shellable nonpure complexes and posets. II*, Trans. Amer. Math. Soc. **349** (1997) no. 10, 3945–3975.
[5] Robert J. Blattner, Miriam Cohen, and Susan Montmery, *Crossed products and inner actions of Hopf algebras*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 671–711. MR 87k:16012
[6] Miklós Bóna, *A walk through combinatorics*, with a foreword by Richard Stanley, World Scientific Publishing Co., NJ, 2002. MR 1936456
[7] Christian Brouder and Alessandra Frabetti, *Renormalization of QED with planar binary trees*, Europ. Phys. J. C **19** (2001), 715–741.
[8] ______, *QED Hopf algebras on planar binary trees*, J. Algebra **267** (2003), no. 1, 298–322. MR 1 993 478
[9] Alain Connes and Dirk Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys. **199** (1998), no. 1, 203–242.
[10] Loïc Foissy, *Les algèbres de Hopf des arbres enracinés décorés. I*, Bull. Sci. Math. **126** (2002), no. 3, 193–239. MR 2003d:16049
[11] ______, *Les algèbres de Hopf des arbres enracinés décorés. II*, Bull. Sci. Math. **126** (2002), no. 4, 249–288. MR 2003e:16047
[12] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, and Jean-Yves Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), no. 2, 218–348. MR 96e:05175
[13] Ira M. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and algebra (Boulder, Colo., 1983) (Providence, RI), Amer. Math. Soc., 1984, 289–317. MR 86k:05007
[14] Robert Grossman and Richard G. Larson, *Hopf-algebraic structure of families of trees*, J. Algebra **126** (1989), no. 1, 184–210. MR 90j:16022
[15] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon, *Un analogue du monoïde plaxique pour les arbres binaires de recherche*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 7, 577–580. MR 2003j:68108
[16] ______, *Sur quelques propriétés de l'algèbre des arbres binaires*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 9, 565–568. MR 2 017 727
[17] ______, *The Algebra of Binary Search Trees*, 2004, `math.CO/0401089`.

[18] Michael E. Hoffman, *Combinatorics of rooted trees and Hopf algebras*, Trans. Amer. Math. Soc. **355** (2003), 3795–3811.
[19] Ralf Holtkamp, *Comparison of Hopf algebras on trees*, Arch. Math. (Basel) **80** (2003), no. 4, 368–383. MR 2004f:16067
[20] Dirk Kreimer, *On the Hopf Algebra structure of perturbative quantum field theories*, Adv. Theory. Math. Phys. **2** (1998), 303–334.
[21] Jean-Louis Loday, *Dialgebras*, Dialgebras and Related Operads, Lecture Notes in Mathematics, no. 1763, Springer-Verlag, 2001, 7–66.
[22] ———, *Realization of the Stasheff polytope*, `math.AT/0212126`, to appear in Arch. Math. (Basel).
[23] Jean-Louis Loday and María O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), no. 2, 293–309. MR 99m:16063
[24] ———, *Order structure on the algebra of permutations and of planar binary trees*, J. Alg. Combinatorics **15** (2002), 253–270.
[25] Claudia Malvenuto, *Produits et coproduits des fonctions quasi-symétriques et de l'algèbre des descents*, no. 16, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Univ. du Québec à Montréal, Montréal, 1994.
[26] Claudia Malvenuto and Christophe Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982. MR 97d:05277
[27] Susan Montgomery, *Hopf algebras and their actions on rings*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. MR 94i:16019
[28] Patricia Palacios, *Una generalización a operads de la construcción de Hopf de Connes-Kreimer*, Tesis de licenciatura, Universidad de Buenos Aires, Argentina, 2002
[29] Nathan Reading, *Cambrian lattices*, 2004, `math.CO/0402086`.
[30] ———, *Lattice congruences, fans and Hopf algebras*, 2004, `math.CO/0402063`.
[31] Victor Reiner, *Equivariant fiber polytopes*, Doc. Math. **7** (2002), 113–132 (electronic).
[32] Christophe Reutenauer, *Free Lie algebras*, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications. MR 94j:17002
[33] Gian-Carlo Rota, *On the foundations of combinatorial theory. I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368 (1964), Reprinted in *Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries* (Joseph P.S. Kung, Ed.), Birkhäuser, Boston, 1995. MR 30 #4688
[34] Richard P. Stanley, *Ordered structures and partitions*, American Mathematical Society, Providence, R.I., 1972, Memoirs of the American Mathematical Society, No. 119. MR 48 #10836
[35] ———, *Enumerative combinatorics. Vol. 1*, Cambridge University Press, Cambridge, 1997, with a foreword by Gian-Carlo Rota, corrected reprint of the 1986 original. MR 98a:05001
[36] ———, *Enumerative combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 1999, with a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR 2000k:05026
[37] Andy Tonks, *Relating the associahedron and the permutohedron*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995) (Providence, RI), Contemp. Math., vol. 202, Amer. Math. Soc., 1997, 33–36. MR 98c:52015
[38] Pepijn van der Laan, *Some Hopf Algebras of Trees*, 2002, `math.QA/0106244`.
[39] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, 152, Springer-Verlag, 1995.

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