Formation of an interface by competitive erosion

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Abstract

In 2006, the fourth author of this paper proposed a graph-theoretic model of interface dynamics called **competitive erosion**. Each vertex of the graph is occupied by a particle that can be either red or blue. New red and blue particles alternately get emitted from their respective bases and perform random walk. On encountering a particle of the opposite color they kill it and occupy its position. We prove that on the cylinder graph (the product of a path and a cycle) an interface spontaneously forms between red and blue and is maintained in a predictable position with high probability.
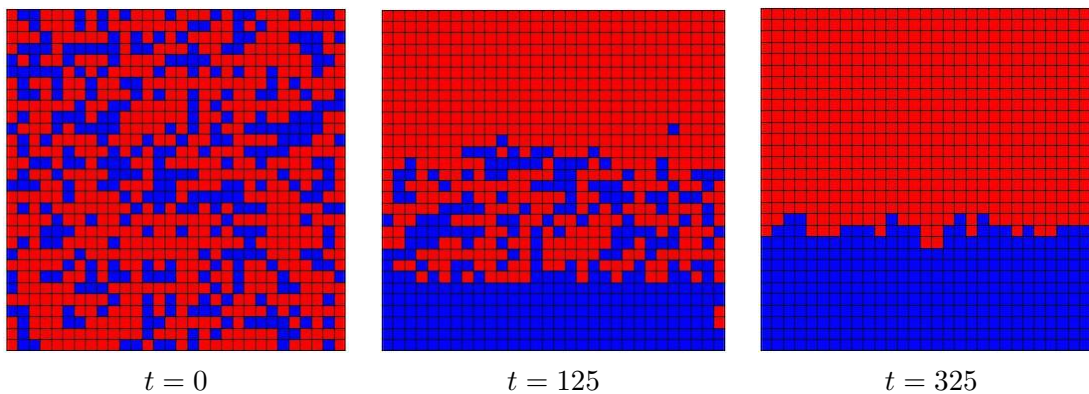
$t = 0$ $\qquad\qquad$ $t = 125$ $\qquad\qquad$ $t = 325$

Figure 1: Competitive erosion spontaneously forms an interface. In the initial state $S(0)$ each vertex of the 30×30 cylinder graph is independently colored blue with probability .33 and red otherwise. After 325 time steps of competitive erosion, an interface has formed between red and blue.

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1 An Interface In Equilibrium

We introduce a graph-theoretic model of a random interface maintained in equilibrium by equal and opposing forces on each side of the interface. Our model can also be started from a heterogeneous state with no interface, in which case an interface forms spontaneously. Here are the data underlying our model, which we call *competitive erosion*:

- a finite connected graph on vertex set V;

- probability measures μ_1 and μ_2 on V; and

- an integer $0 \leq k \leq \#V - 1$.

Competitive erosion is a discrete-time Markov chain $(S(t))_{t \geq 0}$ on the space of all subsets of V of size k. One time step is defined by

$$S(t+1) = (S(t) \cup \{X_t\}) - \{Y_t\} \tag{1}$$

where X_t is the first site in $S(t)^c$ visited by a simple random walk whose starting point has distribution μ_1, and Y_t is the first site in $S(t) \cup \{X_t\}$ visited by an independent simple random walk whose starting point has distribution μ_2.

For a concrete metaphor, one can imagine that $S(t)$ and its complement represent the territories of two competing species. We will call the vertices in $S(t)$ "blue" and those in $S(t)^c$ "red". According to (1), a blue individual beginning at a random vertex with distribution μ_1 wanders until it encounters a red vertex X_t and inhabits it, evicting the former inhabitant. The latter returns to an independent random vertex with distribution μ_2 and from there wanders until it encounters a blue vertex Y_t and inhabits it, evicting the former inhabitant.

If the distributions μ_1 and μ_2 have well-separated supports, one expects that these dynamics resolve the graph into coherent red and blue territories separated by an interface that, although microscopically rough, appears in a macroscopically predictable position with high probability. The purpose of this article is to prove this assertion in the case of the *cylinder graph* $\mathrm{Cyl}_n = C_n \times P_n$, where C_n is a cycle of length n and P_n is a path of length n. Here $\times$ denotes the Cartesian (box) product of graphs. We identify C_n with $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$ and P_n with $(\frac{1}{n}\mathbb{Z}) \cap [0,1]$. We take μ_1 and μ_2 to be the uniform distributions on $C_n \times \{0\}$ and $C_n \times \{1\}$, respectively: blue particles are released from the base of the cylinder and red particles from the top.

Taking $k = \alpha n^2$ for some $0 < \alpha < 1$, it is natural to guess that the stationary distribution of the Markov chain $S(t)$ assigns high probability to the event

$$\mathcal{A}_{\epsilon,n} := \{S \subset \mathrm{Cyl}_n : C_n \times [0, \alpha - \epsilon] \subset S \subset C_n \times [0, \alpha + \epsilon]\}, \tag{2}$$

that is, the event that sites below the line $y = \alpha - \epsilon$ are all blue and that sites above the line $y = \alpha + \epsilon$ are all red. Our main result confirms this guess.

Theorem 1. *Given $\epsilon > 0$ there exists a positive constant $d = d(\epsilon)$ such that*

$$\pi_n(\mathcal{A}_{\epsilon,n}) \geq 1 - e^{-dn} \tag{3}$$

where π_n is the stationary distribution of the competitive erosion chain on Cyl_n.

The proof of Theorem 1 will show that the predicted interface forms quickly, even if the initial state $S(0)$ is heterogeneous (such as a checkerboard of red and blue, or a random initial state like the one shown in Figure 1).

Theorem 2. *Let $\tau_\epsilon = \inf\{t \,:\, S(t) \in \mathcal{A}_{\epsilon,n}\}$. For any $\epsilon > 0$ there exist positive constants $c = c(\epsilon), d = d(\epsilon), N = N(\epsilon)$ such that for all $n > N$ and all subsets $S \subset \mathrm{Cyl}_n$ of cardinality $\lfloor \alpha n^2 \rfloor$ we have*

$$\mathbb{P}(\tau_\epsilon > dn^2 \mid S(0) = S) < e^{-cn}.$$

See the last section for some related models exhibiting interface formation. We also mention a superficially similar process called "oil and water" [4] which has an entirely different behavior: the two species do not form a macroscopic interface at all.

1.1 Comparison with IDLA

Internal diffusion limited aggregation (IDLA) is a fundamental model of a random interface moving in a *monotone* (outward) fashion. IDLA involves only one species with an ever-growing territory

$$I(t+1) = I(t) \cup \{X_t\}$$

where X_t is the first site in $I(t)^c$ visited by a simple random walk whose starting point has distribution μ_1. Competitive erosion can be viewed as a symmetrized version of IDLA: whereas $I(t)$ and $I(t)^c$ play asymmetric roles, $S(t)$ and $S(t)^c$ play symmetric roles in (1).

IDLA on a finite graph is only defined up to the finite time t when $I(t)$ is the entire vertex set. For this reason, the IDLA is usually studied on an infinite graph and the theorems about IDLA are limit theorems: asymptotic shape [11], order of the fluctuations [1,2,6], and distributional limit of the fluctuations [7]. In contrast, competitive erosion on a finite graph is defined for all times, so it is natural to ask about its stationary distribution. To appreciate the difference in character between IDLA and competitive erosion, note that the stationary distribution of the latter assigns tiny but positive probability to configurations that look nothing like the predicted horizontal interface. Competitive erosion will occasionally form these exceptional configurations: for example, at a tiny but positive fraction of times t the boundary of the set $S(t)$ is a *vertical* line! The proof of Theorem 1 is delicate since there are so many exceptional configurations: in terms of cardinality, the desired set $\mathcal{A}_{\epsilon,n}$ is an exponentialy small fraction of the set of all recurrent configurations.

1.2 Idea of the proof

To prove that the stationary distribution concentrates on the small set $\mathcal{A} = \mathcal{A}_{\epsilon,n}$, we identify a *Lyapunov function* h on the state space which attains its global maximum in $\mathcal{A}$ and increases in expectation

$$\mathbb{E}(h(S(1)) - h(S(0))) \geq a > 0 \tag{4}$$

provided $S(0)$ is sufficiently far from $\mathcal{A}$. The function h is as simple as one could hope for: the sum of the heights of the red vertices. The heart of the proof is Theorem 5, which uses an electrical resistance argument to establish the drift (4) for a suitable notion of "sufficiently far from $\mathcal{A}$". Since the function $h(\sigma)$ is bounded by n^2, we then use Azuma's inequality to argue that the process $h(S(t))$ spends nearly all its time in a neighborhood of its maximum. This establishes Theorem 4 (a statistical version of Theorem 1).

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The main remaining difficulty lies in showing that if $S(0)$ is sufficiently *close* to $\mathcal{A}$, then the chain $S(t)$ hits $\mathcal{A}$ quickly with high probability, establishing Theorem 2. This is done using stochastic domination arguments involving IDLA on the cylinder. These ingredients together with a general estimate relating hitting times to stationary distributions (Lemma 4) establish Theorem 1.

1.3 The level set heuristic

Before restricting to the cylinder we mention a heuristic that predicts the location of the competitive erosion interface for well-separated measures μ_1, μ_2 on an arbitrary finite connected graph. Let g be a function on the vertices satisfying

$$\Delta g = \mu_1 - \mu_2 \tag{5}$$

where Δ denotes the Laplacian

$$\Delta g(x) := \frac{1}{\deg(x)} \sum_{y \sim x} (g(x) - g(y))$$

and the sum is over vertices y neighboring x. Since the graph is assumed connected, the kernel of Δ is one-dimensional consisting of the constant functions, so that equation (5) determines g up to an additive constant.

Consider a partition of the vertex set into subsets S_1 and S_2 (think of S_1 and S_2 as the blue and red territories respectively) of the sort we might expect to see in equilibrium, and let B denote the boundary between the two territories. (We do not aim to make anything precise in this section. The reader may for the time being imagine that B consists of all vertices in one set with a neighbor in the other; we will write $\partial S_1 = B = \partial S_2$).

Let g_i for $i = 1, 2$ be the Green function for random walk started according to μ_i and stopped on exiting S_i. These functions satisfy

$$\begin{aligned} \Delta g_i &= \mu_i \quad \text{on } S_i, \\ g_i &= 0 \quad \text{on } S_i^c. \end{aligned}$$

The probability that simple random walk started according to μ_i first exits S_i at $x \in B$ is $-\Delta g_i(x)$.

To maintain equilibrium in competitive erosion, we seek a partition such that $\Delta g_1 \approx \Delta g_2$ on B, that is

$$\Delta(g_1 - g_2) \approx \mu_1 - \mu_2.$$

(Exact equality holds except on B.) Thus by (5), the function $g - (g_1 - g_2)$ is approximately constant. Since g_i vanishes on B, the equilibrium interface B should have the property that

$$g \text{ is approximately constant on } B.$$

The partition that comes closest to achieving this goal takes S_1 to be the level set

$$S_1 = \{x \,:\, g(x) < K\} \tag{6}$$

for a cutoff K chosen to make $\#S_1 = k$. An application of the maximum principle shows that for this choice of S_1, the maximum and minimum values of $g - (g_1 - g_2)$ differ by at most

$$\max_{x \in S_1, y \notin S_1, x \sim y} |g(x) - g(y)|,$$

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suggesting that the right notion of "well-separated" measures μ_1 and μ_2 is that the resulting function g has small gradient.

1.3.1 Mutually annihilating fluids

The divisible sandpile [13, 14] is a deterministic analogue of IDLA. In a competitive version of the divisible sandpile, the edges of our graph form a network of pipes containing a red fluid and a blue fluid. The two fluids are injected at respective rates μ_1 and μ_2 and annihilate upon contact. One can convert the above heuristic into a proof that this deterministic model has its interface exactly at a level line of g (where g is interpolated linearly on edges).

1.4 Diffusive sorting

We define here a process called *diffusive sorting*, introduced by the fourth author, which couples the competitive erosion chains on a given graph for all values of k, using a single random walk to drive them all. We will not use the coupling in this paper, but find it to be of intrinsic interest.

It is convenient to imagine that the blue and red random walks start from vertices v_1 and v_2 respectively that are external to the finite connected graph G and have directed weighted edges into G: each edge (v_i, v) has weight $\mu_i(v)$.

The state space of diffusive sorting consists of bijective label lings λ of the vertex set of G by the integers $\{1, \ldots, N\}$, where N is the number of vertices. We imagine that the labels are detachable from the vertices and can be carried temporarily by the random walker. Consider an initial labeling λ_0. A random walker starts at v_1 carrying the label 0. Whenever it comes to a vertex whose label exceeds the label the walker is currently carrying, the walker swaps labels with that vertex, dropping its old (smaller) label and stealing the new (larger) label for itself. Eventually the walker will visit the vertex labeled N and acquire the label N for itself, at which point we may stop the walk, since the particle will carry the label N forever after and no labels will change. The vertex labels are now $\{0, ..., N-1\}$. We now increase all those labels by 1, and call the result $\lambda_{1/2}$. (We have noticed that this process bears a strong resemblance to an algorithm in algebraic combinatorics called promotion [16], but we have not explored this connection.) To get from $\lambda_{1/2}$ to λ_1, we apply the same process from the other side, releasing a random walker from v_2 bearing the label $N+1$, and letting it swap labels with any vertex it encounters whose label is smaller than its own, until it acquires the label 1; then we decrease the labels of all vertices by 1, obtaining λ_1.

Diffusive sorting is the Markov chain $(\lambda_t)_{t \geq 0}$ on labelings, each of whose transitions from λ_t to λ_{t+1} is as described in the previous paragraph. To see how this chain relates to competitive erosion, for each integer $k = 0, \ldots, N-1$ let

$$S_k(t) = \{v \,:\, \lambda_t(v) \leq k\}.$$

Then it is not hard to check that S_k has the law of the competitive erosion chain. (The key observation is that the vertices labeled 1 through k in λ_0 must be a subset of the vertices labeled 1 through $k+1$ in $\lambda_{1/2}$, and similarly for going from $\lambda_{1/2}$ to λ_1.)

The following result shows that the stationary distribution of diffusive sorting on the cylinder graph is concentrated on labelings that are uniformly close to the function $n^2 y$.

Theorem 3. *For each $\epsilon > 0$ there is a constant $d = d(\epsilon) > 0$ such that*

$$\hat{\pi}_n \left\{ \lambda : \sup_{(x,y) \in \mathrm{Cyl}_n} \left| \frac{\lambda(x,y)}{n^2} - y \right| > \epsilon \right\} < e^{-dn}.$$

where $\hat{\pi}_n$ is the stationary distribution of the diffusive sorting chain $(\lambda_t)_{t \geq 0}$ on Cyl_n.

The level set heuristic (§1.3) makes a prediction for diffusive sorting on a general graph: if the gradient of g is not too large and we label the vertices by $\{g(v) : v \in V\}$ instead of by $\{1, \ldots, N\}$, then $\hat{\pi}_n$ should concentrate on labelings not too far from the function g.

We mention, as an aside, that if instead of alternating between blue walkers and red walkers we use walkers of just one color, we get a sorting version of ordinary IDLA. For any $k \geq 1$ and $t \geq 0$ the law of $S_k(t + k)$ is that of the IDLA cluster with k particles. (This is trivial for $k = 1$ and the rest follows by induction on k.)

2   Connectivity properties of competitive erosion dynamics

Implicit in the statement of Theorem 1 is the claim that competitive erosion has a unique stationary distribution. In this section we formally define the competitive erosion chain and prove this claim.

2.1   Formal definition of competitive erosion

The graph Cyl_n is a discretization of the cylinder $(\mathbb{R}/\mathbb{Z}) \times [0,1]$ with mesh size $\frac{1}{n}$. To avoid degeneracies, we will always assume $n \geq 2$. Let $P_n = (\frac{1}{n}\mathbb{Z}) \cap [0,1]$ be the path graph with edges $\frac{k}{n} \sim \frac{k+1}{n}$ for $k = 0, 1, \ldots, n-1$. Let $C_n = (\frac{1}{n}\mathbb{Z})/\mathbb{Z}$ be the cycle graph obtained by gluing together the endpoints 0 and 1 of P_n. Let

$$\mathrm{Cyl}_n = C_n \times P_n \tag{7}$$

with edges $(x,y) \sim (x',y')$ if either $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$. We also add a self-loop at each point in $C_n \times \{0\}$ and $C_n \times \{1\}$, so that Cyl_n is a 4-regular graph. We will use Cyl_n to denote both this graph and its set of vertices.

For $t = 0, 1, 2, \ldots$ let $(X_s^{(t)})_{s \geq 0}$ and $(Y_s^{(t+\frac{1}{2})})_{s \geq 0}$ be independent simple random walks in Cyl_n with

$$P(X_0^{(t)} = (x,0)) = \frac{1}{n} = P(Y_0^{(t+\frac{1}{2})} = (x,1))$$

for all $x \in C_n$. That is, each walk $X^{(t)}$ starts uniformly on $C_n \times \{0\}$ and each walk $Y^{(t+\frac{1}{2})}$ starts uniformly on $C_n \times \{1\}$. Given the state $S(t)$ of the competitive erosion chain at time t, we build the next state $S(t+1)$ in two steps as follows.

$$S\left(t + \frac{1}{2}\right) = S(t) \cup \{X_{\tau(t)}^{(t)}\}$$

where $\tau(t) = \inf\{s \geq 0 : X_s^{(t)} \notin S(t)\}$. Let

$$S(t+1) = S\left(t + \frac{1}{2}\right) - \{Y_{\tau(t+\frac{1}{2})}^{(t+\frac{1}{2})}\}$$

where $\tau(t + \frac{1}{2}) = \inf\{s \geq 0 : Y_s^{(t+\frac{1}{2})} \in S(t + \frac{1}{2})\}$.

To highlight the symmetrical roles played by $S(t)$ and its complement, we will often think of the states as 2-colorings rather than sets: let

$$\sigma_t(x) = \begin{cases} 1 & x \in S(t), \\ 2 & x \notin S(t). \end{cases} \tag{8}$$

Also define for $i = 1, 2$

$$B_i := B_i(\sigma) := \{x \in \mathrm{Cyl}_n : \sigma(x) = i\}. \tag{9}$$

Thus

$$S(t) = B_1(\sigma_t).$$

We will use color 1 and "blue" interchangeably and similarly color 2 and "red". Now for each α (the proportion of blue sites) strictly between 0 and 1, competitive erosion defines a Markov chain on the space

$$\Omega := \{\sigma \in \{1, 2\}^{\mathrm{Cyl}_n}, \#\{x \in \mathrm{Cyl}_n : \sigma(x) = 1\} = \lfloor \alpha |\mathrm{Cyl}_n| \rfloor\}. \tag{10}$$

We also set

$$\Omega' := \{\sigma \in \{1, 2\}^{\mathrm{Cyl}_n}, \#\{x \in \mathrm{Cyl}_n : \sigma(x) = 1\} = \lfloor \alpha |\mathrm{Cyl}_n| \rfloor + 1\}. \tag{11}$$

A single time step of competitive erosion consists of a step from Ω to Ω' followed by a step from Ω' back to Ω.

2.2 Notational conventions

We will often use the same letter (generally C, D, c or d) for a constant whose value may change from line to line (or sometimes even within one line); this convention obviates the need for distracting subscripts and should cause no confusion.

For any process and a subset A of the corresponding state space $\tau(A)$ will denote the hitting time of that set. When ω is a state in the space and A is a set of states, we will write $\{\tau(A) \leq K \mid \omega\}$ to denote the event that, starting from ω, the process hits A in at most K steps. Finally, in all the notation the dependence on n will be often suppressed.

2.3 Blocking sets and transient states

Definition 1. Call a subset $A \subset \mathrm{Cyl}_n$ **blocking** if $\mathrm{Cyl}_n \setminus A$ is disconnected and the subsets $C_n \times \{0\} \setminus A$ and $C_n \times \{1\} \setminus A$ lie in different components.

We next prove that the competitive erosion chain has exactly one irreducible class and hence has a well-defined stationary measure. We start with a definition.

Definition 2. For two disjoint blocking subsets $A, B \subset \mathrm{Cyl}_n$ we say that A is **over** B if

1. A and $C_n \times \{0\} \setminus B$ lie in different components of $\mathrm{Cyl}_n \setminus B$, and

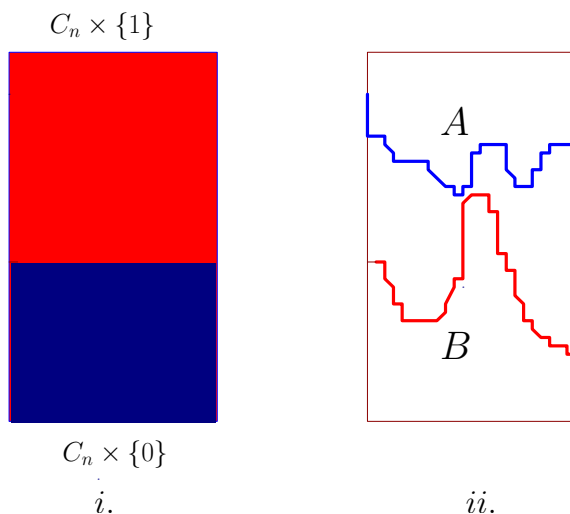2. B and $C_n \times \{1\} \setminus A$ lie in different components of $\mathrm{Cyl}_n \setminus A$.

Figure 2: *i*. The configuration σ_*. *ii*. A configuration with a blue blocking subset over a red blocking subset.

Lemma 1. *The competitive erosion chain has exactly one irreducible class. Moreover any $\sigma \in \Omega$ that has a blue blocking set over a red blocking set is transient.*

Proof. Consider the configuration σ_* in Figure 2 where the lowest $\lfloor \alpha n(n+1) \rfloor$ vertices are colored blue. To prove the first statement notice that from any σ one can reach σ_*. Since in the target configuration there is exactly one blue component, we look at the closest vertex with σ value 2 from $C_n \times \{0\}$. Since there is a blue path from $C_n \times \{0\}$ to that point the Markov chain allows us to change it to 1 and similarly at the other end. Thus we are done by repeating this.

To prove the second statement we prove that starting from σ_0 one cannot reach any configuration like the one on the right that has one blue blocking set over another red blocking set. We formally prove this by contradiction. Let σ be a configuration with a blue blocking set B over a red blocking set A. Now the competitive erosion chain evolves as

$$\sigma_0, \sigma_{1/2}, \sigma_1, \sigma_{3/2} \ldots$$

where for every non-negative integer k, $\sigma_k \in \Omega$ and $\sigma_{k+1/2} \in \Omega'$. Assume now that there is a path

$$\sigma_0, \sigma_{1/2}, \sigma_1, \sigma_{3/2} \ldots, \sigma_t = \sigma.$$

Let τ' and τ'' be the last times along the path such that at least one vertex of A is blue and similarly B contains at least one red vertex respectively. If such a time does not exist let us call it $-\infty$. Since σ_0 does not have a blue blocking set over a red blocking set,

$$\max(\tau', \tau'') > -\infty.$$

τ' must be a half integer (since at half integers there is one more blue particle) and similarly τ'' must be an integer. Thus

$$\tau' \neq \tau''.$$

Next we see that we cannot have $\tau' < \tau''$. For, suppose otherwise. Then for all times greater than τ', A is entirely red. Thus no blue walker crosses A at any time greater than τ'. So B

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must already be blue at τ' and stays blue through till t implying that $\tau'' < \tau'$. By a similar argument we cannot have $\tau' > \tau''$. Hence we arrive at a contradiction. $\qquad\square$

2.4 Organization of the proofs

In Section 3 we state a weak "statistical" version of the main result, Theorem 4, and provide its proof using hitting time estimates whose proof appear in Section 5 . A key step in the proof, the construction of a suitable Lyapunov function, appears in Section 4. This section uses the theory of electrical networks. A short review of a few basic facts about electrical networks that are assumed in this section appears in Appendix B. In Section 6 we deduce the stronger Theorem 1. As a simple corollary we obtain Theorem 3. This last section uses an estimate for IDLA on the cylinder proved in Appendix A. We also apply a hitting time estimate for submartingales several times throughout the paper. The proof of the estimate is included in Appendix C. The final section discusses some related models and future directions.

3 Statistical version of the main theorem

Given $\epsilon > 0$ we define the set

$$\mathcal{G}_\epsilon := \left\{ \sigma \in \Omega : |\sigma^{-1}(1) \triangle \{y \le \alpha\}| \le \epsilon n^2 \right\}. \tag{12}$$

where $\triangle$ denotes the symmetric difference of sets, $A \triangle B := (A^c \cap B) \cup (A \cap B^c)$.

Theorem 4. *Given $\epsilon > 0$ there exists $c > 0$ such that for large enough n*

$$\pi_n(\mathcal{G}_\epsilon) \ge 1 - e^{-cn}.$$

This theorem says that, after running the competitive erosion chain for a long time, one is unlikely to see many blue sites below the line $y = \alpha$ or many red sites above it, which (see Remark 1 iii. below) is a weaker statement than Theorem 1.

3.1 The height function

We now introduce a function on the state space that will appear throughout the rest of the article and will be used as a Lyapunov function, along lines sketched in (4). The interested reader can learn more about Lyapunov functions in [5]. Given $\sigma \in \Omega \cup \Omega'$ (recall 10 and 11), let us define the height function of σ as

$$h(\sigma) = \sum_{(x,y) \in B_1(\sigma)} (1 - y) \tag{13}$$

where $B_1(\sigma)$ is defined in (9). Clearly

$$\sup_{\sigma \in \Omega} h(\sigma) = h(\sigma_{max}) = \alpha(1 - \frac{\alpha}{2})n^2 + O(n), \tag{14}$$

where σ_{max} is any element of σ with $\sigma(x,y) = 1$ for the lowest $\lfloor \alpha n(n+1) \rfloor$ vertices. Also notice that for all t
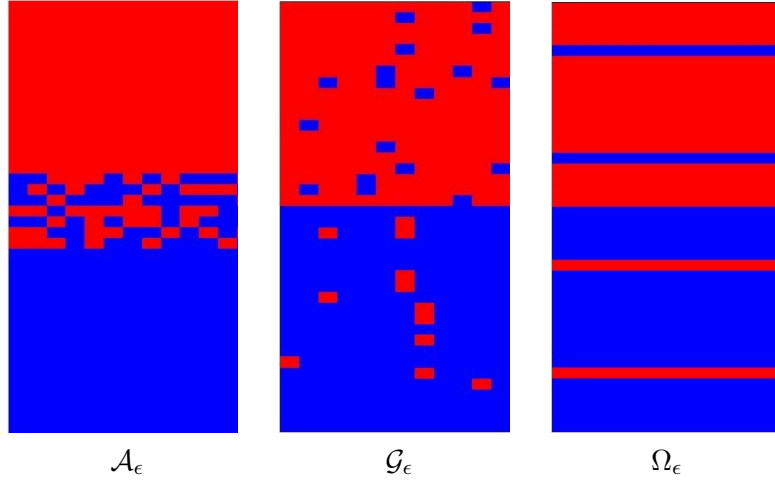
$$|h(\sigma_t) - h(\sigma_{t-1})| \le 1. \tag{15}$$

$$\mathcal{A}_\epsilon \qquad\qquad \mathcal{G}_\epsilon \qquad\qquad \Omega_\epsilon$$

Figure 3: Examples of the various good sets defined in Table 1. Γ_ϵ is defined for technical purposes and is not included in the figure since it essentially looks like $\mathcal{G}_\epsilon$ by Remark 1.

Set	Description	Defined in
$\mathcal{A}_\epsilon$	all sites outside an ϵ-band are the right color	(2)
$\mathcal{G}_\epsilon$	at most ϵn^2 sites are the wrong color	(12)
Γ_ϵ	height function within ϵn^2 of its maximum	(16)
Ω_ϵ	at most ϵn rows have sites of the wrong color	(24)

Table 1: Four different kinds of good sets.

Definition 3. Throughout the rest of the article given $\sigma \in \Omega \cup \Omega'$ we say that a vertex $v = (x, y)$ has the *wrong color* if either $y < \alpha$ and $\sigma(v)$ is red or if $y > \alpha$ and $\sigma(v)$ is blue.

Definition 4. For $\epsilon > 0$ let

$$\Gamma_\epsilon := \{\sigma \in \Omega : h(\sigma) > \big(\alpha(1 - \tfrac{\alpha}{2}) - \epsilon\big)n^2\}. \tag{16}$$

where $h(\cdot)$ is the height function defined in (13).

Remark 1. One easily observes the following inclusions:

1. $\Gamma_{\epsilon^2} \subset \mathcal{G}_\epsilon \subset \Gamma_\epsilon$

2. $\mathcal{A}_\epsilon \subset \Omega_\epsilon$

3. $\mathcal{A}_\epsilon \subset \mathcal{G}_{2\epsilon}$.

The following lemma states that starting from any configuration σ the process takes $O(n^2)$ steps to hit Γ_ϵ.

Lemma 2. *Given any small enough $\epsilon > 0$ there exist positive constants $c = c(\epsilon), d = d(\epsilon), N = N(\epsilon)$ such that for all $n > N$ and $\sigma \in \Omega$*

$$\mathbb{P}_\sigma(\tau(\Gamma_\epsilon) > 2dn^2) \ \leq \ e^{-cn^2}. \tag{17}$$

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The next lemma asserts that once the process has hit Γ_ϵ it is likely to stay in a neighborhood $\Gamma_{2\epsilon}$ for a long time.

Lemma 3. *Given $\epsilon > 0$ there exist positive constants $c = c(\epsilon), d = d(\epsilon), N = N(\epsilon)$ such that for all $n > N$ and $\sigma \in \Gamma_\epsilon$*

$$\mathbb{P}_\sigma(\tau(\Omega \setminus \Gamma_{2\epsilon}) > e^{cn^2}) \quad \geq \quad 1 - e^{-dn^2}. \tag{18}$$

The remainder of this section deduces Theorem 4 from Lemmas 2 and 3. The Lemmas themselves are proved in Section 5.

3.2 Bounding stationary measure in terms of hitting times

We now prove a general lemma on Markov chains relating hitting times to the stationary distribution, which will be used in the proofs of both Theorems 1 and 4. Roughly, it says that if one can identify subsets $A \subset B$ of the state space of a Markov chain such that (1) A is hit quickly from all starting points, and (2) it takes a long time to escape B from all starting points inside A, then the stationary distribution assigns high probability to B. This along with Lemmas 2 and 3 would then allow us to conclude that $\Gamma_{2\epsilon}$ has high stationary measure.

Lemma 4. *Let $\omega(\cdot)$ be an irreducible Markov chain on a finite state space $\mathcal{M}$. Suppose $A \subset B \subset \mathcal{M}$. Let t_1, t_2, p_1, p_2 be such that*

$$\max_{\omega \in \mathcal{M}}\big\{\mathbb{P}_\omega(\tau(A) \geq t_1)\big\} \quad \leq \quad p_1 \tag{19}$$

$$\min_{\omega \in A}\big\{\mathbb{P}_\omega(\tau(B^c) \geq t_2)\big\} \quad \geq \quad 1 - p_2. \tag{20}$$

Then

$$\pi(B^c) \leq \frac{t_1}{t_2} + p_1 + p_2,$$

where π is the stationary distribution of the Markov chain $\omega(\cdot)$ on $\mathcal{M}$.

Proof. First we note that by the ergodic theorem for Markov chains for all $\omega \in \mathcal{M}$

$$\pi(B^\mathbf{c}) = \lim_{t \to \infty} \frac{\mathbb{E}_\omega \sum_{k=1}^t \mathbf{1}(\omega(k) \in B^\mathbf{c})}{t}. \tag{21}$$

The proof of the above is standard. For example, it directly follows from [12, Theorem 4.9]. We now claim that the following bound is true:

$$\sup_{\omega \in \mathcal{M}} \mathbb{E}_\omega \sum_{k=1}^{t_2} \mathbf{1}(\omega(k) \in B^\mathbf{c}) \leq t_1 + (p_1 + p_2)(t_2).$$

To see this decompose the sum on the left hand side:

$$\mathbb{E}_\omega \sum_{k=1}^{t_2} \mathbf{1}(\omega(k) \in B^\mathbf{c}) = \mathbb{E}_\omega \sum_{k=1}^{\tau(A) \wedge t_2} \mathbf{1}(\omega(k) \in B^\mathbf{c}) + \mathbb{E}_\omega \sum_{k=\tau(A) \wedge t_2}^{t_2} \mathbf{1}(\omega(k) \in B^\mathbf{c}).$$

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The first sum is at most t_1 on the event that $\tau(A) \leq t_1$ and at most t_2 on the complement. The second sum is at most t_2 on the event that after $\tau(A)$ the process exits B in fewer than t_2 steps and 0 otherwise. Hence adding these up we get the upper bound

$$\sup_{\omega \in \mathcal{M}} \mathbb{E}_\omega \sum_{k=1}^{t_2} \mathbf{1}(\omega(k) \in B^{\mathbf{c}}) \leq t_1 + p_1 t_2 + p_2 t_2. \tag{22}$$

We now decompose the sum $\sum_{k=1}^{t} \mathbf{1}(\omega(k) \in B^{\mathbf{c}})$ into blocks of length t_2, i.e.

$$\sum_{k=1}^{t} \mathbf{1}(\omega(k) \in B^{\mathbf{c}}) = \sum_{k=1}^{t_2} \mathbf{1}(\omega(k) \in B^{\mathbf{c}}) + \sum_{k=t_2+1}^{2t_2} \mathbf{1}(\omega(k) \in B^{\mathbf{c}}) + \dots.$$

The theorem now follows from (21) by using (22) for each of the above sums on the right hand side. $\qquad\square$

3.3 Proof of Theorem 4

Notice that it suffices to prove that for given small enough ϵ for all large enough n

$$\pi(\Gamma_{2\epsilon}) \geq 1 - e^{-cn^2} \tag{23}$$

for some $c = c(\epsilon) > 0$. By the lower containment in Remark 1 this proves that

$$\pi(\mathcal{G}_{\sqrt{2\epsilon}}) \geq 1 - e^{-cn^2}.$$

The proof now follows immediately from Lemma 4 with the following choices of the parameters:

$$
\begin{aligned}
A &= \Gamma_\epsilon \\
B &= \Gamma_{2\epsilon} \\
t_1 &= d_1 n^2 \\
t_2 &= e^{d_2 n} \\
p_1 &= e^{-c_1 n^2} \\
p_2 &= e^{-c_2 n^2}
\end{aligned}
$$

where c_1, c_2, d_1, d_2 are chosen such that the hypotheses (19) and (20) of Lemma 4 are satisfied. Lemmas 2 and 3 allow us to do that. $\qquad\square$

4 The main technical result

We now build towards the proofs of Lemmas 2 and 3. The next result is one of the key technical ingredients of the paper. Recall (10) and (11). Given $\sigma \in \Omega \cup \Omega'$ for any $a \in \{0, \frac{1}{n}, \dots, 1\}$ let us say that the line $y = a$ is a *bad level* if there exists $\mathbf{x} = (x, a)$ such that $\sigma(\mathbf{x})$ has the wrong color. We now define the set of configurations σ that have very few bad levels.
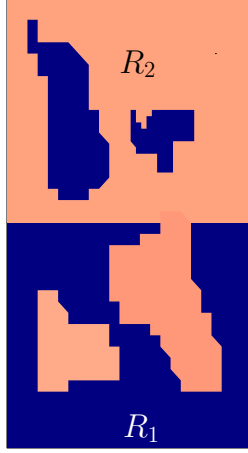
Figure 4: The sets R_1 and R_2 of Definition 6. Note that the two blue islands and the red island are not included in either set.

Definition 5. Given $\epsilon > 0$ for all n let Ω_ϵ be the set of all configurations $\sigma \in \Omega$ such that

$$\#\{y : \sigma((j,y)) = 2 \text{ for some } (j,y) \in \mathrm{Cyl}_n, \text{ with } y \leq \alpha\} \leq \epsilon n \quad \text{and} \quad (24)$$
$$\#\{y : \sigma((j,y)) = 1 \text{ for some } (j,y) \in \mathrm{Cyl}_n, \text{ with } y \geq \alpha\} \leq \epsilon n. \quad (25)$$

Define Ω'_ϵ similarly.

Recall the competitive erosion chain σ_t defined in (8). This section is devoted entirely to the proof of the following theorem:

Theorem 5. (Lyapunov Function) *Given $\epsilon > 0$ there exists $a = a(\epsilon) > 0$ such that for all n sufficiently large and for all $\sigma \in \Omega_\epsilon^\complement$*

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \geq a$$

where $h(\cdot)$ is the height function defined in (13).

For $A \subset \mathrm{Cyl}_n$ and $v \in \mathrm{Cyl}_n$ define

$$H_A(v) = \mathbb{E}_v[y_{\tau(A^c)}] \quad (26)$$

where $(x_{\tau(A^c)}, y_{\tau(A^c)})$ is the point at which the random walk exits A and $\mathbb{E}_v$ is the expectation under the random walk measure on Cyl_n with starting point v.

Definition 6. Given $\sigma \in \Omega \cup \Omega'$ define $R_1 = R_1(\sigma) \subset \mathrm{Cyl}_n$ to be the set of all points of color 1 in Cyl_n reachable by a monochromatic path of color 1 in σ from a point in $C_n \times \{0\}$.

Similarly let $R_2 = R_2(\sigma)$ be the set of all points of color 2 in Cyl_n reachable by a monochromatic path of color 2 from a point in $C_n \times \{1\}$. See Figure 4.

Lemma 5. *For $\sigma \in \Omega$,*

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) = \frac{1}{n} \mathbb{E}_{\sigma_0} \left[\sum_{w \in C_n \times \{1\}} H_{R_2(\sigma_{1/2})}(w) \right] - \frac{1}{n} \sum_{v \in C_n \times \{0\}} H_{R_1(\sigma_0)}(v) \quad (27)$$

$$(28)$$

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where

$$\sigma_0, \sigma_{1/2}, \sigma_1$$

form the steps of the competitive erosion chain.

Proof. The first walker of color 1 starts uniformly on $C_n \times \{0\}$ and stops on first hitting a site of color 2, that is, on first exiting $R_1(\sigma)$. Writing

$$h(\sigma_1) - h(\sigma_0) = [h(\sigma_{1/2}) - h(\sigma_0)] + [h(\sigma_1) - h(\sigma_{1/2})],$$

we see that the expected value of the first term in brackets on the right side is

$$1 - \frac{1}{n} \sum_{v \in C_n \times \{0\}} H_{R_1(\sigma_0)}(v).$$

Now a walker of color 2 starts uniformly on $C_n \times \{1\}$ and stops on first exiting $R_2(\sigma_{1/2})$, so the expected value of the second term in brackets is

$$\frac{1}{n} \mathbb{E}_{\sigma_0} \left[\sum_{w \in C_n \times \{1\}} H_{R_2(\sigma_{1/2})}(w) \right] - 1$$

(this term appears with an expectation because $\sigma_{1/2}$ is random). $\qquad\square$

4.1   Energy and flows

Next we relate the expression on the the right side of (27) to energy of flows. We first recall some standard notation in electrical network theory. On a graph $G = (V, E)$ let $w \sim v$ signify that w is a neighbor of v. Also let $\vec{E}$ be the set of directed edges where each edge in E corresponds to two directed edges in $\vec{E}$, one in each direction (except for self-loops, which correspond to just one directed edge in $\vec{E}$). A flow is an antisymmetric function $f : \vec{E} \to \mathbb{R}$ (i.e. a function satisfying $f(w, v) = -f(v, w)$). Note that by definition the value of a flow on a self-loop is 0. Define the energy of a flow f by

$$\mathcal{E}(f) = \frac{1}{2} \sum_{(v,w) \in \vec{E}} f(v, w)^2. \tag{29}$$

For any flow $f : \vec{E} \to \mathbb{R}$ define the divergence div $f : V \to \mathbb{R}$ by

$$\text{div} f(v) = \sum_{w \sim v} f(w, v). \tag{30}$$

Note that for any flow

$$\sum_{x \in V} \text{div} f = \sum_{\substack{x,y \in V \\ y \sim x}} f(x, y) + f(y, x) = 0$$

since f is antisymmetric. For disjoint subsets $A, B \subset V$ and a flow f we say that the flow is from A to B if div $f(x) = 0$ for all vertices x except vertices of A and B and the sum of the divergences across vertices of A and B are non-negative and non-positive respectively. For more about flows see [12, Chapter 9].

Now we state a key lemma relating Lemma 5 to energy of flows on Cyl_n.

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Lemma 6. *For any $A \subset \mathrm{Cyl}_n$ such that $A \cap (C_n \times \{1\}) = \emptyset$,*

$$\frac{1}{n} \sum_{v \in C_n \times \{0\}} H_A(v) = \inf_f \mathcal{E}(f)$$

where the infimum is taken over all flows from $A \bigcap (C_n \times \{0\})$ to A^c such that for $(x, y) \in A$

$$\mathrm{div}(f)(x, y) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for any $B \subset \mathrm{Cyl}_n$ such that $B \cap (C_n \times \{0\}) = \emptyset$,

$$1 - \frac{1}{n} \sum_{v \in C_n \times \{1\}} H_B(v) = \inf_f \mathcal{E}(f)$$

where the infimum is taken over all flows from $(C_n \times \{1\}) \cap B$ to B^c such that for $(x, y) \in B$

$$\mathrm{div}(f)(x, y) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is deferred to Appendix B. We now briefly sketch how it helps in proving Theorem 5.

By (27) and Lemma 6 one sees that

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) = 1 - \mathcal{E}(f_1) - \mathbb{E}[\mathcal{E}(f_2)],$$

where f_1 and f_2 are flows on $R_1(\sigma)$ and $R_2(\sigma_{1/2})$ satisfying the divergence conditions mentioned in Lemma 6 and have minimal energy. Note that the second term on the RHS has an extra expectation to average over the random intermediate random configuration $\sigma_{1/2}$.

It turns out that the flow with minimal energy on R_1 is the "natural" flow induced by random walk started uniformly on $C_n \times \{0\}$ and killed on exiting R_1 and similarly for R_2.

Theorem 5 is now proved by estimating $\mathcal{E}(f_1)$ and $\mathcal{E}(f_2)$. Consider the voltage function on Cyl_n that is linear in the height. $\mathcal{E}(f_1)$ can be interpreted as the voltage difference between $C_n \times \{0\}$ and the boundary of R_1 with glued boundary condition. The above heuristics are formalized in Appendix B. Now if the boundary is not horizontal enough, gluing vertices at different potentials causes a voltage drop. We estimate this voltage drop en route to proving Theorem 5 by constructing suitable flows on R_1 and R_2.

4.2 Towards the proof of Theorem 5

For technical purposes we introduce a few definitions. Recall R_1 and R_2 from Definition 6.

Definition 7. Given $\sigma \in \Omega \cup \Omega'$ let $\tilde{R}_1$ be the set of all points in Cyl_n reachable by a monochromatic path of color 1 in σ from a point in $C_n \times \{0\}$ which does not hit the set $C_n \times \{1\}$.

Similarly define $\tilde{R}_2$. See Figure 6.

For $i = 1, 2$ define

$$\tau_i = \tau(R_i^c) \tag{31}$$

to be the exit time from R_i. Also let

$$\tilde{\tau}_1 = \tau(\tilde{R}_1^c) = \tau_1 \wedge \tau(C_n \times \{1\}) \tag{32}$$
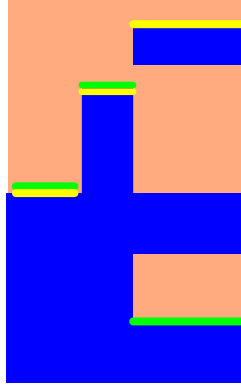
and similarly $\tilde{\tau}_2$.

Figure 5: The green and yellow segments depict y_k^* and y_k^{**} respectively for $k \in C_n$.

Remark 2. Recalling the notation in (26) we see that

$$H_{R_i(\sigma)}(v) \leq H_{\tilde{R}_i(\sigma)}(v)$$

since on the event $\tilde{\tau}_1 < \tau_1$, $y_{\tilde{\tau}_1} = 1$ and hence

$$y_{\tau_1} < y_{\tilde{\tau}_1} = 1.$$

For notational simplicity we write

$$H_1(\cdot) = H_{1,\sigma}(\cdot) = H_{R_1(\sigma)}(\cdot)$$
$$H_2(\cdot) = H_{2,\sigma}(\cdot) = H_{R_2(\sigma)}(\cdot)$$
$$\tilde{H}_1(\cdot) = \tilde{H}_{1,\sigma}(\cdot) = H_{\tilde{R}_1(\sigma)}(\cdot)$$
$$\tilde{H}_2(\cdot) = \tilde{H}_{2,\sigma}(\cdot) = H_{\tilde{R}_2(\sigma)}(\cdot).$$

We use the first notation when the underlying σ is clear from context.

For every $k \in C_n, \sigma \in \Omega \cup \Omega'$ let

$$y_k^*(\sigma) = \inf\{y : (k, y) \notin \tilde{R}_1\} \tag{33}$$
$$y_k^{**}(\sigma) = \sup\{y : (k, y) \notin \tilde{R}_2\}. \tag{34}$$

See Figure 5.

As usual we will suppress the dependence on σ and denote $y_k^*(\sigma)$, $y_k^{**}(\sigma)$ by y_k^* and y_k^{**} respectively when σ is clear from the context.

Lemma 7. *For $\sigma \in \Omega \cup \Omega'$*

$$\frac{1}{n} \sum_{k \in C_n} H_1(k, 0) \leq \frac{1}{n} \sum_{k \in C_n} y_k^* \tag{35}$$

$$\frac{1}{n} \sum_{k \in C_n} H_2(k, 0) \geq \frac{1}{n} \sum_{k \in C_n} y_k^{**}. \tag{36}$$

Proof. We will only prove (35) since the arguments for (36) are symmetric. By Remark 2 it suffices to prove that

$$\sum_{k \in C_n} \tilde{H}_1(k, 0) \leq \sum_{k \in C_n} y_k^*. \tag{37}$$
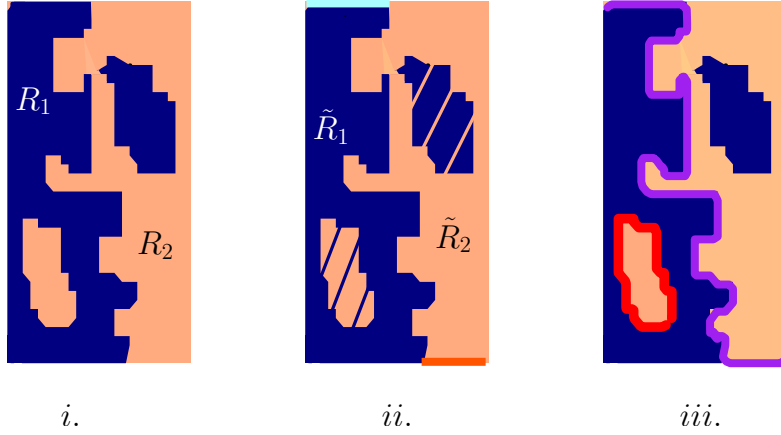
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Figure 6: i. A general configuration with R_1 and R_2. Note than in ii. the shaded red region and the top light-blue line $(R_1 \cap C_n \times \{1\})$ are outside $\tilde{R}_1$. Similarly the shaded blue region and the bottom orange line are outside $\tilde{R}_2$. Notice the difference between R_i and $\tilde{R}_i$. In iii. $\partial^{out}\tilde{R}_1$ is represented by the purple curve. The rest of $\partial\tilde{R}_1$ (bottom left) is represented by the red curve.

Note that $\tilde{R}_1$ satisfies the hypothesis of Lemma 6. Thus using the lemma we will prove the above by finding a flow f^* from $\tilde{R}_1 \cap (C_n \times \{0\})$ to $\tilde{R}_1^c$ such that for all $(k, y) \in \tilde{R}_1$

$$\text{div}(f^*)(k, y) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{E}(f^*) = \frac{1}{n} \sum_{k \in C_n} y_k^*. \tag{38}$$

Let

$$f^*((k, y), (k, y + \frac{1}{n})) = -\frac{1}{n} \text{ for } (k, y) \in \tilde{R}_1, \ 0 \le y < y_k^* \tag{39}$$

and let every other edge have zero flow. It is easy to check that f^* satisfies the required divergence conditions and (38). See Figure 7 i. $\qquad\square$

The next lemma can be thought of as a weaker version of Theorem 5. It shows that at all times the height function has an "almost" non-negative drift whereas Theorem 5 states that outside Ω_ϵ the drift becomes significantly stronger.

Lemma 8. *For all $\sigma \in \Omega$*

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \ge -\frac{1}{n}$$

where $h(\cdot)$ is the height function defined in (13).

Note that the above lemma is sharp for $\sigma_0 = \sigma_{\max}$ (14), up to a constant factor.

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Proof. By Lemma 5

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) = \frac{1}{n}\mathbb{E}\left[\sum_{w \in C_n \times \{1\}} H_{2,\sigma_{1/2}}(w)\right] - \frac{1}{n}\sum_{v \in C_n \times \{0\}} H_{1,\sigma_0}(v).$$

Thus by Lemma 7

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \geq \frac{1}{n}\mathbb{E}\left[\sum_{k \in C_n} y_k^{**}(\sigma_{1/2})\right] - \frac{1}{n}\sum_{k \in C_n} y_k^{*}(\sigma_0) \tag{40}$$

where $y_k^{*}(\cdot)$ and $y_k^{**}(\cdot)$ are defined in (33) and (34) respectively. We make the following two easy observations: For all $\sigma_0 = \sigma \in \Omega$ and $k \in C_n$,

$$y_k^{**}(\sigma_{1/2}) \geq y_k^{**}(\sigma_0) \tag{41}$$

$$y_k^{**}(\sigma_0) - y_k^{*}(\sigma_0) \geq -\frac{1}{n}. \tag{42}$$

The first one is a straightforward consequence of the fact that $\sigma_{1/2}$ has one more blue vertex than σ_0. To see the second one, notice that for all $k \in \frac{1}{n}\mathbb{Z}/n\mathbb{Z}$, either $y_k^{*}(\sigma_0) = 0$ or

$$\{(k,0), \ldots, (k, y_k^{*}(\sigma_0) - \frac{1}{n})\} \in \tilde{R}_1.$$

In the first case (42) follows since $y_k^{**}(\sigma_0) \geq 0$. In the second case, $\tilde{R}_1$ and $\tilde{R}_2$ are disjoint and therefore $y_k^{*}(\sigma_0) - \frac{1}{n} \notin \tilde{R}_2$. Hence by definition (42) follows.

Thus

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \geq \frac{1}{n}\mathbb{E}\left[\sum_{k \in C_n} y_k^{**}(\sigma_{1/2})\right] - \frac{1}{n}\sum_{k \in C_n} y_k^{*}(\sigma_0) \tag{43}$$

$$\geq \frac{1}{n}\sum_{k \in C_n} y_k^{**}(\sigma_0) - \frac{1}{n}\sum_{k \in C_n} y_k^{*}(\sigma_0) \tag{44}$$

$$\geq -\frac{1}{n}. \tag{45}$$

The first inequality is (40), the second and third follow immediately from the above two observations. Hence we are done. $\square$

4.3 Bending flows

We prove Theorem 5 in this subsection by improving upon Lemma 8 which was proved using the trivial flow on the cylinder graph that sends all the mass along the vertical edges. However if the boundary of R_1 (Definition 6) is not horizontal then flows that send some mass along the horizontal edges are more optimal. This is because the horizontal boundary edges have significant harmonic measure and the random walk started from $C_n \times \{0\}$ has a significant chance of exiting R_1 through these edges. Thus to prove Theorem 5 we modify the trivial flow by sending some mass along the horizontal edges. See Figure 7. We begin with a few definitions.
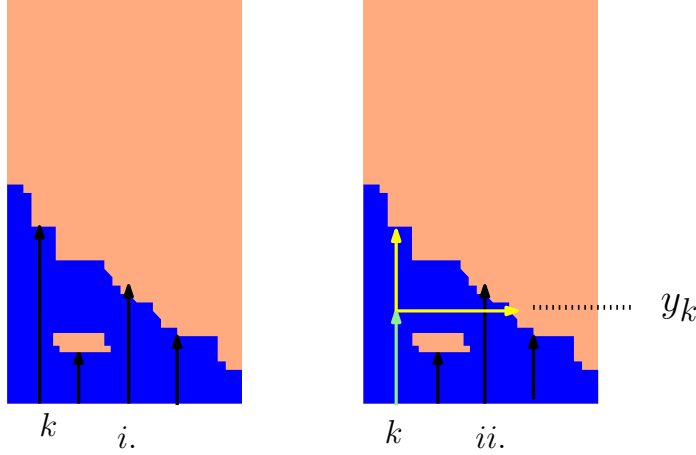
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Figure 7: Bending the trivial flow to improve upon Lemma 8. In $i.$ every vertical edge carries mass $\frac{1}{n}$. In $ii.$ the flow carries mass $\frac{1}{n}$ up to height y_k where it splits into $\frac{d_k}{n}$ (vertically) and $1 - \frac{d_k}{n}$ (horizontally). The d_k's are chosen to be $\frac{1}{1+y_k^*-y_k}$. Also note the flows are split only for the columns where y_k^* and y_k^{**} are close to each other to beat (42) used in Lemma 8.

For any $m \in (0,1)$ let us define the following sets:

$$W = \{\ell : y_\ell^{**} \geq m\} \tag{46}$$
$$W' = \{\ell : y_\ell^{*} \leq m\}. \tag{47}$$

Also let us define the following properties for $c > 0$ and $m \in (0,1)$:

Property 1. *There exist $cn/4$ horizontal lines below the level $y = m - c/4$ that intersect $\mathrm{Cyl}_n \setminus \tilde{R}_1$.*

Property 2. *There exist $cn/4$ horizontal lines above the level $y = m + c/4$ that intersect $\mathrm{Cyl}_n \setminus \tilde{R}_2$.*

The statement of the next lemma roughly says that if there are sufficiently many bad levels below or above a certain horizontal line then the height function experiences a strong drift. The proof uses the idea of bending flows outlined above.

Lemma 9. *Given $c, d > 0$ suppose $\sigma \in \Omega$ is such that one of the following holds: for some m,*

  i. Property 1 holds for c and $|W| \geq dn$.

  ii. Property 2 holds for c and $|W'| \geq dn$.

Then for large enough n,
$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \geq a \tag{48}$$
where $a = a(c, d)$.

Proof. Let m be as in the statement of the lemma. Because of the obvious symmetry between the two conditions we will only discuss the proof of case i. Without loss of generality we can assume $c = d$.
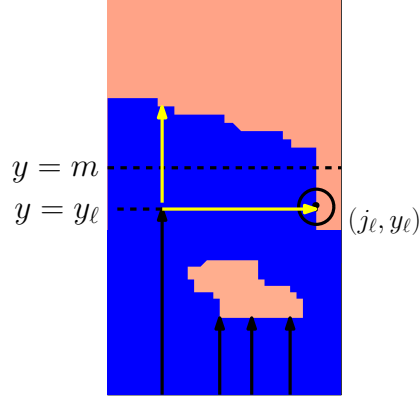
Figure 8: Proof of Lemma 9 in case of Property 1.

Let W_1 be an arbitrary subset of W of size $cn/6$. We choose an arbitrary pairing of points in W_1 with horizontal lines mentioned in Property 1, i.e. for every ℓ in W_1 we associate a distinct horizontal line $y = y_\ell$ that intersects $\partial \tilde{R}_1$ and $y_\ell \leq m - c/4$. Also let

$$(j_\ell, y_\ell) \tag{49}$$

be the point on the line $y = y_\ell$ closest to the point (ℓ, y_ℓ) such that $(j_\ell, y_\ell) \in \partial \tilde{R}_1$. j_ℓ exists since the line $y = y_\ell$ intersects $\partial \tilde{R}_1$ (see Figure 8). Now for all $\ell \in W_1$,

$$y_\ell^{**} - y_\ell > c/4 \tag{50}$$

as $W_1 \subset W$.

Let

$$W_2 = \{\ell \in W_1 : y_\ell^* \geq m - \frac{c}{8}\}. \tag{51}$$

Hence for all $\ell \in W_1 \setminus W_2$

$$y_\ell^{**} - y_\ell^* \geq \frac{c}{8}. \tag{52}$$

Thus we have

$$\frac{1}{n} \sum_{k \in C_n} [y_k^{**} - y_k^*] \geq \frac{1}{n} \sum_{k \in W_1 \setminus W_2} \frac{c}{8} - \frac{1}{n}.$$

The extra $-\frac{1}{n}$ term follows from (42).

Also for all $\ell \in W_2$,

$$y_\ell^* - y_\ell \geq \frac{c}{8}. \tag{53}$$

We now consider the two following sub-cases:

(i) $|W_2| \leq |W_1|/2$.

Then since $|W_1| = \frac{cn}{6}$ and hence $|W_1 \setminus W_2| \geq \frac{cn}{12}$, by (52) we have

$$\frac{1}{n} \sum_{k \in C_n} [y_k^{**} - y_k^*] \geq \frac{c^2}{96} - \frac{1}{n}. \tag{54}$$

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Thus in this case by (43),

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \geq \frac{c^2}{96} - \frac{1}{n}$$

and hence we are done.

(ii) $|W_2| \geq |W_1|/2$.

The proof in this case is more involved than the previous case. We first claim that in this case there exists a flow f_1 from $C_n \times \{0\} \cap \tilde{R}_1$ to $\mathrm{Cyl}_n \setminus \tilde{R}_1$ such that for all $(k, y) \in \tilde{R}_1$

$$\mathrm{div}(f_1)(k, y) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

and

$$\mathcal{E}(f_1) \leq \frac{1}{n} \sum_{k \in C_n} y_k^* - \frac{c^3}{1536}. \tag{56}$$

Before proving the above claim we show why it suffices and implies (48). By taking the set A in Lemma 6 to be $\tilde{R}_1$ followed by using Remark 2 we get

$$\frac{1}{n} \sum_{v \in C_n \times \{0\}} H_{1,\sigma_0}(v) \;\; \leq \;\; \frac{1}{n} \sum_{v \in C_n \times \{0\}} \tilde{H}_{1,\sigma_0}(v) \tag{57}$$

$$\leq \;\; \frac{1}{n} \sum_{k \in C_n} y_k^* - \frac{c^3}{1536}.$$

By Lemma 5

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) = \frac{1}{n} \mathbb{E}[\sum_{w \in C_n \times \{1\}} H_{2,\sigma_{1/2}}(w)] - \frac{1}{n} \sum_{v \in C_n \times \{0\}} H_{1,\sigma_0}(v). \tag{58}$$

The first term is bounded from below by $\dfrac{1}{n} \sum_{k \in C_n} y_k^{**}(\sigma_0)$ (as in the proof of (43)) and the second term by (57) is upper bounded by

$$\frac{1}{n} \sum_{k \in C_n} y_k^*(\sigma_0) - \frac{c^3}{1536}.$$

Hence we get

$$\mathbb{E}(h(\sigma_1) - h(\sigma_0) \mid \sigma_0 = \sigma) \geq \left[\frac{1}{n} \sum_{k \in C_n} y_k^{**}(\sigma_0) - \frac{1}{n} \sum_{k \in C_n} y_k^*(\sigma_0) \right] + \frac{c^3}{1536}$$

$$\geq \frac{c^3}{1536} - \frac{1}{n}.$$

Note that the term in the brackets is greater than $-\frac{1}{n}$ by (42). Thus the proof of Lemma 9 is complete in this case except for the proof of the initial claim. We now prove the claim by defining the flow f_1 as shown in Figure 8. Recall (51). For $k \notin W_2$ let

$$f_1((k, y), (k, y + \frac{1}{n})) = -\frac{1}{n} \text{ for all } (k, y) \in \tilde{R}_1 \text{ and } y < y_k^*.$$

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For $k \in W_2$

$$f_1((k,y),(k,y+\frac{1}{n})) = \begin{cases} -\frac{1}{n} & \text{if} \quad y < y_k \\ \\ -\frac{d_k}{n} & \text{if} \quad y_k \leq y < y_k^* \end{cases}$$

$$f_1((j,y_k),(j+\frac{1}{n},y_k)) = -\frac{1-d_k}{n} \; \forall j = k, k+\frac{1}{n}, \ldots j_k$$

where j_k was defined in (49) and $0 < d_k < 1$ are to be specified later. f_1 is 0 on all other edges. It is easy to see that the flow f_1 satisfies the divergence conditions (55).

To see why (56) is true first observe that

$$\mathcal{E}(f_1) \leq \frac{1}{n} \sum_{k \in W_2} [y_k + d_k^2(y_k^* - y_k) + (1-d_k)^2] + \frac{1}{n} \sum_{k \notin W_2} y_k^*.$$

This is because by construction:

- for $k \notin W_2$ the flow along the line $x = k$ is $\frac{1}{n}$ for ny_k^* edges.

- for $k \in W_2$

  - the flow along the line $x = k$ is $\frac{1}{n}$ for ny_k edges and $\frac{d_k}{n}$ for $n(y_k^* - y_k)$ edges.
  - the number of edges with flow $\frac{1-d_k}{n}$ on the line $y = y_k$ is at most n.

Plugging in $d_k = \frac{1}{1+y_k^*-y_k}$ we get that the expression on the RHS equals

$$\frac{1}{n} \sum_{k \in W_2} \left[y_k + \frac{(y_k^* - y_k)}{(y_k^* - y_k) + 1} \right] + \frac{1}{n} \sum_{k \notin W_2} y_k^*$$

$$= \frac{1}{n} \sum_{k \in W_2} \left[y_k^* + (y_k - y_k^*) + \frac{(y_k^* - y_k)}{(y_k^* - y_k) + 1} \right] + \frac{1}{n} \sum_{k \notin W_2} y_k^*,$$

$$= \frac{1}{n} \sum_{k \in W_2} y_k^* + + \frac{1}{n} \sum_{k \notin W_2} y_k^* + \frac{1}{n} \sum_{k \in W_2} \left[(y_k - y_k^*) + \frac{(y_k^* - y_k)}{(y_k^* - y_k) + 1} \right]$$

$$= \frac{1}{n} \sum_{k \in C_n} y_k^* - \frac{1}{n} \sum_{k \in W_2} \left[\frac{(y_k^* - y_k)^2}{(y_k^* - y_k) + 1} \right].$$

Now since $|W_1| \geq \frac{cn}{6}$ and we are considering the case $|W_2| \geq \frac{|W_1|}{2}$, we have $|W_2| \geq \frac{cn}{12}$. Also for $k \in W_2$ by (53)

$$\frac{c}{8} \leq y_k^* - y_k \leq 1.$$

Plugging these in the expression

$$\frac{1}{n} \sum_{k \in W_2} \left[\frac{(y_k^* - y_k)^2}{(y_k^* - y_k) + 1} \right]$$

gives us (56). Thus the proof of Lemma 9 for case $i.$ is complete. $\qquad \square$

The proof of Theorem 5 involves considering a few cases and showing that the hypotheses of Lemma 9 are satisfied in each case. We start with some definitions.

Definition 8. If A is a set of vertices in a graph $G = (V, E)$, define the "boundary" of A to be the set of vertices in $V \setminus A$ connected to a vertex in A by an edge in E.

Let Cyl_n^+ be the slightly larger graph $C_n \times P_{n+2}$ where we identify the path P_{n+2} with

$$\{\frac{a}{n} \; : \; a = -1, 0, 1, \ldots, n+1\}.$$

The following is a technical definition needed for convenience (see Figure 6):

Definition 9. Let $\partial \tilde{R}_1$ be the boundary of $\tilde{R}_1 \cup \left(C_n \times \{\frac{-1}{n}\} \right)$ in Cyl_n^+. Define $\partial^{out} \tilde{R}_1$ to be the subset of $\partial \tilde{R}_1$ that is visible from $C_n \times \{1\}$, i.e., the set of all points in $\partial \tilde{R}_1$ that are connected to $C_n \times \{1\}$ in the subgraph induced by $\mathrm{Cyl}_n \setminus \tilde{R}_1$. As usual we have similar definitions for $\partial \tilde{R}_2$, $\partial^{out} \tilde{R}_2$.

Note that since $\tilde{R}_1 \cap \left(C_n \times \{1\} \right) = \emptyset$,

$$\partial \tilde{R}_1 \subset \mathrm{Cyl}_n.$$

Define the graph Cyl_n^* to be the graph Cyl_n along with the additional diagonal edges

$$\left\{ \left(\frac{i}{n}, \frac{j}{n} \right), \left(\frac{i+1}{n}, \frac{j+1}{n} \right) \right\}$$
$$\left\{ \left(\frac{i}{n}, \frac{j}{n} \right), \left(\frac{i-1}{n}, \frac{j+1}{n} \right) \right\}$$

for all $i = 0, \ldots n-1$, $j = 0 \ldots n-1$ where the addition in the first coordinate is in $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$. Call a subset $B \subset \mathrm{Cyl}_n$, $*$-connected if it is a connected set in the graph Cyl_n^*.

Remark 3. It follows from [17, Lemma 2] that for $i = 1, 2$, $\partial^{out} \tilde{R}_i$ is a $*$-connected set.

Let

$$\mathcal{Y} \quad = \quad \{y : (k, y) \in \partial^{out} \tilde{R}_1 \text{ for some } k \in C_n\}. \tag{59}$$

Note that $\mathcal{Y}$ is a connected subset of $[0, \frac{1}{n} \ldots, 1]$ since $\partial^{out} \tilde{R}_1$ is $*$-connected. Also for every $k \in C_n$ let

$$y_{(k)} \quad = \quad \inf\{y : (k, y) \in \partial^{out} \tilde{R}_1\} \tag{60}$$
$$y^{(k)} \quad = \quad \sup\{y : (k, y) \in \partial^{out} \tilde{R}_1\}.$$

See Figure 9. Recall the definition of y_k^* and y_k^{**} from (33),(34). Note that for every $k \in C_n$,

$$y_k^* \leq y_{(k)}. \tag{61}$$

We make another simple observation: For any $k \in C_n$ there exists $\ell \in \{k - \frac{1}{n}, k, k + \frac{1}{n}\}$ such that

$$y^{(k)} - \frac{1}{n} \leq y_\ell^{**}. \tag{62}$$

This follows since the point $(k, y^{(k)}) \in \partial^{out} \tilde{R}_1$ and hence either $y^{(k)} = 0$, or $(k, y^{(k)})$ has a neighbor colored 1 in Cyl_n.

We are now ready to prove Theorem 5. As we remarked earlier it suffices to show that the hypotheses of Lemma 9 are satisfied.

Proof of Theorem 5. By hypothesis $\sigma \in \Omega \setminus \Omega_\epsilon$. Also observe that for any $\sigma \in \Omega$ one of the following three conditions holds:
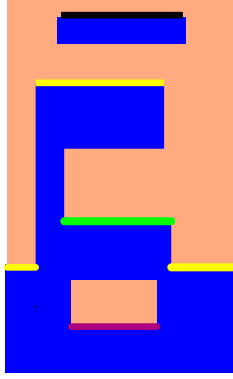
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Figure 9: The yellow segment depicts $y^{(k)}$ for $k \in C_n$. The green segment depicts $y_{(k)}$ for those k for which $y^{(k)} \neq y_{(k)}$. Note that $y^{(k)} = y_{(k)}$ unless $\partial^{out}\tilde{R}_1$ intersects the column $x = k$ at more than one point. Also we include the purple segment at the bottom and the black segment at the top depicting y_k^* and y_K^{**} respectively to illustrate the differences between the four definitions.

(i) $\sup \mathcal{Y} - \inf \mathcal{Y} > \epsilon/20$.

(ii) $\sup \mathcal{Y} - \inf \mathcal{Y} \leq \epsilon/20$ and $|\sup \mathcal{Y} - \alpha| > \epsilon/10$

(iii) $\sup \mathcal{Y} - \inf \mathcal{Y} \leq \epsilon/20$ and $|\sup \mathcal{Y} - \alpha| \leq \epsilon/10$

where the set $\mathcal{Y}$ was defined in (59).

Case (i): Let $m = \frac{1}{2}(\sup \mathcal{Y} + \inf \mathcal{Y}), c = \frac{\epsilon}{20}$. By hypothesis $\sup \mathcal{Y} - \inf \mathcal{Y} > c$. We first show under this hypothesis both Properties 1 and 2 hold.

By Remark 3, $\mathcal{Y}$ is connected. Thus there exist $cn/4$ values in $\mathcal{Y}$ less than $m - c/4$. In other words:

Property 1 holds with m and c as defined above.

Again using connectedness of $\mathcal{Y}$, $cn/4$ values in $\mathcal{Y}$ are greater than $m + c/4$. Also notice that for every $k < \sup \mathcal{Y}$ the line $y = k$ must intersect R_1 since the line $y = \sup \mathcal{Y}$ has a vertex lying on the boundary of R_1 and hence we have a monochromatic path of color 1 from the level $y = 0$ to $y = \sup \mathcal{Y} - \frac{1}{n}$. Thus there exist $cn/4$ horizontal lines above the level $y = m + c/4$ that intersect $\text{Cyl}_n \setminus R_2$. Hence

Property 2 holds with m and c as defined above.

To verify the rest of the hypotheses of Lemma 9 we have to show that either the set W or W' is large. Recall $y^{(k)}$ from (60). Clearly at least one of the inequalities

$$|\{k : y^{(k)} > m\}| \geq n/2, \tag{63}$$

$$|\{k : y^{(k)} \leq m\}| \geq n/2 \tag{64}$$

is true. Recall from (46) and (47)

$$\begin{aligned} W &= \{\ell : y_\ell^{**} \geq m\} \\ W' &= \{\ell : y_\ell^* \leq m\} \end{aligned}$$

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When (63) holds, by (62)

$$|W| \geq \frac{n}{6}.$$

Similarly when (64) holds by (61),

$$|W'| \geq \frac{n}{2}.$$

Thus all the hypotheses of Lemma 9 are satisfied in case (i) and hence we are done.

To verify the hypotheses of Lemma 9 in the remaining cases we start by making an observation. Let

$$
\begin{aligned}
W'' &:= \{\ell : y_\ell^{**} \geq \inf \mathcal{Y} - \frac{1}{n}\} \\
W''' &:= \{\ell \in C_n : y_\ell^{*} \leq \sup \mathcal{Y}\}.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
|W''| &\geq \frac{n}{3}, & (65) \\
|W'''| &= n & (66)
\end{aligned}
$$

(the first follows from (62) and the second from (61)). Thus by choosing $m = \inf \mathcal{Y}$ or $\sup \mathcal{Y}$ the conditions for sets W and W' respectively in the hypotheses of Lemma 9 are always satisfied .

The remainder of the proof shows that in cases (ii) and (iii) either Property 1 or 2 holds by choosing $m = \inf \mathcal{Y}$ or $\sup \mathcal{Y}$. This would then complete the proof by Lemma 9.

Both cases (ii) and (iii) have two subcases, arguments for which are symmetric. We will discuss only one of the cases. In case (ii) we will discuss the case

$$\sup \mathcal{Y} - \alpha \geq \epsilon/10$$

and skip the case $\sup \mathcal{Y} - \alpha \leq -\epsilon/10$. In the case $\sup \mathcal{Y} - \alpha \geq \epsilon/10$

$$\inf \mathcal{Y} - \alpha \geq \frac{\epsilon}{10} - \frac{\epsilon}{20} = \frac{\epsilon}{20}.$$

Now this implies that there are at least $\frac{\epsilon}{40}n^2$ red vertices below the line $y = \inf \mathcal{Y} - \frac{\epsilon}{40}$ since

$$\inf \mathcal{Y} - \frac{\epsilon}{40} \geq \alpha + \frac{\epsilon}{40}.$$

In particular there exist at least $\frac{\epsilon}{40}$ disjoint horizontal lines below the line $y = \inf \mathcal{Y} - \frac{\epsilon}{40}$ with at least one red vertex, i.e. Property 1 is true for $m = \inf \mathcal{Y}$ and $c = \frac{\epsilon}{40}$.

In case (iii) we use the fact that $\sigma \in \Omega \setminus \Omega_\epsilon$ and hence there are at least ϵn horizontal lines with a red vertex below the line $y = \alpha$ or ϵn horizontal lines with a blue vertex above the line $y = \alpha$. We will discuss only the first case. Now by hypothesis in (iii)

$$\inf \mathcal{Y} \geq \alpha - \frac{\epsilon}{10} - \frac{\epsilon}{20} \geq \alpha - \frac{\epsilon}{5}.$$

Hence there exist at least $\frac{\epsilon}{20}$ disjoint horizontal lines below the line $y = \inf \mathcal{Y} - \frac{\epsilon}{20}$ with at least one red vertex, i.e. Property 1 is true for $m = \inf \mathcal{Y}$ and $c = \frac{\epsilon}{40}$.

The symmetric arguments which we skip will show that Property 2 holds in the other cases with $m = \sup \mathcal{Y}$ and $c = \frac{\epsilon}{40}$. Thus we are done. □

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5 Proofs of hitting time results

In this section we provide the proofs of Lemmas 2 and 3. We first state the following general lemma about hitting times of submartingales. The statement involves a few parameters and can be slightly difficult to follow. However it will be useful in subsequent applications. Let $\omega(t)$ be a stochastic process taking values in an abstract set $\mathcal{D}$. Also let $g : \mathcal{D} \to \mathbb{R}$ be a real-valued function. Let $\mathcal{F}_t$ be the filtration generated by the process $\omega(t)$ up to time t. Also define $X_t := g(\omega(t))$.

Lemma 10. *Let $A_1, A_2, a_1 > 0$. Suppose*

$$
\begin{aligned}
|g(\omega)| &\leq A_1 \text{ for all } \omega \in \mathcal{D} &\qquad (67) \\
|X_t - X_{t-1}| &\leq A_2 \text{ for all } t. &\qquad (68)
\end{aligned}
$$

Also suppose that $B \subset \mathcal{D}$ is such that for any time t

$$
\mathbb{E}(X_t - X_{t-1}|\mathcal{F}_{t-1}) \geq a_1 \mathbf{1}\left(\omega_{t-1} \notin B\right). \qquad (69)
$$

Then:

    i. For any $a_2 > 0$,

$$
P_\omega(\tau(B) \geq T) \leq \exp\left(-\frac{(a_2 - a_1 T)^2}{4A_2^2 T}\right)
$$

    for all $\omega \in \mathcal{D}$ such that $g(w) \geq A_1 - a_2$ and any T such that $a_2 - a_1 T < 0$.

    ii. Now consider the special case when B is a level set, i.e. suppose for some $a_4 > 2A_2$, $B = \{\omega : g(w) \geq A_1 - a_4\}$ and $B' = \{\omega : g(w) \leq A_1 - 2a_4\}$. Then for all $\omega \in B$ and all $T > \frac{2A_2}{a_1}$

$$
\mathbb{P}_\omega(\tau(B') \geq T') \geq 1 - \left[\exp\left(-\frac{a_4^2}{32A_2^2 T}\right) + \exp\left(-\frac{a_1^2 T^2}{32A_2^2 T}\right)\right],
$$

    where $T' = \exp\left(\frac{\min(a_4^2, a_1^2 T^2)}{32A_2^2 T}\right)$.

The proof of the above lemma follows easily from the standard Azuma-Hoeffding inequality for submartingales. We defer it to Appendix C. However we immediately see some applications. Recall the definition of Ω_ϵ from Table 1.

Lemma 11. *Given any $\epsilon > 0$ there exist positive constants $c = c(\epsilon), d = d(\epsilon), N = N(\epsilon)$ such that for all $n > N$ and $\sigma \in \Omega$*

$$
\mathbb{P}_\sigma(\tau(\Omega_\epsilon) > dn^2) \leq e^{-cn^2}. \qquad (70)
$$

Proof. The proof follows from Lemma 10 i. The stochastic process we consider is the competitive erosion chain σ_t and $X_t = h(\sigma_t)$ is the height function defined in (13). We make

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the following choice of parameters:

$$B = \Omega_\epsilon$$
$$A_1 = n^2$$
$$A_2 = 1$$
$$a_1 = a(\epsilon) \text{ appearing in Theorem } 5$$
$$a_2 = n^2$$
$$T = \frac{2n^2}{a_1}.$$

Clearly (19) and (20) are satisfied by (14) and (15). Thus by Lemma 10 i.

$$\mathbb{P}_\sigma\left(\tau(\Omega_\epsilon) \geq \frac{2n^2}{a_1}\right) \leq e^{-\frac{a_1 n^2}{8}}.$$

$\square$

5.1 Proof of Lemma 2

The proof follows from Lemma 11 and the containment (Remark 1) $\Omega_\epsilon \subset \Gamma_\epsilon$. $\square$

5.2 Proof of Lemma 3

The proof follows from Lemma 10 ii. Let

$$X(t) = h(\sigma_t)$$

where $h(\cdot)$ is the height function defined in (13). We make the following choice of parameters:

$$B = \Gamma_\epsilon$$
$$B' = \Gamma_{2\epsilon}$$
$$A_1 = \alpha(1 - \alpha/2)n^2$$
$$a_4 = \epsilon n^2$$
$$A_2 = 1$$
$$a_1 = a(\epsilon) \text{ appearing in Theorem } 5$$
$$T = n^2.$$

The second containment in Remark 1 along with Theorem 5 satisfy the drift condition (69) with $a_1 = a(\epsilon)$. Now by the above choice of parameters $T' = \exp\left(\frac{\min(\epsilon^2, a_1^2)n^2}{32}\right)$. Thus by Lemma 10 ii. for all $\sigma \in \Gamma_\epsilon$,

$$\mathbb{P}_\sigma(\tau(\Gamma_{2\epsilon}) \geq T') \geq 1 - [\exp\left(-\frac{\epsilon^2 n^2}{32}\right) + \exp\left(-\frac{a_1^2 n^2}{32}\right)].$$

Thus the proof is complete. $\square$

6   Proof of the main result

The proof of Theorem 1 has the same structure as the proof of Theorem 4. We first state
and prove the following two results analogous to Lemmas 2 and 3.

Lemma 12. *Given any $\epsilon > 0$ there exist positive constants $c = c(\epsilon), d = d(\epsilon)$, such that for
all large enough n and $\sigma \in \Omega_\epsilon$*

$$\mathbb{P}_\sigma(\tau(\mathcal{A}_{\sqrt{\epsilon}}) > dn^2) \quad \leq \quad e^{-cn}. \tag{71}$$

Lemma 13. *Given $\epsilon > 0$ there exist positive constants $\delta, b, d > 0$ such that for all large
enough n and $\sigma \in \mathcal{A}_\epsilon \cap \Gamma_{\delta/2}$*

$$\mathbb{P}_\sigma(\tau(\mathcal{A}^c_{\sqrt{\epsilon}}) > e^{dn}) > 1 - e^{-bn}.$$

For the definition of the sets appearing in the above statements see Table 1. The proofs
of the above lemmas are significantly more delicate than the proofs of Lemmas 2 and 3. We
now proceed to the proofs. However for that we need some estimates about internal DLA
on cylinder graphs.

6.1   IDLA on the cylinder

Recall the definition of internal DLA from Subsection 1.1. In this section we discuss the
case when the underlying graph is the infinite cylinder $C_n \times \mathbb{Z}_{\geq 0}$ and the starting locations
of the particles are uniformly distributed on $C_n \times \{0\}$. By way of comparison with the
results stated in [8], we will show a cruder bound on the fluctuation but we will show that
the failure probability is exponentially small. We will also need to consider slightly more
general starting configurations, as described below.

We now formally define a generalized IDLA process. Consider the graph

$$\mathcal{C}_n = C_n \times \mathbb{Z}_{\geq 0}. \tag{72}$$

For any integer $j \geq 0$ we also define the set

$$Y_{j,n} = C_n \times \{j\}. \tag{73}$$

Definition 10. For integers $t \geq 0$ the sequence of random subsets (cluster) $I(t)$ is defined
inductively. Let $I(0)$ be any arbitrary subset of $\mathcal{C}_n$. Now given $I(t-1)$ start a random walk
uniformly on the set $Y_{0,n}$.
$$I(t) \setminus I(t-1)$$
consists of the site at which the random walk exits $I(t-1)$ for the first time.

Remark 4. For the purposes of this article we will consider the initial cluster to be a union
of rows, i.e.
$$I(0) = C_n \times \mathcal{I}_n,$$

with $\mathcal{I}_n = \{i_1, i_2, \ldots i_m\}$ for non-negative integers $\{i_1, i_2, \ldots i_m\}$ with m allowed to depend
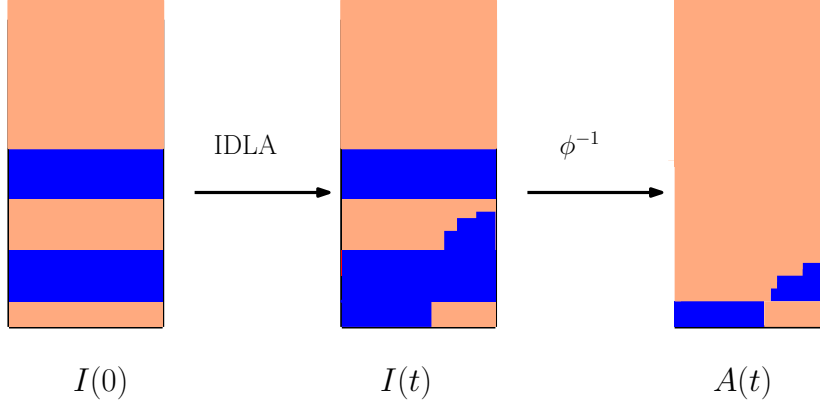on n.

Figure 10: The IDLA cluster $I(t)$, and the cluster $A(t) = \phi^{-1}(I(t) \setminus I(0))$.

For convenience we introduce the following notation. Let

$$\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \setminus \mathcal{I}_n$$

be the bijection that preserves order. Abusing notation slightly we write

$$\phi : \mathcal{C}_n \to \mathcal{C}_n \setminus I(0)$$

to denote the map that is identity in the first co-ordinate and ϕ in the second coordinate. Also we define

$$A(t) := \phi^{-1}(I(t) \setminus I(0)).$$

with $A(0) = \emptyset$ i.e. $A(t)$ is the set of new sites in the growth cluster under the map ϕ^{-1}. See Figure 10. We now state the result about the growth cluster in the generalized version of IDLA on the cylinder to be proved in Appendix A.

Theorem 6. *Given positive number $k < 1$ for any small enough ϵ there exists a positive number $c = c(k, \epsilon)$ such that for all large enough n and all $\mathcal{I}_n$ with $m \leq n$*

$$\mathbb{P}\Big(C_n \times [0, (1 - \epsilon)kn] \subset A(kn^2) \subset C_n \times [0, (1 + \epsilon)kn]\Big) \geq 1 - e^{-cn}.$$

6.2   Proof of Lemma 12

We start by defining some notations and making a few remarks. Recall from (9) the definition of $B_i(\sigma)$ for $i = 1, 2$. Given $\sigma \in \Omega \cup \Omega'$ for $i = 1, 2$, let $I_i(t)$ be the blue and red IDLA clusters respectively with initial clusters $I_i(0) = B_i(\sigma)$. Note that for the process $I_2(t)$ the particles start from $C_n \times \{1\}$.

Also for $i = 1, 2$ let $\tilde{I}_i(t)$ be variants of the processes $I_i(t)$ where the blue and red random walks are killed on hitting the lines $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ and $y = \alpha + \frac{\sqrt{\epsilon}}{2}$ respectively. We will refer to these as the *killed IDLA processes*. One immediately notices that starting from the same initial condition the trivial coupling (using the same random walks for both the processes) gives the inclusions

$$\tilde{I}_i(t) \quad \subseteq \quad I_i(t), \tag{74}$$
$$B_i(\sigma_t) \quad \subseteq \quad I_i(t) \tag{75}$$

for $i = 1, 2$ and all times t. Now suppose that $\sigma^{(1)}$ and $\sigma^{(2)}$ are two initial conditions with $B_1(\sigma^{(1)}) \subseteq B_1(\sigma^{(2)})$. Then again under the trivial coupling

$$
\begin{align}
I_1^{(1)}(t) &\subseteq I_1^{(2)}(t), \tag{76}\\
\tilde{I}_1^{(1)}(t) &\subseteq \tilde{I}_1^{(2)}(t) \tag{77}
\end{align}
$$

where for $j = 1, 2$, $I_1^{(j)}(\cdot)$ and $\tilde{I}_1^{(j)}(\cdot)$ are the blue IDLA and killed IDLA clusters with initial cluster $B_1(\sigma^{(j)})$. Clearly similar statements hold for the red cluster. We omit them for brevity.

Throughout the rest of the proof we will think of the competitive erosion processes $B_1(\sigma_t), B_2(\sigma_t)$ and the IDLA processes $I_1(\cdot), I_2(\cdot)$ starting from $B_1(\sigma_0), B_2(\sigma_0)$ respectively to be all coupled under the trivial coupling. For neatness we will say that an event holds *with overwhelming probability* if the probability of occurrence of the event is at least $1 - e^{-cn}$ for some $c > 0$ not depending on n. Also let

$$
T = \frac{\sqrt{\epsilon}}{10} n^2
$$

where ϵ appears in the statement of Lemma 12.

Proof of Lemma 12. By symmetry it suffices to show that at time T with overwhelming probability all vertices below the line $y < \alpha - \sqrt{\epsilon}$ are blue.

Define the event

$$
\mathcal{B}_\epsilon := \left\{ B_2(\sigma_t) \text{ does not intersect the line } y = \alpha - \frac{\sqrt{\epsilon}}{2}, \text{ for } t \leq T \right\}.
$$

Now given $\sigma_0 = \sigma \in \Omega_\epsilon$,

$$
\mathcal{B}_\epsilon \text{ occurs with overwhelming probability.} \tag{78}
$$

To see this notice that there are at least $(\frac{\sqrt{\epsilon}}{2} - \epsilon)n$ lines between the levels $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ and $y = \alpha$ that have no red vertices. Thus by the upper bound in Theorem 6 with overwhelming probability $I_2(t)$ will not intersect $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ for $t \leq T$. (78) now follows from (75).

We now show that on the event $\mathcal{B}_\epsilon$ with overwhelming probability at time T all the initial red vertices below the line $y = \alpha - \sqrt{\epsilon}$ in the competitive erosion process become blue, i.e.

$$
C_n \times [0, \alpha - \sqrt{\epsilon}] \subset B_1(\sigma_T).
$$

The strategy to show this is to compare the processes $B_1(\sigma_t)$ and $\tilde{I}_1(t)$. Since on the event $\mathcal{B}_\epsilon$ no new red particles hit the line $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ up to time T the blue competitive erosion cluster dominates the blue killed IDLA cluster. That is,

$$
\mathcal{B}_1 := \left\{ \tilde{I}_1(t) \subset B_1(\sigma_t) \text{ for } t \leq T \right\} \text{ occurs with overwhelming probability.} \tag{79}
$$

Note however that the domination works in the reverse direction for the actual blue IDLA cluster by (75) and hence the need for the killed process. We would be done at this point if the lower bound in Theorem 6 was stated for $\tilde{I}_1(t)$. However since Theorem 6 is for $I_1(t)$ we need to argue a little more.

Now given the initial condition σ consider a modification σ' obtained by making all the rows between the lines $y = \alpha - \sqrt{\epsilon}$ and line $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ completely red. Then by (76) for all t

$$\tilde{I}'_1(t) \subset \tilde{I}_1(t) \tag{80}$$

where the LHS and the RHS are the killed blue IDLA clusters with initial conditions σ' and σ respectively.

Also define $I'_1(t)$ to be the blue IDLA cluster with starting condition σ'. By the upper bound in Theorem 6 with overwhelming probability $I'_1(t)$ does not intersect the line $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ for $t \leq T$. This is because in σ' all the $\frac{\sqrt{\epsilon}}{2} n^2$ vertices between $y = \alpha - \sqrt{\epsilon}$ and line $y = \alpha - \frac{\sqrt{\epsilon}}{2}$ are colored red. Note that this implies that both the blue IDLA and the killed IDLA clusters starting from σ' are exactly the same for the first T steps. That is,

$$\mathcal{B}_2 := \left\{ \tilde{I}'_1(t) = I'_1(t) \text{ for } t \leq T \right\} \text{ occurs with overwhelming probability.} \tag{81}$$

Now by the lower bound in Theorem 6

$$\mathcal{B}_3 := \left\{ C_n \times [0, \alpha - \sqrt{\epsilon}] \subset I'_1(T) \right\} \text{ occurs with overwhelming probability} \tag{82}$$

since σ' has at most ϵn lines containing red vertices below the line $\alpha - \sqrt{\epsilon}$. Lastly notice that (80) implies the following containment of events:

$$\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \subseteq \left\{ C_n \times [0, \alpha - \sqrt{\epsilon}] \subset B_1(\sigma_T) \right\}.$$

Thus we are done since by (79),(81) and (82) all the event on the LHS occur with overwhelming probability.

$\square$

Having proved Lemma 12 we state and prove a few preliminary lemmas required for the proof of Lemma 13.

Lemma 14. *For small enough $\epsilon > 0$ for any $\sigma \in \Omega_\epsilon \cap \mathcal{A}_{\sqrt{\epsilon}}$*

$$\mathbb{P}_\sigma(\tau(\Omega \setminus \mathcal{A}_{\sqrt{\epsilon}}) \geq \epsilon n^2) \geq 1 - e^{-cn} \tag{83}$$

for some $c = c(\epsilon) > 0$. For the definitions of $\mathcal{A}_\epsilon$ and Ω_ϵ refer to Table 1.

Proof. As $\sigma \in \Omega_\epsilon$ there are at least $(\sqrt{\epsilon} - \epsilon)n$ rows with no red vertex between the lines $y = \alpha - \sqrt{\epsilon}$ and α and similarly at least $(\sqrt{\epsilon} - \epsilon)n$ rows with no blue vertex between the lines $y = \alpha$ and $\alpha + \sqrt{\epsilon}$. The proof now follows from the upper bound in Theorem 6 which implies that the IDLA processes $I_1(t), I_2(t)$ require at least ϵn^2 rounds with probability at least $1 - e^{-cn}$ for some positive $c = c(\epsilon)$ before a blue vertex is seen above the line $y = \alpha + \sqrt{\epsilon}$ and a red vertex is seen below the line $y = \alpha - \sqrt{\epsilon}$ or a . Thus we are done by (75) which says that the same holds for the competitive erosion clusters $B_i(t)$ for $i = 1, 2$. $\square$

For any set $A \subset \Omega$ define the positive return time $\tau^+(A)$ to be

$$\inf\{t \geq 1 : \sigma_t \in A\}.$$

Lemma 15. *Given positive numbers ϵ, c there exist positive constants δ, D such that for any $\sigma \in \Omega_\epsilon \cap \Gamma_\delta$*

$$\mathbb{P}_\sigma(\tau^+(\Omega_\epsilon) \geq cn^2) < e^{-Dn^2}. \tag{84}$$

For the definition of the sets see Table 1.

Proof. The proof follows from Lemma 10 *i*. First let $\sigma' = \sigma_1$ for the competitive erosion chain starting from $\sigma_0 = \sigma$. If $\sigma' \in \Omega_\epsilon$ we are done. Otherwise by (15) and the fact that by hypothesis $\sigma \in \Gamma_\delta$,

$$x := h(\sigma') \geq \alpha(1 - \alpha/2)n^2 - \delta n^2.$$

Now let

$$X(t) = h(\sigma_{t+1})$$

where $h(\cdot)$ is the height function defined in (13). Thus $X(0) = x$. We make the following choice of parameters:

$$
\begin{aligned}
B &= \Omega_\epsilon \\
A_1 &= \alpha(1 - \alpha/2)n^2 \\
A_2 &= 1 \\
a_1 &= a(\epsilon) \text{ appearing in Theorem 5} \\
a_2 &= \delta n^2 \text{ with } \delta < ac/2 \\
T &= cn^2.
\end{aligned}
$$

The choice of a_2 works since $\sigma \in \Gamma_\delta$ by hypothesis. Also the choice of δ ensures that $a_2 - a_1 T < 0$ and hence the hypothesis of Lemma 10 *i*. is satisfied. Thus for $\delta = \frac{ac}{2}$ by Lemma 10 *i*. for all $\sigma \in \Omega_\epsilon \cap \Gamma_\delta$,

$$\mathbb{P}_{\sigma'}(\tau(\Omega_\epsilon) \geq T) \leq \exp\left(-\frac{a_1^2 cn^2}{16}\right).$$

Since $\tau^+(\Omega_\epsilon)$ starting from σ is one more than $\tau(\Omega_\epsilon)$ starting from σ' we are done. $\qquad \square$

Lemma 16. *Given $\epsilon > 0$ there exist positive constants δ, b such that for large enough n and any $\sigma \in \Omega_\epsilon \cap \Gamma_\delta \cap \mathcal{A}_{\sqrt{\epsilon}}$,*

$$\mathbb{P}_\sigma(\tau^+(\Omega_\epsilon) \leq \tau(\Omega \setminus \mathcal{A}_{\sqrt{\epsilon}})) > 1 - e^{-bn}.$$

Proof. Since $\sigma \in \Omega_\epsilon \cap \mathcal{A}_{\sqrt{\epsilon}}$ by Lemma 14

$$\mathbb{P}_\sigma(\tau(\Omega \setminus \mathcal{A}_{\sqrt{\epsilon}}) \geq \epsilon n^2) \geq 1 - e^{-hn}$$

for some $h = h(\epsilon) > 0$. Now using $c = \epsilon/2$ in Lemma 15 we get that we can choose δ such that for any $\sigma \in \Omega_\epsilon \cap \Gamma_\delta \cap \mathcal{A}_{\sqrt{\epsilon}}$

$$\mathbb{P}_\sigma(\tau^+(\Omega_\epsilon) \geq \epsilon n^2/2) < e^{-Dn^2}. \tag{85}$$

for some positive constant D. Thus for such a δ

$$
\begin{aligned}
\mathbb{P}_\sigma(\tau^+(\Omega_\epsilon) \leq \tau(\Omega \setminus \mathcal{A}_{\sqrt{\epsilon}})) &\geq \mathbb{P}_\sigma(\tau^+(\Omega_\epsilon) \leq \epsilon/2n^2) - \mathbb{P}_\sigma(\tau(\Omega \setminus \mathcal{A}_{\sqrt{\epsilon}}) \leq \epsilon n^2) \\
&\geq 1 - e^{-hn} - e^{-Dn^2}.
\end{aligned}
$$

Hence we are done by choosing $b = \frac{h}{2}$. $\qquad \square$

We are now ready to prove Lemma 13.

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6.3 Proof of Lemma 13

We will specify δ later. However notice that for any small enough δ if $\sigma \in \Gamma_{\delta/2}$ then by Lemma 3 there exist positive constants c, d depending on δ such that for large enough n

$$\mathbb{P}_\sigma(\tau' > e^{cn^2}) \geq 1 - e^{-dn^2} \tag{86}$$

where $\tau' = \tau(\Omega \setminus \Gamma_\delta)$. Now notice that by hypothesis $\sigma \in \mathcal{A}_\epsilon$ and hence $\sigma \in \Omega_\epsilon$. Let $\tau_{(1)}, \tau_{(2)} \ldots$ be successive return times to Ω_ϵ, i.e. $\tau_{(1)} = 0$ and for all $i \geq 0$,

$$\tau_{(i+1)} = \inf\{t : t > \tau_{(i)}, \sigma_t \in \Omega_\epsilon\}.$$

The following containment is true for any positive b': Let

$$s := e^{b'n}.$$

Then

$$\{\tau(A^c_{\sqrt{\epsilon}}) \leq s\} \subset \{\exists i \leq s \text{ such that } \sigma_i \notin \Gamma_\delta\} \cup \left(\bigcup_{j=0}^{s} \left\{ \{\tau_{(j)} < \tau(A^c_{\sqrt{\epsilon}}) \leq \tau_{(j+1)}\} \cap \{\sigma_{\tau_{(j)}} \in \Gamma_\delta\} \right\} \right).$$

The above follows by first observing whether there exists any $i < e^{b'n}$ such that $\sigma_i \notin \Gamma_\delta$. Also let j be the first index such that

$$\tau_{(j)} < \tau(A^c_{\sqrt{\epsilon}}) \leq \tau_{(j+1)}.$$

Notice that on the event $\{\tau(A^c_{\sqrt{\epsilon}}) \leq s\}$,

$$j \leq s.$$

Also by definition on the event $\left\{ \sigma_i \in \Gamma_\delta \text{ for all } i \leq s \right\}$,

$$\sigma_{\tau_{(j)}} \in \Gamma_\delta \cap \Omega_\epsilon \cap \mathcal{A}_{\sqrt{\epsilon}}.$$

Thus by the union bound,

$$\begin{aligned}
\mathbb{P}\big(\tau(A^c_{\sqrt{\epsilon}}) \leq e^{b'n}\big) \quad &\leq \quad \mathbb{P}(\exists i \leq s \text{ such that } \sigma_i \notin \Gamma_\delta) \\
&+ \sum_{j=1}^{e^{b'n}} \mathbb{P}\left(\{\sigma_{\tau_{(j)}} \in \Gamma_\delta \cap \Omega_\epsilon \cap \mathcal{A}_{\sqrt{\epsilon}}\} \cap \{\tau(A^c_{\sqrt{\epsilon}}) \leq \tau_{(j+1)}\} \right).
\end{aligned}$$

Recall that by hypothesis $\sigma \in \Gamma_{\delta/2}$. Thus by (86) for any small enough δ there exists $d = d(\delta) > 0$ such that for any $b' > 0$ the first term is less than e^{-dn^2} for large enough n. Also notice that by Lemma 16 we can choose δ such that every term in the sum is at most e^{-bn} for some constant $b > 0$. Thus for such a δ, putting everything together we get that for any $b' > 0$ and large enough n,

$$\mathbb{P}(\tau(A^c_{\sqrt{\epsilon}}) \leq e^{b'n}) \leq e^{-dn^2} + \sum_{i=1}^{e^{b'n}} e^{-bn}.$$

Hence by choosing $b' < b$ we get that for large enough n,

$$\mathbb{P}(\tau(A^c_{\sqrt{\epsilon}}) \leq e^{b'n}) \leq 2e^{(b'-b)n}.$$

The proof is thus complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.4 Proof of Theorem 2

We will use Lemma 4 and prove something stronger which will be used in the proof of Theorem 1. However for brevity we first need some notation. We start by recalling the sets in Table 1. Now given any positive ϵ_1 and ϵ_2 define the set

$$\mathcal{C}_{\epsilon_1,\epsilon_2} = \mathcal{A}_{\epsilon_1} \cap \Gamma_{\epsilon_2}.$$

We claim that for small enough $\epsilon, \delta > 0$ there exists constants $D, D' > 0$ such that for any $\sigma \in \Omega$,

$$\mathbb{P}_\sigma\big(\tau(\mathcal{C}_{\sqrt{\epsilon},\delta/2}) \geq Dn^2\big) \leq e^{-D'n}. \tag{87}$$

Clearly this proves Theorem 2 since $\mathcal{C}_{\sqrt{\epsilon},\delta/2} \subset \mathcal{A}_{\sqrt{\epsilon}}$. Let

$$\begin{aligned}
\tau' &= \tau(\Gamma_{\delta/4}) \\
\tau'' &= \tau(\Gamma_{\delta/2}^{\mathsf{c}}) \\
\tau''' &= \tau(\Omega_\epsilon) \\
\tau'''' &= \tau(\mathcal{A}_{\sqrt{\epsilon}}).
\end{aligned}$$

Recall the notational convention made in Subsection 2.2. Now to show (87) we notice the following containment of events:

$$\big[\{\tau' \leq An^2\} \cap \{\tau'' \geq e^{cn} \mid \sigma_{\tau'}\} \cap \{\tau''' \leq An^2 \mid \sigma_{\tau'}\} \cap \{\tau'''' \leq An^2 \mid \sigma_{\tau'''}\}\big] \subset \{\tau(\mathcal{C}_{\sqrt{\epsilon},\delta/2}) \leq 3An^2\}.$$

To see why this containment holds we first notice that

$$\Gamma_{\delta/4} \subset \Gamma_{\delta/2}.$$

Thus the first two events imply that the process hits the set $\Gamma_{\delta/2}$ in An^2 steps and stays inside for an exponential (in n) amount of time, and in particular stays in $\Gamma_{\delta/2}$ from time An^2 through time $3An^2$. In addition, the third and fourth events together imply that regardless of where the chain is at time An^2, the chain enters Ω_ϵ by time $2An^2$ and then enters $\mathcal{A}_{\sqrt{\epsilon}}$ by time $3An^2$. Hence in the intersection of the four events, the hitting time of $\mathcal{C}_{\sqrt{\epsilon},\delta/2} = \mathcal{A}_{\sqrt{\epsilon}} \cap \Gamma_{\delta/2}$ is at most $3An^2$. Let $D = 3A$. Now for a large enough constant A, there exists $D' > 0$ such that the probabilities of of all the four events on the left hand side are at least $1 - e^{-D'n}$. This follows by Lemmas 2, 3, 11 and 12 respectively. Hence by the union bound (87) follows for a slightly smaller value of D'. $\square$

6.5 Proof of Theorem 1

Note that (87) is true for all small enough choices of ϵ and δ. However given ϵ by Lemma 13 there exist $\delta, b, d > 0$ such that for $\sigma \in \mathcal{C}_{\sqrt{\epsilon},\delta/2}$

$$\mathbb{P}_\sigma(\tau(\mathcal{A}_{\epsilon^{1/4}}^c) > e^{dn}) > 1 - e^{-bn}. \tag{88}$$

For such a δ the proof of Theorem 1 follows by using Lemma 4 with the following choices

of parameters:

$$
\begin{aligned}
A &= \mathcal{C}_{\sqrt{\epsilon}, \delta/2} \\
B &= \mathcal{A}_{\epsilon^{1/4}} \\
t_1 &= Dn^2 \\
t_2 &= e^{dn} \\
p_1 &= e^{-D'n} \\
p_2 &= e^{-bn}.
\end{aligned}
$$

The above choice of parameters satisfy the hypotheses of Lemma 4 by (87) and (88). $\qquad\square$

6.6 Proof of Theorem 3

The proof is a simple corollary of Theorem 1. Let $k = \frac{2}{\epsilon}$. For $i = 1 \dots k$, let

$$
\alpha_i = \frac{\epsilon}{2} i.
$$

Now define

$$
\Lambda_i = \{(x,y) \in \mathcal{C}_n : \lambda(x,y) \le \alpha_i n^2\}.
$$

Choose $\delta = \epsilon^{100}$. By Theorem 1 for all $i = 1, 2, \dots k$ there exists β_i such that

$$
\begin{aligned}
\hat{\pi}_n \left(\forall (x,y) \in \Lambda_i : y \le \alpha_i + \delta \right) &\ge 1 - e^{-\beta_i n} \qquad (89) \\
\hat{\pi}_n \left(\forall (x,y) \notin \Lambda_i : y \ge \alpha_i - \delta \right) &\ge 1 - e^{-\beta_i n}.
\end{aligned}
$$

By union bound there exists β such that the events on the LHS of (89) simultaneously hold for all i with failure probability at most $e^{-\beta n}$. Clearly this completes the proof. To see this suppose that all the events hold and there exists $(x,y) \in \mathcal{C}_n$ such that $\frac{\lambda(x,y)}{n^2} - y \ge \epsilon$. Let $1 \le j \le k$ be such that

$$
\alpha_{j-1} n^2 < \lambda(x,y) \le \alpha_j n^2
$$

(take $\alpha_0 = 0$). Now by the second event in (89) $y \ge \alpha_{j-1} - \delta$ which implies that

$$
\frac{\lambda(x,y)}{n^2} - y \le \alpha_j - \alpha_{j-1} + \delta = \frac{\epsilon}{2} + \epsilon^{100} < \epsilon
$$

which is a contradiction. Similarly one can show $\frac{\lambda(x,y)}{n^2} - y > -\epsilon$. Details are omitted. $\quad\square$.

Appendix

A   IDLA on the cylinder

The proof of Theorem 6 follows by adapting the ideas of the proof appearing in [11]. The proof in [11] follows from a series of lemmas which we now state in our setting. Recall (73). Let $\tau_z, \tilde{\tau}_{kn}$ be the hitting times of $\phi(z)$ and $\phi(Y_{kn,n})$ respectively.

Lemma 17. *For any $z = (x, y) \in \mathcal{C}_n$ with $y \leq kn$*

$$kn^2 \mathbb{P}_{Y_{0,n}}(\tau_z < \tilde{\tau}_{kn}) \geq \sum_{w \in \mathcal{C}_n \times [0, kn)} \mathbb{P}_{\phi(w)}(\tau_z < \tilde{\tau}_{kn})$$

where $\mathbb{P}_{\phi(w)}$ and $\mathbb{P}_{Y_{0,n}}$ are the random walk measures on $\mathcal{C}_n$ with starting point $\phi(w)$ and uniform over $Y_{0,n}$ respectively.

Proof. By symmetry in the first coordinate under $\mathbb{P}_{Y_{0,n}}$ for any j the distribution of the random walk when it hits the set $Y_{j,n}$ is uniform over the set $Y_{j,n}$. Hence by the Markov property the chance that random walk hits $\phi(z)$ before $Y_{\phi(kn),n}$ after reaching the line $Y_{\phi(j),n}$ is

$$\frac{1}{n} \sum_{w \in Y_{\phi(j),n}} \mathbb{P}_w(\tau_z < \tilde{\tau}_{kn}).$$

Thus clearly for any $j < kn$

$$\mathbb{P}_{Y_{0,n}}(\tau_z < \tilde{\tau}_{kn}) \geq \frac{1}{n} \sum_{w \in Y_{\phi(j),n}} \mathbb{P}_w(\tau_z < \tilde{\tau}_{kn}). \tag{90}$$

The lemma follows by summing over j from 0 through $kn - 1$. $\qquad\square$

Lemma 18. *Given positive numbers k and ϵ with $\epsilon < 1$, then there exists $\beta = \beta(k, \epsilon)$ such that for all $z = (x, y)$ with $y \leq (1 - \epsilon)kn$*

$$\mathbb{P}_{Y_{0,n}}(\tau_z < \tilde{\tau}_{kn,n}) \geq \frac{\beta}{\ln n}.$$

Proof. Since $\phi(kn) - \phi(y) \geq \epsilon kn$ the semi-disc of radius $\min(\epsilon k, 1/2)n$ around z lies below the line $Y_{\phi(kn),n}$. The random walk starting uniformly on $Y_{o,n}$ hits the interval $(z - \min(\epsilon k/2, 1/4)n, z + \min(\epsilon k/2, 1/4)n)$ with probability at least $\min(\epsilon k, 1/2)$. Now the lemma follows by the standard result that the random walk starting within radius $n/2$ has $\Omega(\frac{1}{\log n})$ chance of returning to the origin before exiting the ball of radius n in $\mathbb{Z}^2$. This fact can be found in [10, Prop 1.6.7]. $\qquad\square$

The next result is the standard Azuma-Hoeffding inequality stated for sums of indicator variables.

Lemma 19. *For any positive integer n if X_i $i = 1, 2 \ldots n$ are independent indicator variables then*

$$\mathbb{P}(|\sum_{i=1}^{n} X_i - \mu| \geq t) \leq 2e^{-\frac{t^2}{4n}}$$

where $\mu = \mathbb{E} \sum_{i=1}^{n} X_i$.

A.1 Hitting estimates

Consider the simple random walk $(Y(t))_{t \geq 0}$ on $\mathbb{Z}^2$.

Lemma 20. *For $(x, y) \in \mathbb{Z}^2$, let $h(x, y) = \mathbb{P}_{(x,y)}\{Y(\tau(\mathbb{Z} \times \{0\})) = (0, 0)\}$ be the probability of first hitting the x-axis at the origin. Then*

$$h(x, y) = \frac{y}{\pi(x^2 + y^2)} + O(\frac{1}{x^2 + y^2}).$$

Proof. Let

$$\widetilde{h}(x, y) = \begin{cases} h(x, y), & y > 0 \\ 0, & y = 0 \\ -h(x, y), & y < 0. \end{cases}$$

The discrete Laplacian

$$\Delta \widetilde{h}(x, y) = \frac{\widetilde{h}(x + 1, y) + \widetilde{h}(x - 1, y) + \widetilde{h}(x, y + 1) + \widetilde{h}(x, y - 1)}{4} - \widetilde{h}(x, y)$$

vanishes except when $(x, y) = (0, \pm 1)$, and $\Delta \tilde{h}(0, \pm 1) = \mp \frac{1}{4}$. Since $\tilde{h}$ vanishes at ∞ it follows that

$$\tilde{h}(x, y) = \frac{a(x, y + 1) - a(x, y - 1)}{4} \tag{91}$$

where

$$a(x, y) = \frac{1}{\pi} \log(x^2 + y^2) + \kappa + O(\frac{1}{x^2 + y^2})$$

is the recurrent potential kernel for $\mathbb{Z}^2$ (see [9]). Here κ is a constant whose value is irrelevant because it cancels in the difference (91). $\qquad \square$

Let $X(\cdot)$ be the simple symmetric random walk on the half-infinite cylinder $\mathcal{C}_n = C_n \times \mathbb{Z}_{\geq 0}$.

Lemma 21. *For any positive integers $j < k$, with $\Delta = k - j < n$:*

i. for any $w \in Y_{k,n}$,

$$\mathbb{P}_w(\tau(j) < \tau^+(k)) < \frac{1}{\Delta}$$

where $\tau(j)$ and $\tau^+(k)$ are the hitting and positive hitting times of $Y_{j,n}$ and $Y_{k,n}$ respectively for $X(\cdot)$.

ii. there exists a constant J such that for $w \in Y_{j,n}$ and any subset $B \subset Y_{k,n}$,

$$\mathbb{P}_w(X(\tau(k)) \in B) < J|B|/\Delta.$$

Proof. i. is the following standard result about one-dimensional random walk: starting from 1 the probability of hitting Δ before 0 is $\frac{1}{\Delta}$.
Now we prove *ii.* Clearly it suffices to prove it in the case when B consists of a single element. It is easy to check that if $Y(t) = (Y_1(t), Y_2(t))$ is the simple random walk on $\mathbb{Z}^2$, then

$$X(t) = (Y_1(t) \bmod n, |Y_2(t)|) \tag{92}$$

is distributed as the simple random walk on $\mathcal{C}_n$. For any $\ell \in \mathbb{Z}$ let $\tau_1(\ell)$ be the hitting time of the line $y = \ell$, for $Y(t)$. Clearly by (92) for $w = (0, j), z = (z_1, k) \in \mathcal{C}_n$,

$$\mathbb{P}_w(X(\tau(k)) = z) = \sum_{i=-\infty}^{\infty} P_{(0,j)} \left\{ Y(\tau_1(k) \wedge \tau_1(-k)) \in \{(z_1 + in, k), (z_1 + in, -k)\} \right\}.$$

By union bound the RHS is at most

$$\sum_{i=-\infty}^{\infty} P_{(0,j)} \left\{ Y(\tau_1(k)) = (z_1 + in, k) \right\} + \sum_{i=-\infty}^{\infty} P_{(0,j)} \left\{ Y(\tau_1(-k)) = (z_1 + in, -k) \right\}.$$

Using the notation in Lemma 20 we can write the above sum as

$$\sum_{i=-\infty}^{\infty} [h(-z_1 + in, \Delta) + h(-z_1 + in, k + j)]$$

By Lemma 20 the above sum is

$$O(\frac{1}{\Delta}) + O(\frac{1}{k+j}) = O(\frac{1}{\Delta}).$$

Hence we are done. □

A.2   Proof of Theorem 6

Equipped with the results in the previous subsection the proof of Theorem 6 will now be completed by following the steps in [11] .

Lower bound: It suffices to show $C_n \times [0, (1 - \epsilon)kn] \subset A((1 + \epsilon)kn^2)$. Fix $z \in C_n \times (1 - \epsilon)kn$ For any positive integer i we associate the following stopping times to the ith walker:

- σ^i: the stopping time in the IDLA process

- τ_z^i: the hitting time of $\phi(z)$

- $\tau_{kn,n}^i$: the hitting time of the set $\phi(Y_{kn,n})$.

Now we define the random variables

$$N = \sum_{i=1}^{(1+\epsilon)kn^2} \mathbf{1}_{(\tau_z^i < \sigma^i)}, \text{ the number of particles that visit } \phi(z) \text{ before stopping}$$

$$M = \sum_{i=1}^{(1+\epsilon)kn^2} \mathbf{1}_{(\tau_z^i < \tau_{kn,n}^i)}, \text{ the number of particles that visit } \phi(z) \text{ before reaching } \phi(Y_{kn,n})$$

$$L = \sum_{i=1}^{(1+\epsilon)kn^2} \mathbf{1}_{(\sigma^i < \tau_z^i < \tau_{kn,n}^i)}, \text{ the number of particles that visit } \phi(z) \text{ before reaching } \phi(Y_{kn,n})$$
but after stopping.

Thus

$$N \geq M - L.$$

Hence

$$\mathbb{P}\Big(z \notin A((1+\epsilon)kn^2)\Big) = \mathbb{P}(N=0) \le \mathbb{P}(M < a) + \mathbb{P}(L > a). \tag{93}$$

where the last inequality holds for any a. Now by definition

$$\mathbb{E}(M) = (1+\epsilon)kn^2 \, \mathbb{P}_{Y_{0,n}}(\tau_z < \tau_{kn,n}). \tag{94}$$

We now bound the expectation of L. Define the following quantity: let independent random walks start from each $w \in \phi(C_n \times [0, kn])$ and let

$$\tilde{L} = \sum_{w \in \phi(C_n \times [0,kn])} \mathbf{1}(\tau_z < \tau_{kn,n} \text{ for the walker starting at } w).$$

Clearly $L \le \tilde{L}$. Hence the RHS of (93) can be upper bounded by $\mathbb{P}(M < a) + \mathbb{P}(\tilde{L} > a)$. Now

$$\mathbb{E}(\tilde{L}) = \sum_{w \in \phi(C_n \times [0,kn])} \mathbb{P}_w(\tau_z < \tau_{kn,n}).$$

Also by Lemma 17 and (94)

$$\Big(1 + \frac{\epsilon}{2}\Big)\mathbb{E}(\tilde{L}) \le \mathbb{E}(M).$$

Choose $a = (1 + \epsilon/4)\max\big(\frac{\beta kn^2}{\ln n}, \mathbb{E}(\tilde{L})\big)$ where the β appears in Lemma 18. Now using Lemma 19 we get

$$
\begin{aligned}
\mathbb{P}(\tilde{L} > a) &\le \exp(-dn) \\
\mathbb{P}(M < a) &\le \exp(-dn)
\end{aligned}
$$

for some constant $d = d(\epsilon, k) > 0$. Thus in (93) we get

$$\mathbb{P}(M < a) + \mathbb{P}(L > a) \le 2\exp(-dn).$$

The proof of the lower bound now follows by taking the union bound:

$$
\begin{aligned}
\mathbb{P}(C_n \times [0, (1-\epsilon)kn] \subset A((1+\epsilon)kn^2) &\le \sum_{z \in C_n \times [0,(1-\epsilon)kn]} \mathbb{P}\Big(z \notin A((1+\epsilon)kn^2)\Big) \\
&\le \sum_{z \in C_n \times [0,(1-\epsilon)kn]} 2\exp(-dn) \\
&\le \exp(-cn).
\end{aligned}
$$

where the last inequality holds for large enough n when c is smaller than d.

Upper bound: In [11] the upper bound is proven by showing that the growth of the cluster above level $(1+\epsilon)kn$ is dominated by a multitype branching process. However here we slightly modify the proof to take into account that in our situation the initial cluster is not empty. We define some notation. Let us denote the particles making it out of level $\phi(Y_{kn,n})$ by $w_1, w_2, \ldots$ and define

$$\tilde{A}(j) = A(w_j).$$

Choose $k_0 = k(1 + \sqrt{\epsilon})n$. We define

$$\tilde{Y}_{\ell,n} := Y_{k_0+\ell,n}. \tag{95}$$

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Given the above notation let

$$Z_\ell(j) := \tilde{A}(j) \cap \tilde{Y}_{\ell,n}.$$

Define

$$\mu_\ell(j) := \mathbb{E}(Z_\ell(j)).$$

Lemma 22. [11, Lemma 7] *There exists a universal $J_1 > 0$ such that for all $k, \epsilon \in (0,1)$, $n \geq N(k,\epsilon)$ and all positive integers j, ℓ*

$$\mu_\ell(j) < kn\left(J_1 \frac{j}{\ell} \frac{1}{\sqrt{\epsilon}kn}\right)^\ell.$$

We include the proof of Lemma 22 for completeness. However first we show how it implies the upper bound in Theorem 6. Let us define the event

$$F := C_n \times (1-\epsilon)kn \subset A(kn^2).$$

Now let $B = B(k) > 0$ be a constant to be specified later. Then

$$\mathbb{P}\big(A(kn^2) \not\subset C_n \times [0, k(1+B\sqrt{\epsilon})n] \cap F\big) \;\leq\; P(Z_{n'}(2k\epsilon n^2) > 1)$$
$$\leq\; \mu_{n'}(2k\epsilon n^2)$$

where $n' = (kB)\sqrt{\epsilon}n - 1 - k\sqrt{\epsilon}n$. To see why these inequalities are true first note that the set $\tilde{Y}_{n',n}$ is at height less than $k(1+B\sqrt{\epsilon}n)$. Hence the cluster at time kn^2 should intersect $\tilde{Y}_{n',n}$ to grow beyond height $k(1+B\sqrt{\epsilon}n)n$. However on the event F at most $2\epsilon kn^2$ particles out of the first kn^2 move beyond height kn. Hence the size of the intersection of $\tilde{Y}_{n',n}$ and the cluster is at most $Z_{n'}(2k\epsilon n^2)$. Thus we get the first inequality. The second inequality follows trivially from the fact that for a non-negative integer-valued random variable the expectation is at least as big as the probability of the random variable being positive. Using Lemma 22 we get

$$\mu_{n'}(2k\epsilon n^2) \;\leq\; kn\left(J_1 \frac{2k\epsilon n^2}{n'} \frac{1}{\sqrt{\epsilon}kn}\right)^{n'}$$
$$=\; kn\left(J_1 \frac{4k\epsilon n^2}{k(B-1)\sqrt{\epsilon}n} \frac{1}{\sqrt{\epsilon}kn}\right)^{k(B-1)\sqrt{\epsilon}n}$$
$$=\; kn\left(\frac{4J_1}{(B-1)k}\right)^{k(B-1)\sqrt{\epsilon}n}.$$

Thus

$$\mathbb{P}\big(A_{kn^2} \notin [0,n] \times [0, k(1+B\sqrt{\epsilon})n] \cap F\big) \leq kn\left(\frac{4J_1}{(B-1)k}\right)^{(B-1)\sqrt{\epsilon}kn}.$$

Now by choosing B such that $4J_1 < (B-1)k$ we are done. $\qquad\square$

Proof of Lemma 22. The rate at which $\tilde{Y}_{\ell,n}$ grows is at most the rate at which a particle exiting height kn reaches the occupied sites in $\tilde{Y}_{\ell-1,n}$. Thus if $X(t)$ is the random walk on C_n defined in (72) then for any m

$$\mu_\ell(m+1) - \mu_\ell(m) \;\leq\; \sup_{y \in Y_{kn,n}} \mathbb{P}_y\left[X(\tau_{\tilde{Y}_{\ell-1,n}}) \in \tilde{A}(m)\right] \qquad (96)$$

$$\leq\; J\frac{\mu_{\ell-1}(m)}{kn\sqrt{\epsilon}} \qquad (97)$$

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where the second inequality follows by Lemma 21 *ii*. Summing over $m = 0, 1 \ldots j$ we get

$$\mu_\ell(j) \le \frac{J}{kn\sqrt{\epsilon}} \sum_{m=0}^{j-1} \mu_{\ell-1}(m).$$

Iterating the above relation in ℓ with fixed j gives us

$$\mu_\ell(j) \le \left(J \frac{1}{\sqrt{\epsilon}kn}\right)^{\ell-1} \frac{j^\ell}{\ell!}.$$

The lemma follows by using the inequality

$$\ell! \ge \ell^\ell e^{-\ell}.$$

$\qquad\square$

B   Green's function and flows

We prove Lemma 6. We start by discussing some properties of the ordinary random walk on Cyl_n (defined in (7)). For any $v \in \text{Cyl}_n$ define

$$G_n(v) = \frac{1}{4n} \mathbb{E}_w \big[\# \text{ visits to the line } y = 0 \text{ before } \tau(C_n \times \{1\}) \big]. \tag{98}$$

Lemma 23. *For any point $(x, y) \in \text{Cyl}_n$*

$$G_n(x, y) = 1 - y.$$

Proof. Consider the lazy symmetric random walk on the interval $[0, n]$ where at 0 the chance that it moves to 1 is $\frac{1}{4}$ and everywhere else the chance that it jumps is $\frac{1}{2}$. By symmetry of Cyl_n in the x-coordinate it is clear that for all $(x, y) \in \text{Cyl}_n$, $4nG_n(x, y)$ is the expected number of times that the above one-dimensional random walk starting from ny hits 0 before hitting n. The above quantity is easy to compute and is $4n(1 - y)$. $\qquad\square$

Remark 5. Thus for any $\sigma \in \Omega \cup \Omega_n'$

$$h(\sigma) = \sum_{(x,y) \in B_1} G_n(x, y) \tag{99}$$

where $h(\cdot)$ is defined in (13).

We now define the stopped Green's function. For any $A \subset \text{Cyl}_n$ and $v \in \text{Cyl}_n$ define

$$G_A(v) = \frac{1}{4n} \mathbb{E}_v \big[\# \text{ visits to the line } y = 0 \text{ before } \tau(A^c) \big]. \tag{100}$$

Lemma 24. *Given $A \subset \text{Cyl}_n$ such that $A \cap (C_n \times \{1\}) = \emptyset$, for all (x, y) in Cyl_n we have*

$$G_A(x, y) = H_A(x, y) - y$$

where $H_A(\cdot)$ was defined in (26).

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Proof. Let y_t be the height of the walk at time $t \leq \tau(A^c)$. Consider the following telescopic series:

$$y_{\tau(A^c)} - y_0 = \sum_{t=0}^{\infty} (y_{t+1} - y_t)\mathbf{1}(\tau(A^c) > t). \tag{101}$$

Notice that since $A \cap (C_n \times \{1\}) = \emptyset$, $t < \tau_{A^c}$ implies $y_t < 1$. We make the following simple observation:

$$\mathbb{E}\big[(y_{t+1} - y_t)\mathbf{1}(\tau(A^c) > t)|\mathcal{F}_t\big] = \begin{cases} \frac{1}{4n}\mathbf{1}(\tau(A^c) > t) & y_t = 0 \\ 0 & y_t > 0 \end{cases} \tag{102}$$

where $\mathcal{F}_t$ is the filtration generated by the random walk up to time t. Taking expectations on both sides of (101), we get

$$\mathbb{E}_{(x,y)}[y_{\tau(A^c)}] - y = \frac{1}{4n}\mathbb{E}_{(x,y)}\sum_{t=0}^{\infty}\mathbf{1}(y_t = 0)\mathbf{1}(\tau(A^c) > t) = G_A(x,y)$$

and hence we are done. $\qquad\square$

Remark 6. Note that the above lemma implies for any $(x,y) \in \mathrm{Cyl}_n$,

$$\mathbb{E}_{(x,y)}[y_{\tau(A^c)}] \geq y$$

since the Green's function is a non-negative quantity.

Next we relate the Green's function to the solution of a variational problem. The results are well known and classical even though our setup is slightly different. Hence we choose to include the proofs for clarity. As defined in subsection 4.1 let $\vec{E}$ denote the set of directed edges of Cyl_n.

For any function $F : \mathrm{Cyl}_n \to \mathbb{R}$ define the gradient $\nabla F : \vec{E} \to \mathbb{R}$ by

$$\nabla F(v,w) = F(w) - F(v)$$

and the discrete Laplacian $\Delta F : \mathrm{Cyl}_n \to \mathbb{R}$ by

$$\Delta F(v) = F(v) - \frac{1}{4}\sum_{w \sim v} F(w). \tag{103}$$

Note that the graph Cyl_n is 4-regular.

Recall the definition of energy from subsection 4.1. The next result is a standard summation-by-parts formula.

Lemma 25. *For any function $F : \mathrm{Cyl}_n \to \mathbb{R}$*

$$\mathcal{E}(\nabla F) = 4\sum_{v \in \mathrm{Cyl}_n} F(v)\Delta F(v).$$

The proof follows by definition and expanding the terms.

For a subset $A \subset \mathrm{Cyl}_n$ recalling the definition of stopped Green's function let

$$f_A := \nabla G_A. \tag{104}$$

Also recall the definition of divergence (30).

Lemma 26. *For any $(x, y) \in A$*

$$\mathrm{div}(f_A)(x, y) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For any $v = (x, y) \in A$ by definition

$$\mathrm{div}(f_A)(v) = 4\Delta G_A(v) = 4G_A(v) - \sum_{w \sim v} G_A(w) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{105}$$

The last equality follows by the definition of G_A in (100) by looking at the first step of the random walk started from v. $\qquad\square$

We now prove that the random walk flow f_A on a set A is the flow with minimal energy.

Lemma 27.

$$\mathcal{E}(f_A) = \inf_f \mathcal{E}(f)$$

where the infimum is taken over all flows from $(C_n \times \{0\}) \bigcap A$ to A^c such that for $(x, y) \in A$

$$\mathrm{div}(f)(x, y) = \begin{cases} \frac{1}{n} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof follows by standard arguments, see [12, Theorem 9.10]. We sketch the main steps. One begins by observing that the flow f_A satisfies the cycle law, i.e. the sum of the flow along any cycle is 0. To see this notice that for any cycle

$$x_1, x_2, \ldots x_k = x_1$$

where $x_i's \in \mathrm{Cyl}_n$,

$$\sum_{i=1}^{k-1} f_A(x_i, x_{i+1}) = \sum_{i=1}^{k-1} (G_A(x_{i+1}) - G_A(x_i)) = 0.$$

The proof is then completed by first showing that the flow with the minimum energy must satisfy the cycle law, followed by showing that there is an unique flow satisfying the given divergence conditions and the cycle law.

$\qquad\square$

Now suppose $A \subset \mathrm{Cyl}_n \setminus (C_n \times \{1\})$. Then

$$\mathcal{E}(f_A) = \mathcal{E}(\nabla G_A) = \sum_{v \in A} G_A(v) 4\Delta G_A(v) = \frac{1}{n} \sum_{k \in C_n} G_A(k, 0) = \frac{1}{n} \sum_{k \in C_n} H_A(k, 0). \tag{106}$$

The first equality is by definition. The second equality follow from Lemma 25 and the fact that G_A is 0 outside A. The third equality is by (105). The last equality is by Lemma 24 since by hypothesis $A \cap (C_n \times \{1\}) = \emptyset$.

Proof of Lemma 6. The proof now follows from (106) and Lemma 27. $\qquad\square$

C Proof of Lemma 10

We first prove i. Looking at the process $\omega(t)$ started from $\omega(0) = \omega$ we see by (69) that the process

$$Z_t = X_{t \wedge \tau(B)} - c[t \wedge \tau(B)]$$

with $X_0 = g(\omega)$ is a submartingale with respect to the filtration $\mathcal{F}_t$. Also by hypothesis $|Z_{t+1} - Z_t| \leq 2A_2$.

Now by the standard Azuma-Hoeffding inequality for submartingales, for any time $t > 0$ such that $a_2 - a_1 t < 0$ we have

$$\mathbb{P}\left(Z_t - Z_0 \leq -(a_1 t - a_2)\right) < e^{-\frac{(a_1 t - a_2)^2}{4A_2^2 t}}. \tag{107}$$

Let T be as in the hypothesis of the lemma. We observe that the event $\{X_0 \geq A_1 - a_2\} \cap \{\tau(B) > T\}$ implies that

$$Z_T - Z_0 \leq -(a_1 T - a_2).$$

This is because by hypothesis $Z_0 = X_0 \geq A_1 - a_2$. Hence on the event $\tau(B) > T$

$$Z_T - Z_0 = X_T - a_1 T - X_0 < a_2 - a_1 T$$

since $X_T \leq A_1$ by (67). Thus by (107)

$$\mathbb{P}(\tau(B) > T \mid X_0 > A_1 - a_2) \leq e^{-\frac{(a_1 T - a_2)^2}{4A_2^2 T}}.$$

To prove ii. let $\omega_0 = \omega(\tau(B^c))$. By hypothesis

$$x := f(\omega_0) \geq A_1 - a_4 - A_2.$$

since by (68) the process cannot jump by more than A_2. Clearly it suffices to show

$$\mathbb{P}_{\omega_0}(\tau(B') \geq T') \geq 1 - e^{-\frac{a_4^2}{32A_2^2 T}} + e^{-\frac{a_1^2 T}{32A_2^2}}.$$

Now consider the submartingale

$$W_t = X_{t \wedge \tau' \wedge \tau''} - a_1[t \wedge \tau' \wedge \tau'']$$

with $W_0 = x$, where $\tau' = \tau(B)$ and $\tau'' = \tau(B')$. We first claim that

$$\mathbb{P}_{\omega_0}(\tau' \wedge \tau'' > T) < e^{-\frac{c^2 T^2}{16C^2 T}}. \tag{108}$$

To see this notice that by the Azuma-Hoeffding inequality it follows that

$$\mathbb{P}(W_T - W_0 < -a_1 T/2) < e^{-\frac{a_1^2 T^2}{16A_2^2 T}}. \tag{109}$$

On the other hand the event $\tau' \wedge \tau'' > T$ implies

$$W_T \leq A_1 - a_4 - a_1 T$$
$$W_0 \geq A_1 - a_4 - A_2.$$

Thus the event $\tau' \wedge \tau'' > T$ implies

$$W_T - W_0 < -\frac{a_1 T}{2},$$

since by hypothesis $T > \frac{2A_2}{a_1}$. (108) now follows from (109).

Now on the event $\{\tau' \wedge \tau'' \leq T\} \bigcap \{\tau'' < \tau'\}$,

$$\begin{aligned} W_T &< A_1 - 2a_4 \\ W_0 &\geq A_1 - a_4 - A_2. \end{aligned}$$

Hence

$$\{\tau' \wedge \tau'' \leq T\} \cap \{\tau'' < \tau'\} \implies W_T - W_0 \leq -\frac{a_4}{2}$$

since by hypothesis $a_4 > 2A_2$. Thus by (109) we have

$$\mathbb{P}\left(\{\tau' \wedge \tau'' \leq T\} \bigcap \{\tau'' \leq \tau'\}\right) \leq e^{-\frac{a_4^2}{16A_2^2 T}}.$$

Observe that

$$\mathbb{P}(\tau' \leq \tau'') \geq \mathbb{P}(\tau' \wedge \tau'' \leq T) - \mathbb{P}\left(\{\tau' \wedge \tau'' < T\} \bigcap \{\tau'' \leq \tau'\}\right).$$

This along with (108) imply that τ'' stochastically dominates a geometric variable with success probability at most

$$e^{-\frac{a_4^2}{16A_2^2 T}} + e^{-\frac{a_1^2 T}{16A_2^2}}.$$

Thus we are done. $\qquad\square$

Future directions and Related models.

Fluctuations

This article establishes that competitive erosion on the cylinder forms a macroscopic interface quickly. A natural next step is to find the order of magnitude of its fluctuations. Theorem 1 only shows that the fluctuations are $o(n)$.

Randomly evolving interfaces

Competitive erosion on the cylinder models a random interface fluctuating around a fixed line. It can also model a moving interface if the measures μ_1 and μ_2 are allowed to depend on time. An interesting example is $\mu_i(t) = \delta_{Z_i(t)}$ where Z_1 and Z_2 are simple random walks independent of everything else in the process: that is, red and blue walkers are alternately released from a red source and blue source that themselves perform random walk on a slower time scale.

Another model of a randomly evolving interface arises in the case of fixed but equal measures $\mu_1 = \mu_2$. Figure 11 shows a variant of competitive erosion in the square grid $\mathbb{Z}^2$. Initially all sites are colored white. Red and blue particles are alternately released
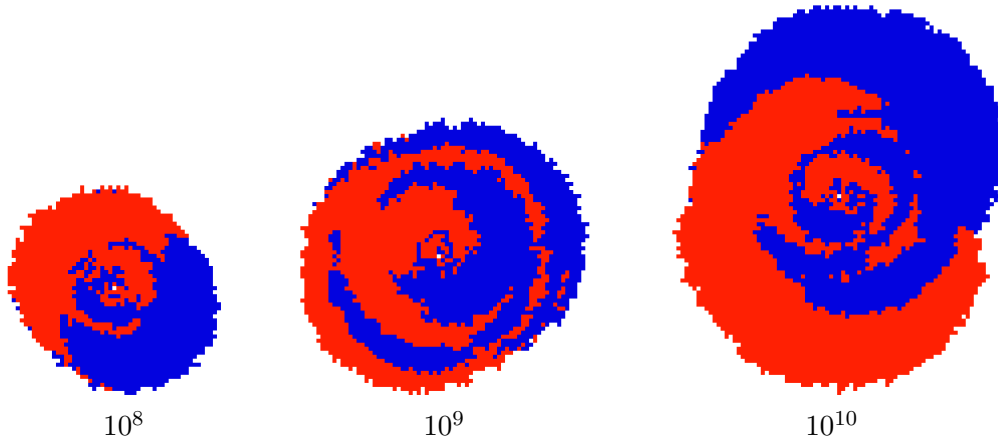
$$10^8 \qquad\qquad 10^9 \qquad\qquad 10^{10}$$

Figure 11: A variant of competitive erosion on $\mathbb{Z}^2$ with $\mu_1 = \mu_2 = \delta_0$. From left to right, the set of all converted sites after $n = 10^8, 10^9, 10^{10}$ particles have been released. Most red particles convert blue sites and vice versa, so that a relatively small number of sites are converted.

from the origin. Each particle performs random walk until reaching a site in $\mathbb{Z}^2 - \{(0,0)\}$ colored differently from itself, and converts that site to its own color. Particles that return to the origin before converting a site are killed. One would not necessarily expect *any* interface to emerge from this process, but simulations show surprisingly coherent red and blue territories.

Conformal invariance

Our choice of the cylinder graph with uniform sources μ_i on the top and bottom is designed to make the function g in the level set heuristic (see (5)) as simple as possible: $g(x,y) = 1-y$. A candidate Lyapunov function for more general graphs is

$$h(S) = \sum_{v \in S^c} g(v)$$

whose maximum over $S \subset V$ of cardinality k is attained by the level set (6).

A case of particular interest is the following: Let $V = D \cap (\frac{1}{n}\mathbb{Z}^2)$ where D is a bounded simply connected planar domain. We take $\mu_i = \delta_{z_i}$ for points $z_1, z_2 \in D$ adjacent to D^c. As the edges of our graph we take the usual nearest-neighbor edges of $\frac{1}{n}\mathbb{Z}^2$ and delete every edge between D and D^c. In the case that D is the unit disk with $z_1 = 1$ and $z_2 = -1$, the level lines of g are circular arcs meeting ∂D at right angles. The location of the interface for general D can then be predicted by conformally mapping D to the disk. Extending the key Theorem 5 to the above setup is a technical challenge we address in a subsequent paper.

Acknowledgments

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