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A Note on Parametric and Nonparametric Uncertainties in Control Systems

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ABSTRACT

It is shown that robust stability and robust performance questions for control systems with nonparametric uncertainties can be turned into those with a particularly simple parametric description. It is pointed out that there exist Routh-like procedures which determine whether a system containing uncertain subsystems is robustly stable (or robustly meets performance requirements).

1. Parametric and Nonparametric Uncertainty

Consider a linear time-invariant (LTI) control system containing uncertain subsystems. Subsystems could have uncertain parameters, e. g.

$$H_{motor}(s) = \frac{a}{s(s+a)} \qquad a_{\min} \le a \le a_{\max} \quad (1)$$

A common method of expressing modeling errors is to specify frequency domain error bounds, as in

$$\left| H_{sensor}(j\omega) - \frac{1}{j\omega+1} \right| \le 10^{-2} \left| \frac{j\omega/3+1}{j\omega/10+1} \right|$$

which we reexpress as

$$\left\| \left(H_{sensor} - \frac{1}{s+1} \right) 100 \frac{s/10+1}{s/3+1} \right\|_{\infty} \le 1 \qquad (2)$$

where $\|H\|_{\infty} \triangleq \sup_{\omega} |H(j\omega)|$ is the usual $\mathbf{H}^{\infty}$ norm.

This is a *nonparametric uncertainty* since the class of all subsystems specified by (2) cannot be parametrized with a finite number of real or complex parameters. One of the purposes of this note is to point out that for the purposes of checking robust stability of a system, the uncertain subsystem (2) can be replaced with a simple parametric one like (1).

2. Robustness Questions

The robust stability question is:

Is our system stable for all possible choices of subsystems?

A harder question is the robust performance question:

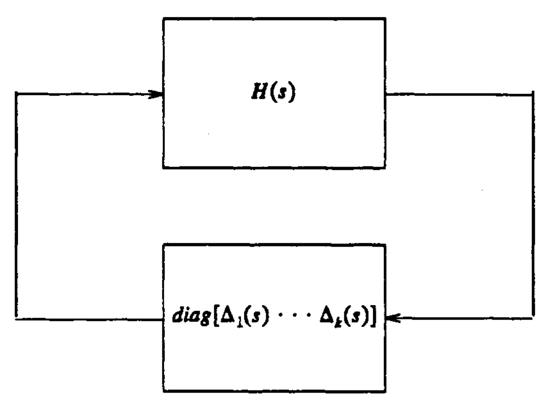
Does our system meet the performance requirements for all possible choices of subsystems?

Doyle[1] has pointed out that if we express our performance requirements as a collection of inequalities like (2), for example,

$$\left\| H_{track\_err} \frac{s+10}{10s} \right\| \le 1 \qquad (3)$$

then the robust performance question can be reduced to a robust stability question for the system augmented by one uncertain subsystem for each performance requirement of the form (3).

Considering for simplicity only the nonparametric uncertainties, we may transform our original system S into the now well known standard form S_{std} for a system with uncertainties of the form (3):



System S_{std}

The robustness of our original system S can be expressed simply as:

Definition: System S is ROBUST iff S_{std} is stable for all $||\Delta_i|| \leq 1$.

Using the notion of structured singular value $\mu(\cdot)$ developed by Doyle and Safonov[1,2] we can say:

$$S \text{ ROBUST iff } \sup_{\omega} \mu(H(j\omega)) < 1 \qquad (4)$$

Much work has focused on the computation of μ. For $k>3$ the computation of $\mu(H)$ for just a single complex matrix H is far from straightforward and is a topic of active research. The problem of verifying that the μ of a whole transfer function lies below one, as required in (4), is of course even harder. Nevertheless we will show that this question can be turned into a simple appearing parameter stability problem, and in fact that there is a decision procedure (finite algorithm) which can answer it.

3. Turning a Nonparametric Uncertainty into a Parametric One

Theorem: S is ROBUST if and only if S_{std} is stable for all Δ_i of the form

$$\Delta_i(s) = \pm \frac{s-d_i}{s+d_i} \text{ or } \pm 1 \qquad (5)$$

for $d_i > 0$.

Note that the third and fourth cases are limiting cases as $d_i \to 0$ or $d_i \to \infty$.

Thus it suffices to check stability with the very special Δ's given by single pole all-passes: if the system S_{std} can be destabilized for some perturbations satisfying $||\Delta_i|| \leq 1$, then it can be destabilized by some single pole allpass perturbations. For a similar observation, see Vidyasagar[3,p282].

4. Some Notes on the Resulting Parameter Stability Problem

Consider the case $\Delta_i(s) = (s-d_i)(s+d_i)^{-1}$ for $i = 1,\ldots k$. Let

$$\dot{x} = Ax + Bu \qquad y = Cx$$

be a realization of H with no hidden unstable modes (we take H strictly proper for simplicity). The Δ's may be realized as

$$\dot{z} = -Dz + y \qquad u = y - 2Dz$$

where $D = diag[d_1, \ldots, d_k]$. Thus our question is whether the system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A+BC & -2BD \\ C & -D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \qquad (6a)$$

$$= \begin{bmatrix} A+BC & -2B \\ C & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \qquad (6b)$$

is stable for all D diagonal and positive.

Note that the parameters enter in a particularly simple form: the matrix in (6) is affine in the parameter vector $[d_1, \ldots, d_k]^T$. The matrices in (6) form a *positive cone*, thus the original robustness question can be transformed to a set of questions of the form:

> *Given matrices $A_1, \ldots, A_m$, determine whether all matrices in the positive cone $\{\sum d_i A_i \mid d_i \geq 0\}$ are stable.*

In our case, $A_2, \ldots, A_m$ have rank 1.

This last problem may be turned into an equivalent problem which appears even simpler, by considering a Lyapunov equation. Provided A_1 is stable, then the cone of matrices described above is stable iff for every matrix A in the cone, the only symmetric solution of $A^T P + PA = 0$ is $P = 0$. Viewing this Lyapunov equation as a homogeneous linear equation in the $n(n+1)/2$ variables P_{ij} with coefficient matrix L, we can restate the problem above as:

> *Given matrices $L_1, \ldots, L_m$, determine whether all matrices in the positive cone $\{\sum d_i L_i \mid d_i \geq 0\}$ are nonsingular.*

Of course, the L's are bigger matrices than the A's, but nonsingularity has been substituted for stability.

4.1. Positivity of a Real Polynomial

In this section we point out that the problem of deciding whether a cone of k matrices is stable can be cast as a problem of deciding whether a real *nonnegative* polynomial of k real variables is in fact *positive*. The cone generated by $A_1, \ldots, A_m$ is stable iff

$$\det(sI - d_1 A_1 \cdots - d_m A_m) \neq 0 \qquad (7a)$$

$$\text{for } \operatorname{Re} s \geq 0,\ d_i \geq 0 \qquad (7b)$$

If the polynomial in (7) vanishes for some $d_i \geq 0$ and s in the closed RHP, then in fact it will do so with the s on the $j\omega$-axis, that is, (7) is equivalent to (8):

$$\det(d_0 j - d_1 A_1 \cdots - d_m A_m) \neq 0 \quad \text{for } d_i \geq 0 \quad (8)$$

Define

$$P(x_0, \ldots, x_m) \triangleq |\det(x_0^2 j - x_1^2 A_1 \cdots - x_m^2 A_m)|^2 \quad (9)$$

Thus our problem is one of determining whether the real nonnegative polynomial P is in fact positive.

4.2. A Decision Procedure

The question of whether the polynomial P in (9) is positive can be determined by a decision procedure, or finite algorithm. In fact, there are a finite number of polynomials of the coefficients of the matrices A_i, such that they are all positive iff (9) is always positive.[4] Thus a Routh-like procedure can determine the robust stability (or performance) of a system with uncertain subsystems. Anderson et al.[5] have pointed out that other hard problems, for example, the stabilization by constant output feedback problem, can also be decided with a Routh-like algorithm.

5. Acknowledgement

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