# Beamforming With Uncertain Weights

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***Abstract*—In this letter, we show that worst-case robust beamforming, with uncertain weights subject to multiplicative variations, can be cast as a convex optimization problem. We interpret this problem as a weighted complex $l_1$-regularization of the nominal beamforming problem, and show that it can be solved with the same computational complexity as nominal beamforming, ignoring the variations. We derive a simple lower bound on how much worse the robust beamformer will be compared to the nominal beamformer solution with no weight uncertainty. We demonstrate the robust approach with a simple narrowband beamformer.**

***Index Terms*—Regularization, robust beamforming, robust optimization, robust sensor array signal processing.**

## I. Beamforming

We consider an array of $n$ sensor elements. Let $a : \Omega \to \mathbf{C}^n$ be the array response to a wave of unit amplitude parametrized by $\theta \in \Omega$, where $\Omega$ is the set of all possible wave parameters. A simple example is an array in a plane, where $\theta \in \Omega = [0, 2\pi]$ corresponds to the arrival angle of a plane wave. In a more complicated example, $\theta$ is a vector that models wave parameters such as wavelength, polarization, range, etc. In the sequel, we refer to the wave parameter $\theta$ as the direction, even though it can be more general and multidimensional. The composite output of the array is a complex weighted sum $w^* a(\theta)$, where $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$ is the vector of weights and $(\cdot)^*$ denotes the conjugate transpose. The magnitude $|w^* a(\theta)|$ of the array output is called the *array gain* or *array sensitivity* in the direction $\theta$.

Next, we describe our beamforming problem. We require a unit array gain in a given desired direction $\theta_{\rm des}$, i.e., $|w^* a(\theta_{\rm des})| = 1$. We also require the array gain to be small for $\theta \in \Omega_{\rm rej}$, where $\Omega_{\rm rej} \subseteq \Omega$ is a given set of directions (not containing $\theta_{\rm des}$), called the *rejection band*. The maximum array gain over the rejection band

$$G(w) = \sup_{\theta \in \Omega_{\rm rej}} |w^* a(\theta)|$$

is called the *rejection level* of the array. The problem of choosing the weight vector to minimize the rejection level, subject to unit array gain in the desired direction can now be formulated as the optimization problem

$$\text{minimize } G(w), \qquad \text{subject to } |w^* a(\theta_{\rm des})| = 1 \tag{1}$$

with variable $w \in \mathbf{C}^n$. A solution of this problem is called a *nominal optimal beamformer*.

The beamforming problem (1) is not a convex optimization problem, since the equality constraint is not linear. However, we can transform it to an equivalent convex problem

$$\text{minimize } G(w), \qquad \text{subject to } \Re(w^* a(\theta_{\rm des})) \geq 1 \tag{2}$$

where $\Re(\cdot)$ denotes the real part. This is a convex problem since the objective $G$ is a convex function (it is a pointwise supremum over an infinite set of convex functions [4, ch. 3]), and the inequality constraint is linear. In particular, when the rejection level is approximated as

$$\hat{G}(w) = \max_{i=1,\ldots,p} |w^* a(\theta_i)|$$

where $\theta_1, \ldots, \theta_p \in \Omega_{\rm rej}$ are sample points in the rejection band, the corresponding approximate beamforming problem becomes a second-order cone program (SOCP), which can be solved with great efficiency [8], [10].

We establish the equivalence of the problems (1) and (2). For any $\alpha \in \mathbf{C}$, we have

$$G(\alpha w) = |\alpha| G(w), \quad |(\alpha w)^* a(\theta_{\rm des})| = |\alpha| \, |w^* a(\theta_{\rm des})| \, .$$

Suppose $w$ is feasible for (2), i.e., $\Re(w^* a(\theta_{\rm des})) \geq 1$. This implies that $|w^* a(\theta_{\rm des})| \geq 1$, so $\tilde{w} = (1/|w^* a(\theta_{\rm des})|) w$ is feasible for (1), and satisfies $G(\tilde{w}) = G(w)/|w^* a(\theta_{\rm des})| \leq G(w)$. Conversely, suppose that $\tilde{w}$ is feasible for (1), i.e., $|\tilde{w}^* a(\theta_{\rm des})| = 1$. Then the point

$$w = \frac{\tilde{w}^* a(\theta_{\rm des})}{|\tilde{w}^* a(\theta_{\rm des})|} \tilde{w}$$

is feasible for (2), and satisfies $G(w) = G(\tilde{w})$. Thus, from any feasible point for either problem, we can construct a feasible point for the other problem, with equal or lower objective value, so we conclude equivalence.

## II. Robust Beamforming With Uncertain Weights

In the beamforming problem (2), we assume that the array response $a$ is perfectly known, and that the weights $w_i$ can be

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A color versions of Figs. 1 is available online at http://ieeexplore.ieee.org.

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implemented with perfect precision. It is well known that the nominal optimal beamformer can be extremely sensitive to variation in $a$ or $w$.

The goal of *robust beamforming* is to choose weights $w$ such that the beamformer performs well despite variations in $a$ or implementation errors in $w$. Robust beamforming has been considered from the beginning of array signal processing [6]. One widely used robust beamforming technique is the diagonal loading method [1], [5], where an $l_2$-regularization term is added to the objective

$$\begin{array}{ll} \text{minimize} & G(w) + \eta \sum_{i=1}^{n} \sigma_i^2 |w_i|^2 \\ \text{subject to} & \Re(w^* a(\theta_{\text{des}})) \geq 1. \end{array} \quad (3)$$

Here, $\eta \geq 0$ is the regularization parameter and $\sigma_i^2$ is the uncertainty (noise) power. More recently, ideas from the (worst-case) robust optimization [2], [7], [3] have been applied to robust beamforming; for example, in robust minimum variance beamforming [11], [12], [9] and in robust array pattern synthesis [13, Sec. IV].

While most researchers have focused on uncertainty in the array response $a$, in this letter, we consider uncertainty in the weights $w$. While the ideas in this letter can be extended to uncertainty in both $a$ and $w$, our goal is not to come up with a general solution, but to point out that the robust beamforming problem with multiplicative uncertainty in the weights can be solved exactly. In addition, this solution can be interpreted as a regularization method.

Thus, we consider robust beamforming with multiplicative uncertainty in the weights

$$\hat{w}_i = w_i(1+\delta_i), \qquad |\delta_i| \leq \rho_i, \qquad i = 1, \ldots, n.$$

Here, $w_i$ are the weights chosen by the designer, and $\hat{w}_i$ are the actual implemented weights. The complex numbers $\delta_i$ are the *relative errors* between the intended and actual weights. The relative error in implementing weight $w_i$ can be as large as $\rho_i$ in magnitude, so $\rho_i$ is a measure of the maximum relative error in implementing $w_i$. For example, $\rho_i = 0.05$ means, roughly, that the implementation error for $w_i$ can be as large as 5%. This corresponds (roughly) to a maximum magnitude variation around $\pm 0.42$ dB, and a maximum phase variation around $\pm 2.86°$. (The actual uncertainty set is a circle in the complex plane.) The set of weight vector uncertainties consistent with this model is denoted

$$\mathcal{D} = \{\delta \in \mathbf{C}^n \mid |\delta_i| \leq \rho_i\}.$$

We take a *worst-case robust optimization* approach to problem (2) with the multiplicative uncertainty model described above: We require the constraint in (2) to hold for all weight vectors $\hat{w}$ consistent with our model, and we judge the objective by its worst-case value over all possible $\delta_i$, i.e., the *worst-case rejection level*

$$G_{\text{wc}}(w) = \sup_{\theta \in \Omega_{\text{rej}}} \sup_{\delta \in \mathcal{D}} |\hat{w}^* a(\theta)|.$$

This leads us to the *robust beamforming problem*

$$\begin{array}{ll} \text{minimize} & G_{\text{wc}}(w), \\ \text{subject to} & \Re(\hat{w}^* a(\theta_{\text{des}})) \geq 1 \quad \text{for all } \delta \in \mathcal{D} \end{array} \quad (4)$$

with variable $w \in \mathbf{C}^n$. Any solution is called a *robust optimal beamformer*.

We note that this problem is convex, since the objective is the pointwise supremum of a family of convex functions, and the constraint is convex, since it consists of a family of linear inequalities, parametrized by $\delta \in \mathcal{D}$. But it appears to be a difficult problem, since the objective is given by a supremum, and the constraint is semi-infinite (given by an infinite number of constraints).

The main goal of this paper is to show that the robust beamforming problem (4) can be reformulated as

$$\begin{array}{ll} \text{minimize} & G(w) + \sum_{i=1}^{n} \mu_i |w_i|, \\ \text{subject to} & \Re(w^* a(\theta_{\text{des}})) \geq 1 + \sum_{i=1}^{n} \gamma_i |w_i| \end{array} \quad (5)$$

where

$$\mu_i = \rho_i \sup_{\theta \in \Omega_{\text{rej}}} |a(\theta)_i|, \qquad \gamma_i = \rho_i |a(\theta_{\text{des}})_i|.$$

This is a convex problem [4, ch. 3]; moreover, after sampling $\theta \in \Omega_{\text{rej}}$, the associated approximate robust beamforming problem is an SOCP that is no harder to solve than the nominal beamforming problem with the same sample points. (The extra terms in the objective and the constraint add new diagonal blocks to the Karush-Kuhn-Tucker (KKT) system. This diagonal structure can be exploited to solve the new system with the same computational complexity required for the nominal problem [4, App. C].)

The equivalence between (4) and (5) follows from

$$G_{\text{wc}}(w) = G(w) + \sum_{i=1}^{n} \mu_i |w_i| \quad (6)$$

and

$$\inf_{\delta \in \mathcal{D}} \Re(\hat{w}^* a(\theta)) = \Re(w^* a(\theta)) - \sum_{i=1}^{n} \rho_i |a(\theta)_i| |w_i| \quad (7)$$

which we establish now. Observe that, for any $\theta \in \Omega$ and $\delta \in \mathcal{D}$, we have

$$\begin{aligned} |\hat{w}^* a(\theta)| &= \left| \sum_{i=1}^{n} a(\theta)_i w_i^* + \sum_{i=1}^{n} a(\theta)_i w_i^* \delta_i^* \right| \\ &\leq \left| \sum_{i=1}^{n} a(\theta)_i w_i^* \right| + \left| \sum_{i=1}^{n} a(\theta)_i w_i^* \delta_i^* \right| \\ &\leq |w^* a(\theta)| + \sum_{i=1}^{n} \rho_i |a(\theta)_i| |w_i|. \end{aligned}$$

Moreover, equality holds here with the choice of $\delta \in \mathcal{D}$ with its $k$th component being

$$\delta_k = \rho_k \frac{a(\theta)_k w_k^*}{|a(\theta)_k w_k^*|} e^{i\phi/n}, \qquad \phi = \angle\left(w^* a(\theta)\right).$$

Therefore, we have

$$\sup_{\delta \in \mathcal{D}} |\hat{w}^* a(\theta)| = |w^* a(\theta)| + \sum_{i=1}^n \rho_i |a(\theta)_i| |w_i|$$

which means that

$$G_{\rm wc}(w) = \sup_{\theta \in \Omega_{\rm rej}} \sup_{\delta \in \mathcal{D}} |\hat{w}^* a(\theta)| = G(w) + \sum_{i=1}^n \mu_i |w_i|.$$

In a similar way, with $\delta_k = -\rho_k\, a(\theta)_k w_k^* / \left(|a(\theta)_k w_k^*|\right)$, we can see that (7) holds.

We can interpret problem (5) in the following way. Its objective consists of the nominal objective $G$ and a weighted complex $l_1$-norm of $w$. This makes sense since we want to penalize large weights, which are more susceptible to multiplicative errors. Adding a positive multiple of a norm to the objective is called *regularization*; in our case, we have a weighted complex $l_1$-regularization of the nominal beamforming problem.

We close by making a comparison between the $l_1$-regularization used in (5) and $l_2$-regularization used in diagonal loading (3). While the $l_1$-regularization is optimal in the worst-case sense for our multiplicative uncertainty model, the $l_2$-regularization has a nice statistical interpretation: it minimizes the power of the additive noise in the system.

## III. A Lower Bound on Optimal Rejection

We derive a simple lower bound on how much worse the robust optimal beamformer will be compared to the nominal beamformer. Let $p^\star_{\rm nom}$ be the optimal value of the nominal beamforming problem (2) and $p^\star_{\rm rob}$ be the optimal value of the robust beamforming problem (5) achieved for a robust beamformer $w^\star_{\rm rob}$. Clearly, $p^\star_{\rm rob} \geq p^\star_{\rm nom}$. We show that

$$p^\star_{\rm rob} - p^\star_{\rm nom} \geq \frac{\mu_i}{(1-\rho_i)|a(\theta_{\rm des})_i|} \tag{8}$$

where $i = \arg\min_{j=1,\ldots,n} \mu_j / ((1-\rho_j)|a(\theta_{\rm des})_j|)$.

The bound is derived by considering the chain of inequalities

$$p^\star_{\rm rob} - p^\star_{\rm nom} \geq G_{\rm wc}(w^\star_{\rm rob}) - G(w^\star_{\rm rob}) \geq p^\star_{\rm aux}$$

where $p^\star_{\rm aux}$ is the desired lower bound and the optimal value of the auxiliary problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \mu_i |w_i| \\ \text{subject to} & \Re(w^* a(\theta_{\rm des})) \geq 1 + \sum_{i=1}^n \gamma_i |w_i|. \end{array}$$

Here, the first inequality holds since $G$ and $G_{\rm wc}$ are always nonnegative, and $G(w^\star_{\rm rob}) \geq p^\star_{\rm nom}$ by the optimality definition and $w^\star_{\rm rob}$ being a feasible point of the nominal problem. The second inequality follows since the optimal value of the auxiliary problem is the minimum value of the difference $G_{\rm wc} - G$ over the feasible set of the robust beamforming problem.

Next we derive an expression for $p^\star_{\rm aux}$. We can assume that $w_i \in \mathbf{R}_+$, since we can rotate each $w$ component by an angle to accomplish this without changing the objective value. The feasible set also stays the same, since the $w_i$ rotation angles can be absorbed into $a(\theta_{\rm des})_i$, which can be further rotated until $a(\theta_{\rm des})_i = |a(\theta_{\rm des})_i| \in \mathbf{R}_+$. Thus, an equivalent problem is

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \mu_i w_i, \\ \text{subject to} & \sum_i |a(\theta_{\rm des})_i| w_i \geq 1 + \sum_{i=1}^n \rho_i |a(\theta_{\rm des})_i| w_i \quad w_i \geq 0. \end{array}$$

This is a linear program with positive variables and data, for which we can explicitly compute the optimal solution and the optimal value (for similar LP problems see [4, ch. 4, p. 192]), which gives $p^\star_{\rm aux}$ and the right-hand side of (8).

As an example, consider the case when $|a(\theta)_i| = 1$ for all $i$ and all $\theta$, and $\rho_i = \rho$. Then the bound (8) gives

$$p^\star_{\rm rob} - p^\star_{\rm nom} \geq \frac{\rho}{(1-\rho)}. \tag{9}$$

For example, if $\rho = 0.05$ (which corresponds to 5% uncertainty in the weights), we find that

$$p^\star_{\rm rob} \geq p^\star_{\rm nom} + 0.0526.$$

In particular, we cannot achieve a worst-case rejection level smaller than $20 \log_{10} 0.0526 = -25.6$ dB, regardless of the array geometry or the number of elements.

## IV. Example

We consider a narrowband beamformer with $n = 36$ elements on a rectangular $6 \times 6$ lattice in a plane. The spacing between the sensor elements is $0.45\lambda$, where $\lambda$ is the wavelength of plane waves arriving from angles $\theta \in [0^\circ, 360^\circ]$. We use a simple model for the array response, $a(\theta)_j = \exp((2\pi i/\lambda)(x_j \cos\theta + y_j \sin\theta))$, where $(x_j, y_j)$ is the location of the $j$th sensor element. We take $\theta_{\rm des} = 60^\circ$ and $\Omega_{\rm rej} = [0^\circ, 40^\circ] \cup [80^\circ, 360^\circ]$. Thus, we want to reject waves arriving from angles more than $20^\circ$ away from the desired direction $\theta_{\rm des}$.

We assume $\rho_i = \rho \geq 0$. We consider a family of problems where we vary $\rho$ from 0 (no uncertainty) to 0.15 (15% relative error). For each value of $\rho$, we compute the rejection levels for the nominal beamformer and the robust beamformer. In addition, for each $\rho$, we solve $l_2$-regularization problem (3) with $\sigma_i^2 = \mu_i$ and with the value of $\eta$ that gives the smallest worst-case rejection level. All solutions are obtained using an SOCP solver, after we sample $\theta$ with $1^\circ$ precision.

The nominal optimal beamformer, a solution of (2), achieves $p^\star_{\rm nom} = 0.0132$, which corresponds to $-37.6$ dB of rejection without uncertainty. However, with uncertainty present, its re-

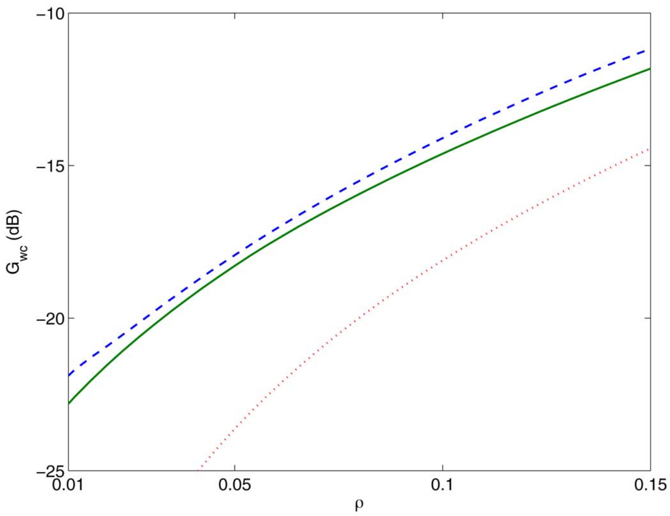


Fig. 1. Worst-case rejection level versus $\rho$ for the robust beamfomer (solid) and the $l_2$-regularized beamformer (dashed). The lower bound on the worst-case rejection level, given in (9), is also shown (dotted).

jection level degrades rapidly; the worst-case rejection level for the nominal design exceeds 0 dB even for $\rho = 0.01$. Thus, the nominal beamformer is useless for weight uncertainty as small as $1\%$. On the other hand, the robust beamformer and the $l_2$-regularized beamformer perform well even with much larger uncertainty. The robust beamformer gives a worst-case rejection level around $-12$ dB even for $15\%$ uncertainty in the weights.

A plot of worst-case rejection level versus $\rho$, for the robust and $l_2$-regularized beamformers is shown in Fig. 1. The plot shows that the worst-case rejection level of the $l_2$-regularized beamformer comes quite close (within around 0.5 dB) to that achieved by the robust beamformer.

For this example, we have $|a(\theta)_i| = 1$ for all $\theta$ and all $i$, so (9) gives the lower bound

$$p^\star_{\rm rob} \geq p^\star_{\rm nom} + \frac{\rho}{(1-\rho)} = 0.0132 + \frac{\rho}{(1-\rho)}.$$

This is plotted as the dotted curve in Fig. 1.

## V. Conclusions

In this letter, we have shown that worst-case robust beamforming with multiplicative uncertainty in the weights can be cast as a tractable convex optimization problem, which can be solved with the same computational cost as nominal beamforming. This globally optimal solution is equivalent to a weighted $l_1$-regularization of the nominal problem. Thus, it is not suprising that $l_2$-regularization of the nominal problem, i.e., diagonal loading, yields beamformer designs that are quite robust to weight variation.

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