Tightness for Non-irreducible Markov Chains

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Abstract

In this paper we develop Foster-type criteria guaranteeing tightness for Markov chains which are not necessarily irreducible. The results include criteria for both tightness of the marginal distributions and tightness of the Cesaro-averaged transition probabilities. In addition, we obtain results guaranteeing boundedness in expectation for real-valued functionals of the chain.

Keywords: Markov chains, stochastic stability, tightness, Lyapunov functions.

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1 Introduction

Let $\boldsymbol{\Phi} = \{\Phi_0, \Phi_1, \ldots\}$ be a time homogeneous Markov chain, and let P be its associated transition kernel, so that

$$P(x, B) := \mathsf{P}\{\Phi_{n+1} \in B \mid \Phi_n = x\}, \qquad x \in \mathsf{X},\ B \in \mathcal{B}(\mathsf{X}).$$

We assume throughout that the state space X is a complete, separable metric space, with Borel σ-field $\mathcal{B}(\mathsf{X})$.

In the stochastic systems literature, *stability* of a Markov chain is frequently equated with *tightness* of the underlying distributions, for each initial condition. This gives rise to the following definition, introduced by Khas'minskii in the sixties [2]. A Markov process is called *bounded in probability* if for each initial condition of the process, there exists a sequence of compact sets $K_n \uparrow \mathsf{X}$ for which

$$\lim_{n\to\infty} \sup_{k\geq 0} \mathsf{P}\{\Phi_k \in K_n^c\} = 0.$$

If the Markov chain $\boldsymbol{\Phi}$ possesses the state transition kernel P, boundedness in probability is equivalent to requiring tightness of the family of probabilities $\{P^k(x, \cdot\,) :$

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$k \geq 0\}$. The Markov chain $\boldsymbol{\Phi}$ is called *bounded in probability on average* if the family of probabilities

$$\Big\{\frac{1}{N}\sum_{k=0}^{N-1} P^k(x,\,\cdot\,) : N \geq 1\Big\} \tag{1}$$

is tight, for any initial condition $x \in \mathsf{X}$.

The most common approach to establishing any form of stability for a Markov chain is to bound the mean of the return time $\tau_A = \min(k \geq 1 : \Phi_k \in A)$, where A is a suitably chosen subset of X. In this paper it is shown that this approach can be extended to the very general framework here, yielding criteria for both tightness and finiteness of moments for the Markov chain under consideration.

The assumptions imposed in this paper are related to the Feller condition that the Markov transition operator maps bounded continuous functions to continuous functions. Feller Markov chains have been treated previously in numerous papers. The main results of [1, 7] imply that if $\sup_{x\in K} \mathsf{E}_x[\tau_K] < \infty$ for a compact subset $K \subset \mathsf{X}$, then an invariant probability π exists, and this implies that the probabilities in (1) will be tight for a.e. $x \in \mathsf{X}$ $[\pi]$. More recently, Lassere [4] has obtained criteria for the existence of invariant probabilities for Feller chains based on a generalization of Farkas' Lemma. Some of the most recent results in both the Feller and ψ-irreducible contexts are described in the monographs [5, 3]. Most of the results in these references concern the existence of an invariant measure for the chain, while in the present paper we seek conditions under which the chain will remain bounded in a probabilistic sense for each initial condition.

The remainder of the paper is organized as follows. In Section 2 we present the main results, establishing tightness of the underlying distributions under bounds related to Foster's criterion. These results are generalized in Section 3 where criteria are developed which guarantee boundedness in expectation for real-valued functionals of the chain. In Section 4 we conclude with a simple example, and a counterexample to show that none of the assumptions imposed here are superfluous.

2 Tightness

We begin with a condition for tightness based upon the return time to a measurable subset $A \in \mathcal{B}(\mathsf{X})$. For such a set A we define the kernel V_A on $\mathsf{X} \times \mathcal{B}(\mathsf{X})$ by

$$V_A(x,S) = \mathsf{E}_x\Big[\sum_{j=0}^{\tau_A-1} \mathbb{1}(\Phi_j \in S)\Big], \qquad x \in \mathsf{X},\ S \in \mathcal{B}(\mathsf{X}).$$

Theorem 2.1 *Suppose that there exists a sequence of compact sets $K_n \uparrow \mathsf{X}$ and $A \in \mathcal{B}(\mathsf{X})$ such that*

$$\lim_{n\to\infty} \sup_{a\in A} V_A(a, K_n^c) = 0. \tag{2}$$

Then,

(a) *the family of probabilities*

$$\Big\{\frac{1}{N}\sum_{k=0}^{N-1} P^k(x,\cdot) : N \geq 1, x \in A\Big\}$$

is tight.

(b) *If in addition to (2) we have*

$$\mathsf{P}_x\{\tau_A < \infty\} = 1, \qquad x \in \mathsf{X}, \tag{3}$$

then the chain is bounded in probability on average.

PROOF Let $T_1 = T_A$, $T_{n+1} = \inf\{k > T_n : \Phi_k \in A\}$, $n \geq 1$. Then for any function $f\colon \mathsf{X} \to \mathbb{R}$ and for $a \in A$,

$$\begin{aligned} \Big|\textstyle\sum_{j=0}^{N-1} P^j f(a)\Big| &\leq \textstyle\sum_{j=0}^{N-1} P^j f(a) \\ &\leq \mathsf{E}_a\Big[\textstyle\sum_{j=0}^{\tau_A-1} f(\Phi_j)\Big] + \textstyle\sum_{\ell=1}^{N-1} \mathsf{E}_a\Big[\textstyle\sum_{j=T_\ell}^{T_{\ell+1}} f(\Phi_j)\Big] \\ &\leq \mathsf{E}_a\Big[\textstyle\sum_{j=0}^{\tau_A-1} f(\Phi_j)\Big] + (N-1)\sup_{x\in A} \mathsf{E}_x\Big[\textstyle\sum_{j=0}^{\tau_A-1} f(\Phi_j)\Big], \end{aligned} \tag{4}$$

where the last inequality follows from the strong Markov property applied to the stopping times $\{T_\ell\}$. On setting $f = \mathbb{1}_{K_n^c}$ we obtain for any $x \in \mathsf{X}$,

$$\frac{1}{N}\sum_{j=0}^{N-1} P^j(x, K_n^c) \leq \frac{1}{N} V_A(x, K_n^c) + \frac{N-1}{N}\sup_{a\in A} V_A(a, K_n^c) \tag{5}$$

Taking the supremum over all $x \in A$ gives

$$\sup_{\substack{a\in A\\ N\geq 1}} \frac{1}{N}\sum_{j=0}^{N-1} P^j(a, K_n^c) \leq \sup_{a\in A} V_A(a, K_n^c).$$

Since the right hand side converges to zero as $n \to \infty$, this proves (a). To establish the conclusions of (b) under (3) we break up the average of P^j as follows:

$$\begin{aligned} \frac{1}{N}\sum_{j=0}^{N-1} P^j(x, K_n^c) &= \frac{1}{N}\sum_{j=0}^{N-1} \mathsf{E}_x[\mathbb{1}(\tau_A \leq j)\mathbb{1}(\Phi_j \in K_n^c)] \\ &\quad + \frac{1}{N}\sum_{j=0}^{N-1} \mathsf{E}_x[\mathbb{1}(\tau_A > j)\mathbb{1}(\Phi_j \in K_n^c)] \\ &\leq \frac{1}{N}\mathsf{E}_x[\sum_{j=0}^{N-1} \mathbb{1}(\tau_A \leq j)\mathbb{1}(\Phi_j \in K_n^c)] \\ &\quad + \frac{1}{N}\sum_{j=0}^{N-1} \mathsf{P}_x\{\tau_A > j\} \\ &\leq \frac{1}{N}\mathsf{E}_x[\sum_{j=\tau_A}^{\tau_A+N-1} \mathbb{1}(\Phi_j \in K_n^c)] \\ &\quad + \frac{1}{N}\sum_{j=0}^{N-1} \mathsf{P}_x\{\tau_A > j\} \end{aligned}$$

From the strong Markov property applied at time τ_A we then obtain

$$\frac{1}{N}\sum_{j=0}^{N-1} P^j(x, K_n^c) \le \frac{1}{N}\Big(\sup_{a\in A} \mathsf{E}_a[\sum_{j=0}^{N-1} \mathbb{1}(\Phi_j \in K_n^c)] + \sum_{j=0}^{N-1} \mathsf{P}_x\{\tau_A > j\}\Big).$$

From (a) and (3) the RHS converges to zero as $N \to \infty$, and then $n \to \infty$, which establishes boundedness in probability on average. □

The corollary below shows that uniform integrability for the return time to a set A is sufficient for tightness of the averaged distributions. Some structure must be imposed on the set A: a minimal requirement is the tightness condition (iii) below. This condition is satsified when $\boldsymbol{\Phi}$ has the Feller property and the set A is compact. To see this, first note that the Feller property is equivalent to continuity of the map $x \mapsto P^n(x, \cdot)$ (from the initial condition x, to the space of probability measures on $\mathcal{B}(\mathsf{X})$, under the topology of weak convergence). Since tightness is equivalent to relative compactness in the this topology, the condition (iii) follows from the fact that the continuous image of a compact set is itself compact.

Corollary 2.1 *Suppose that there exists a subset $A \in \mathcal{B}(\mathsf{X})$ such that*

(i) *The uniform integrability condition holds:*

$$\lim_{m\to\infty} \sup_{a\in A} \mathsf{E}_a[(\tau_A - m)\mathbb{1}(\tau_A > m)] = 0.$$

(ii) *For each x, $\mathsf{P}_x\{\tau_A < \infty\} = 1$.*

(iii) *For each $m \ge 1$, the family of probability measures $\{1/m \sum_{k=1}^m P^k(a, \cdot) : a \in A\}$ is tight.*

Then the chain is bounded in probability on average.

Proof Fix $n > 0$. By Condition (i), there exists $m \ge 1$ such that

$$\sup_{a\in A} \mathsf{E}_a[(\tau_A - m)\mathbb{1}(\tau_A > m)] \quad < \quad 1/(2n).$$

On the other hand, by Condition (iii), there exists a compact set K_n such that

$$\sup_{a\in A} \sum_{j=0}^{m} P^j(a, K_n^c) \quad < \quad 1/(2n).$$

But for $a \in A$,

$$\mathsf{E}_a\Big[\sum_{j=0}^{\tau_A-1} \mathbb{1}(\Phi_j \in K_n^c)\Big] \quad \le \quad \mathsf{E}_a\Big[\sum_{j=0}^{m} \mathbb{1}(\Phi_j \in K_n^c)\Big] + \mathsf{E}_a[(\tau_A - m)\mathbb{1}(\tau_A > m)] < 1/n.$$

Since the left hand side of this bound is $V_A(x, K_n^c)$, it follows that the conditions of Theorem 2.1 are satisfied, which completes the proof. □

The verification of the uniform integrability condition (i) can be subtle. Here is one sufficient condition.

Corollary 2.2 *Suppose that there exists a measurable function $V : \mathsf{X} \to [0,\infty)$ satisfying*

(i) *For some $b < \infty$,*
$$PV \le V - 1 + b\mathbb{1}_A; \tag{6}$$

(ii)
$$\lim_{n\to\infty} \sup_{a\in A} \mathsf{E}_a[V(\Phi_n)\mathbb{1}(\tau_A > n)] = 0. \tag{7}$$

(iii) *For each $m \ge 1$, the family of probability measures $\{1/m \sum_{k=1}^m P^k(a,\cdot) : a \in A\}$ is tight.*

Then the chain is bounded in probability on average.

PROOF If (6) holds then by Theorem 11.3.4 of [5], for all $x \in \mathsf{X}$,
$$\mathsf{E}_x[\tau_A] \le V(x) + b\mathbb{1}_A(x).$$
By the Markov property we then have
$$\begin{aligned}\mathsf{E}_x[(\tau_A - m)\mathbb{1}(\tau_A > m)] &= \mathsf{E}_x[\mathsf{E}[(\tau_A - m) \mid \mathcal{F}_m]\mathbb{1}(\tau_A > m)] \\ &\le \mathsf{E}_x[\mathsf{E}_{\Phi_m}[\tau_A]\mathbb{1}(\tau_A > m)] \\ &\le \mathsf{E}_x[V(\Phi_m)\mathbb{1}(\tau_A > m)].\end{aligned}$$
The result then follows from Corollary 2.1. □

The final result of this section removes the average on P^n to give a criterion for boundedness in probability.

Theorem 2.2 *Suppose that there exists a subset $A \in \mathcal{B}(\mathsf{X})$ such that the strengthened uniform integrability condition holds:*
$$\sum_{n=1}^{\infty} \Big(\sup_{a\in A} \mathsf{P}_a\{\tau_A > n\}\Big) < \infty;$$
and, for each $n \ge 1$, the family of probability measures $\{P^n(a,\cdot) : a \in A\}$ is tight.

Then,

(a) *$\{P^n(a,\cdot) : a \in A, n \ge 1\}$ is tight.*

(b) *If in addition $\mathsf{P}_x\{\tau_A < \infty\} = 1$ for every x, then the chain is bounded in probability.*

PROOF The last exit decomposition [5] may be written
$$P^n(x,B) = \mathsf{P}_x\{\Phi_j \in B; \tau_A \ge n\} + \int_A \sum_{j=1}^{n-1} P^{n-j}(x,dy)\mathsf{P}_y\{\Phi_j \in B; \tau_A \ge j\}$$
It then follows that for any $N \ge 1$,

$$P^n(x,B) \le \mathsf{P}_x\{\tau_A \ge n\} + \sum_{j=1}^{N-1} \sup_{y\in A} \mathsf{P}_y\{\Phi_j \in B\} + \sum_{j=N}^{\infty} \sup_{y\in A} \mathsf{P}_y\{\tau_A \ge j\}$$

The conclusion (a) follows immediately from this bound, on letting B denote the complement of a suitably large compact subset of X. Result (b) then follows from (a) upon conditioning at time τ_A as in the proof of Theorem 2.1. We omit the details. □

A sufficient condition for (i) is that $\sup_A \mathsf{E}_x[\tau_A\{\log(\tau_A)\}^2] < \infty$: Such bounds can be obtained through Lyapunov function arguments, similar to the use of (6) (see [6]). To see why this is sufficient, consider the bounds

$$\sup_{a\in A} \mathsf{P}_a\{\tau_A > n\} \le \sup_{a\in A} \mathsf{E}_a\Big[\frac{\log(\tau_A)^2\tau_A}{\log(n)^2 n}\mathbb{1}(\tau > n)\Big] \le \frac{b}{\log(n)^2 n}$$

where $b = \sup_{a\in A} \mathsf{E}_a[\tau_A\{\log(\tau_A)\}^2]$. The right hand side is summable, so this establishes (i).

3 Bounds on moments

The methods above can be adapted to generate bounds on positive-valued functions of the chain under a strengthened version of Foster's criterion. We suppose throughout this section that $f\colon \mathsf{X} \to \mathbb{R}_+$ is measurable.

Theorem 3.1 *Suppose that there exists $V : \mathsf{X} \to (0,\infty)$ such that*

$$PV(x) \le V(x) - f(x) + b\mathbb{1}_A(x), \qquad x \in \mathsf{X}.$$

Then for all $N \ge 1$ and $x \in \mathsf{X}$,

$$\frac{1}{N}\sum_{j=0}^{N-1} \mathsf{E}_x[f(\Phi_j)] \le b + \frac{1}{N}V(x)$$

Proof This follows from iteration of the inequality $PV \le V - f + b$ (see also the Comparison Theorem of [5]). □

The criterion above may be translated to a condition on the kernel V_A on defining $V_A(x,f) := \mathsf{E}_x\Big[\sum_{j=0}^{\tau_A-1} f(\Phi_j)\Big]$.

Theorem 3.2 *Let $f : \mathsf{X} \to \mathbb{R}$, suppose that there exists $A \in \mathcal{B}(\mathsf{X})$ such that*

$$b := \sup_{a\in A} V_A(a,f) < \infty.$$

Then for all $N \ge 1$ and $x \in \mathsf{X}$,

$$\frac{1}{N}\sum_{j=0}^{N-1} \mathsf{E}_x[f(\Phi_j)] < b + \frac{1}{N}V_A(x,f)$$

Proof This follows from Theorem 3.1 on letting $V(x) = \mathbb{1}_{A^c}(x)V_A(x)$, $x \in \mathsf{X}$: We have for any x,

$$PV = V_A - f \leq V - f + b\mathbb{1}_A.$$

□

Theorem 3.3 *For $f : \mathsf{X} \to \mathbb{R}$, suppose that there exists $A \in \mathcal{B}(\mathsf{X})$ such that*

$$\sum_{j=1}^{\infty} \sup_{a \in A} \mathsf{E}_a[f(\Phi_j); \tau_A \geq j] \quad < \quad \infty.$$

Then,

(a)

$$\sup_{\substack{a \in A \\ n \geq 0}} \mathsf{E}_a[f(\Phi_n)] \quad < \quad \infty.$$

(b) *For any $x \in \mathsf{X}$ satisfying $V_A(x, f) < \infty$,*

$$\sup_{n \geq 0} \mathsf{E}_x[f(\Phi_n)] \quad < \quad \infty.$$

Proof The last exit decomposition yields, for any x,

$$\mathsf{E}_x[f(\Phi_n)] = \mathsf{E}_x[f(\Phi_n); \tau_A \geq n] + \int_A \sum_{j=1}^{n-1} P^{n-j}(x, dy)\mathsf{E}_y[f(\Phi_j); \tau_A \geq j]. \tag{8}$$

Then,

$$\begin{aligned}
\sup_{a \in A} \mathsf{E}_a[f(\Phi_n)] &\leq \sup_{a \in A} \mathsf{E}_a[f(\Phi_n); \tau_A \geq n] + \int_A \sum_{j=1}^{n-1} P^{n-j}(a, dy) \sup_{a \in A} \mathsf{E}_a[f(\Phi_j); \tau_A \geq j] \\
&\leq \sup_{a \in A} \mathsf{E}_a[f(\Phi_n); \tau_A \geq n] + \sum_{j=1}^{n-1} \sup_{a \in A} \mathsf{E}_a[f(\Phi_j); \tau_A \geq j] \\
&\leq \sum_{j=1}^{\infty} \sup_{a \in A} \mathsf{E}_a[f(\Phi_j); \tau_A \geq j],
\end{aligned}$$

which proves (a).

To prove (b) apply the identity (8) and finiteness of $f_{max} = \sup_{\substack{a \in A \\ n \geq 0}} \mathsf{E}_a[f(\Phi_n)]$ to obtain the bound

$$\mathsf{E}_x[f(\Phi_n)] \leq \mathsf{E}_x[f(\Phi_n); \tau_A \geq n] + f_{max}\mathsf{P}_x\{\tau_A < n\}.$$

Under the assumptions of (b) the first term on the RHS of this bound is summable, and this completes the proof. □

Remark: One way to develop Lyapunov criteria for Theorem 3 is to note that if

$$\sup_{z\in A} \mathsf{E}_z\Big[\sum_{j=0}^{\tau_A-1} f(\Phi_j)\varphi(j)\Big] \quad < \quad \infty, \tag{9}$$

then

$$\sup_{z\in A} \mathsf{E}_z[f(\Phi_j);\tau_A\geq j] \quad \leq \quad \frac{1}{\varphi(j)} \sup_{z\in A} \mathsf{E}_z\Big[\sum_{\ell=0}^{\tau_A-1} f(\Phi_\ell)\cdot\varphi(\ell)\Big].$$

So, provided $\sum 1/\varphi(j) < \infty$, Theorem 3.3 holds. Lyapunov criteria for (9) can be derived as in [6].

4 Examples

4.1 A random walk

To see how to apply Corollary 2.2 in operations research models, take the random walk on $\mathbb{R}_+$,

$$\Phi_{n+1} = [\Phi_n + \eta_{n+1}]^+, \qquad n\in\mathbb{Z}_+,$$

with $\mathsf{E}[\eta(1)]<0$, and $\{\eta_k : k\geq 1\}$ is i.i.d.. We have that condition (i) of Corollary 2.2 is satisfied with $A=[0,a_0]$ and $V(x)=cx$, for some $a_0,b,c>0$. To establish (ii), first observe that by (6), the sequence $(M_n,\mathcal{F}_n : n\geq 1)$ is a supermartingale, where $M_n = V(\Phi_n)\mathbb{1}(\tau_A\geq n)$ and $\mathcal{F}_n=\sigma\{\eta_1,\ldots,\eta_n\}$. It follows from the Supermartingale Convergence Theorem that $M_n\to M_\infty=0$ as $n\to\infty$, and by positivity, the supermartingale is uniformly integrable. Hence, for each x,

$$0=\lim_{n\to\infty}\mathsf{E}_x[M_n] = \lim_{n\to\infty}\mathsf{E}_x[V(\Phi_n)\mathbb{1}(\tau_A\geq n)].$$

The random variable $V(\Phi_n)\mathbb{1}(\tau_A\geq n)$ is, for each n, monotone in the initial condition $\Phi_0=x$. It follows that

$$\sup_{a\in A}\mathsf{E}_a[V(\Phi_n)\mathbb{1}(\tau_A\geq n)] = \mathsf{E}_{a_0}[V(\Phi_n)\mathbb{1}(\tau_A\geq n)]\to 0, \qquad n\to\infty.$$

The conditions of Corollary 2.2 are thus satisfied, and hence the random walk is bounded in probability on average. Using similar arguments we may show using Theorem 2.2 that the chain is in fact bounded in probability, and if the sequence $\{\eta_k\}$ possesses a $p+1$-moment, then on redefining $V(x)=c|x|^{p+1}$ we may apply Theorem 3.3 to establish the existence of a bounded pth moment for $\boldsymbol{\Phi}$.

This model is in fact ψ-irreducible, and hence these results follow from the far stronger Harris ergodicity [5]. It is useful however to see how the drift condition (6) leads easily to the uniform bound (7) in this example.

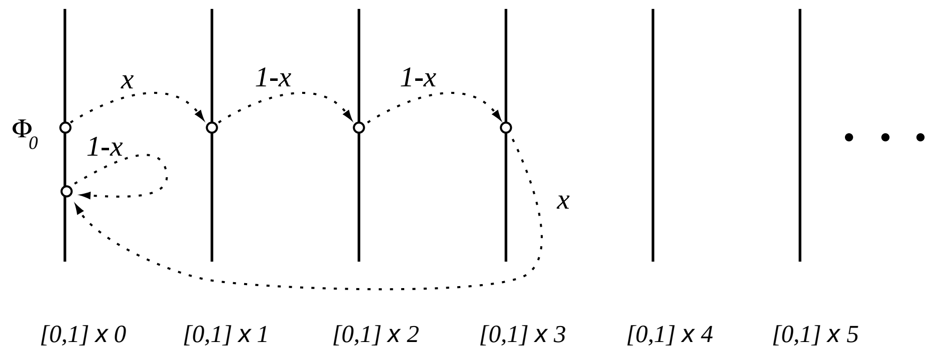

Figure 1. A Typical Sample Path of $\boldsymbol{\Phi}$:
$\Phi_0 = (x,0)$, $\Phi_1 = (x,1)$, $\Phi_2 = (x,2)$, $\Phi_3 = (x,3)$, $\Phi_4 = (x/2,0)$, $\Phi_5 = (x/2,0)$ ...

4.2 A counterexample

One might suspect that the generalization of Foster's criterion (6) alone would be sufficient to establish tightness of the distributions of $\boldsymbol{\Phi}$, so that the analysis of the random walk above could be extended to general models. The following example shows that this is not true: the uniform integrability assumption is necessary in general.

Consider a Markov chain $\boldsymbol{\Phi}$ on $\mathsf{X} = [0,1] \times \mathbb{Z}_+$ whose transition probabilities are defined as follows. Let A denote the compact set $A = [0,1] \times 0$. For initial conditions $(x,0) \in A$, we set

$$\begin{aligned} P((x,0),(x,0)) &= 1-x \\ P((x,0),(x,1)) &= x \end{aligned}$$

For $(x,k) \in \mathsf{X}$, $k \geq 1$, we make the definition

$$\begin{aligned} P((x,k),(x,k+1)) &= 1-x \\ P((x,k),(x/2,0)) &= x \end{aligned}$$

The mean return time to A is easily computed: For any $x \in [0,1]$,

$$\begin{aligned} \mathsf{E}_{(x,0)}[\tau_A] &= \sum_{n=1}^{\infty} \mathsf{P}_{(x,0)}\{\tau_A \geq n\} \\ &= 1 + \sum_{n=0}^{\infty} x(1-x)^n, \end{aligned}$$

which shows that

$$\mathsf{E}_{(x,0)}[\tau_A] = 2 < \infty, \qquad x \in (0,1].$$

Since $(0,0)$ is absorbing, we of course have $\mathsf{E}_{(0,0)}[\tau_A] = 1$. It follows that a solution to Foster's criterion (6) exists: take $V(x) = \mathsf{E}_x[\sigma_A]$, where $\sigma_A := \min(k \geq 0 : \Phi_k \in A)$.

The sample path behavior of the chain is symmetric: The chain stays in the set A for a geometrically distributed time interval, and then remains outside of A for an identically distributed time interval. It follows then that for $x \neq 0$,

$$\lim_{n\to\infty} P^n((x,0),A) = \frac{1}{2};$$

and by the deterministic, explosive nature of the chain when $\boldsymbol{\Phi} \in A^c$, it follows that for any compact subset $K \subset \mathsf{X}$,

$$\limsup_{n\to\infty} P^n((x,0),K) \le \frac{1}{2}.$$

We see that from initial conditions starting outside of $(0,0)$, the distributions $\{P^n((x,k),\cdot\,) : n \ge 0\}$ or even $\{1/N \sum_0^{N-1} P^n((x,k),\cdot\,) : N \ge 1\}$ are *not* tight, even though the mean return time to the compact set A is uniformly bounded. This shows that the uniform integrability condition imposed in Corollary 2.1 is not superfluous.

Note that this is a Feller Markov chain. Given the bound on $\mathsf{E}[\tau_A]$ it must therefore possess at least one invariant probability [5]. The invariant probability for this example is the point mass at $(0,0)$.

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