INERTIA GROUPS AND FIBERS

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Let K be a global field and X, Y two proper, connected K-schemes, with X normal and Y regular. Let $f : X \to Y$ be a finite, flat, generically Galois K-morphism which is tamely ramified along a normal crossings divisor on Y. For closed points $y \in Y$ outside of the branch locus of f and points $x \in f^{-1}(y)$, we use the 'geometric' inertia groups of f and intersection numbers involving y and the branch locus in order to compute the 'arithmetic' inertia groups in $\mathrm{Gal}(K(x)/K(y))$ at all places of $K(y)$ except for those which lie over some fixed finite set of places Σ_f of K, with Σ_f depending only on f. This generalizes a theorem of Beckmann, who considered geometrically connected, generically Galois covers of $\mathbf{P}^1_K$, with K a number field.

Introduction

Let X and Y be proper, normal, connected schemes over a field K, and let $f : X \to Y$ be a finite, flat K-morphism which is generically Galois (i.e., the extension of function fields $K(Y) \hookrightarrow K(X)$ is Galois) with Galois group G. It is well-known that for the Zariski-open complement $U \subseteq Y$ of the branch locus of f, the map $f^{-1}(U) \to U$ is a (right) G-torsor. Thus, for any $y \in U$ and $x \in f^{-1}(y)$, the extension of fields $K(x)/K(y)$ is Galois and the stabilizer in G of x maps isomorphically to the Galois group $\mathrm{Gal}(K(x)/K(y))$. In particular, when the fiber $f^{-1}(y)$ is irreducible, then $\mathrm{Gal}(K(x)/K(y)) = G$.

If K is a global field, it is natural to ask how the injection $\mathrm{Gal}(K(x)/K(y)) \hookrightarrow G$ relates 'arithmetic' inertia groups in $\mathrm{Gal}(K(x)/K(y))$ with 'geometric' inertia groups in G, corresponding to ramification in the map f. The same question can be asked more generally when K is the function field of a connected, normal, noetherian scheme S with positive dimension, where 'arithmetic' ramification in $\mathrm{Gal}(K(x)/K(y))$ corresponds to ramification in $K(x)$ of the valuations on $K(y)$ arising from codimension 1 points of the normalization of S in $K(y)$.

A special case of this question was investigated by S. Beckmann. She considered the case when K is a number field (with integer ring $\mathscr{O}_K$), $Y = \mathbf{P}^1_K$, and X is a *geometrically connected* curve over K. Let $a_1, \dots, a_m$ be the finitely many branch points of f. Since K has characteristic 0, so f is tamely ramified over each a_i, the inertia groups of f over the a_i's are *cyclic* subgroups of G.

For any closed point $y \in \mathbf{P}^1_K$, it is not difficult to show that the scheme-theoretic closure $\overline{\{y\}}$ in $\mathbf{P}^1_{\mathscr{O}_K}$, which is proper over $\mathscr{O}_K$, is also quasi-finite and therefore finite over $\mathscr{O}_K$. For example, if $y, y' \in \mathbf{P}^1_K$ are distinct closed points, then $\overline{\{y\}} \cap \overline{\{y'\}}$ is artinian. In particular, when y is a K-rational point distinct from the a_i's, the intersection $\overline{\{y\}} \cap \overline{\{a_i\}}$ is an artinian closed subscheme of $\overline{\{y\}} \simeq \mathrm{Spec}(\mathscr{O}_K)$. Let $I_{\mathfrak{p}}(y, a_i) \geq 0$ denote the length of the part of $\overline{\{y\}} \cap \overline{\{a_i\}}$ which lies over $\mathfrak{p} \in \mathrm{Spec}(\mathscr{O}_K)$, so obviously $I_{\mathfrak{p}}(y, a_i) = 0$ for all but finitely many $\mathfrak{p}$ (depending on y and a_i).

Let Σ_f denote the finite set of primes $\mathfrak{p}$ of $\mathscr{O}_K$ at which one of the following occurs:

- some $K(a_i)/K$ is ramified at $\mathfrak{p}$,
- $I_{\mathfrak{p}}(a_i, a_j) > 0$ for some $i \neq j$ (i.e., the closures $\overline{\{a_i\}}$ and $\overline{\{a_j\}}$ in $\mathbf{P}^1_{\mathscr{O}_K}$ meet over $\mathfrak{p}$),
- the $\mathfrak{p}[t]$-adic valuation on $K(\mathbf{P}^1_K) = K(t)$ is ramified in $K(X)$,
- $\mathfrak{p}$ divides the degree of f.

Note that Σ_f can be effectively determined and depends only on the the geometry of f and the arithmetic in some of the fibers of f. Beckmann proved the following result:

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Theorem 0.1. [B, Prop 4.2] *For any prime* $\mathfrak{p} \notin \Sigma_f$*, any* K*-rational point* $y \in \mathbf{P}^1_K$ *distinct from the* a_i*'s, and any* $x \in f^{-1}(y)$*,* $\mathfrak{p}$ *is unramified in* $K(x)$ *if* $I_{\mathfrak{p}}(a, a_i) = 0$ *for all* i*, and if some* $I_{\mathfrak{p}}(y, a_{i_0}) > 0$ *then the inertia groups at* $\mathfrak{p}$ *in* $\mathrm{Gal}(K(x)/K)$ *are the subgroups of* $I_{\mathfrak{p}}(y, a_{i_0})$*th powers in the (cyclic) inertia groups of* f *over* a_{i_0} *(these subgroups may be trivial).*

One interesting application of Theorem 0.1 is given in [DG, §3], where it is used to analyze the finiteness of the number of solutions to certain generalized Fermat equations. Another application of Theorem 0.1 is that, in certain cases (e.g., generically Galois coverings of $\mathbf{P}^1_K$ with prime power degree) it allows one to make the Hilbert Irreducibility Theorem effective by using just the Chinese Remainder Theorem rather than an effective version of the Cebotarev Density Theorem. This is explained in [B, 1.2, 1.3].

Beckmann's proof of Theorem 0.1 uses topological considerations over $\mathbf{C}$ (hence the geometric connectedness hypothesis) and algebraic calculations based on Galois theory, Abhyankar's Lemma, and the fact that the base is $\mathbf{P}^1_K$. The calculations use that K has characteristic 0, that the discrete valuations on K have perfect residue fields, and that y is K-rational. Grothendieck's theory of specialization for the tame fundamental group [SGA1, XIII, §2.10ff] does not seem to yield Theorem 0.1, but it suggests that Beckmann's result is best understood via geometry and that such a viewpoint should lead to a similar result for generically Galois, tamely ramified coverings of curves and higher-dimensional varieties over more general base fields (such as global fields with positive characteristic).

The purpose of this paper is to prove such a generalization. The requirement above that Σ_f contains primes dividing the degree of f is used in order to avoid wild ramification. Geometric considerations will show that the other conditions in the definition of Σ_f (which *do* have geometric significance) already take care of this problem. Most of our effort is devoted to reformulating the basic problem in the correct geometric framework. Once this is done, the actual proof of our generalization is very conceptual. For example, we will see that the 'arithmetic' condition that all $K(a_i)/K$ are unramified at $\mathfrak{p} \notin \Sigma_f$ is simply a convenient way to ensure that a certain ramification divisor is a normal crossings divisor over $\mathfrak{p}$.

We now describe our version of Theorem 0.1 only in the case of curves, with K a global field (the essential point is that we can include global fields with positive characteristic). Let G and $f : X \to Y$ be as at the beginning, but assume X and Y are curves. For each branch point $a_i \in Y$ of f, let $I_i \subseteq G$ be an inertia group for f at a_i. Assume that f is *tamely ramified* over each a_i (this is automatic when K has characteristic 0), so the I_i's are cyclic. Let $\mathscr{B}_{X/Y} \hookrightarrow Y$ be the (reduced) branch scheme of f; i.e., the closed subscheme defined by the annihilator of $f_*\Omega^1_{X/Y}$ on Y. The underlying set of $\mathscr{B}_{X/Y}$ is the set of a_i's. A special case of our main result (Theorem 2.4) is the following:

Theorem 0.2. *With the notation as above, choose any closed point* $y \in Y$ *distinct from the* a_i*'s and any* $x \in f^{-1}(y)$*, so* $K(x)/K(y)$ *is a finite Galois extension with* $\mathrm{Gal}(K(x)/K(y)) \subseteq G$*. There exists a finite set of non-archimedean places* Σ_f *of* K*, depending only on* f *(and not on* x *or* y*), so that for any non-archimedean place* v *of* $K(y)$ *not lying over* Σ_f*, we can define an intersection number* $(y, \mathscr{B}_{X/Y})_v \geq 0$ *with the following properties:*

(1) *For all but finitely many* v*, depending on* y*,* $(y, \mathscr{B}_{X/Y})_v = 0$*.*
(2) *If* $(y, \mathscr{B}_{X/Y})_v = 0$*, then* v *is unramified in* $K(x)$*.*
(3) *If* $(y, \mathscr{B}_{X/Y})_v > 0$*, the inertia groups at* v *in* $\mathrm{Gal}(K(x)/K(y))$ *are conjugate in* G *to the subgroup of* $(y, \mathscr{B}_{X/Y})_v$*th powers in one of the cyclic groups* I_i *(this subgroup may be trivial)*

Note that in Theorem 0.2, we do not require y to be a K-rational point on Y (nor do we require that $K(y)/K$ is separable or that Y is smooth over K at y). The definition of $(y, \mathscr{B}_{X/Y})_v$ and the particular i which occurs in the third case of Theorem 0.2 can be described in terms of the geometry of certain integral models of f (as we will see in §2). Theorem 0.2 is a special case of Theorem 2.4 below, in which branch points are allowed to "meet over v" and K can be the fraction field of any noetherian normal domain A with infinitely many height 1 primes (i.e., A has dimension > 1 or A is Dedekind and has infinitely many maximal ideals), provided that either

- A is excellent, or
- the curves X and Y are smooth over K and the residue field extensions $K(a_i)/K$ are separable.

In particular, no excellence hypotheses are needed if K has characteristic 0. The excellent case includes most interesting cases and we can formulate Theorem 0.2 as a result about curves over the fraction fields of excellent Dedekind domains, rather than as a result about curves over global fields. Thus, Theorem 0.1 is an algebro-geometric assertion, not an arithmetic one.

After a review of some relevant background in §1, we begin the setup for the proof of Theorem 0.2 (or rather, a more general result) in §2. This proof consists of two steps. The first step is mostly linguistic and consists of constructing the right kind of 'integral model' of $f : X \to Y$. This is not difficult. Arithmetic ramification in the closed fibers of f can be viewed as geometric data in such models. The second step is to combine this geometric viewpoint with the étale local description of tamely ramified maps (via Abhyankar's Lemma) in order to relate the 'arithmetic' inertia groups in the closed fibers of f with the 'geometric' inertia groups in G of the map f. The desired result follows from this by local calculations, as we explain in §3. In §4, we consider cases in which f is not generically Galois.

Although our motivation is the case of tamely ramified coverings of curves, we formulate most of our discussion in arbitrary dimension for tamely ramified covers with a normal crossings branch divisor. This greater generality should clarify the geometric reasoning.

Terminology. For any local ring $(R, \mathfrak{m})$, we let R^{h} denote the *henselization* of R. The local R-scheme $\mathrm{Spec}(R^{\mathrm{h}})$ is the limit of 'all' pointed étale maps $(X, x) \to (\mathrm{Spec}(R), \mathfrak{m})$ with $k(x) = R/\mathfrak{m}$. If we choose a separable closure $i : R/\mathfrak{m} \hookrightarrow (R/\mathfrak{m})_{\mathrm{sep}}$ and instead require our pointed étale maps to come equipped with an embedding of $k(x)$ into $(R/\mathfrak{m})_{\mathrm{sep}}$ over i, the resulting limit is called a *strict henselization* $R^{\mathrm{sh}}_{\mathfrak{m},i}$ of R. An isomorphism between separable closures of $R/\mathfrak{m}$ uniquely lifts to an isomorphism between the corresponding strict henselizations. If the choice of i does not matter or is clear from context, we write $R^{\mathrm{sh}}_{\mathfrak{m}}$ instead of $R^{\mathrm{sh}}_{\mathfrak{m},i}$.

For basic properties of henselizations and strict henselizations, including universal mapping properties and the compatibility of formation of (strict) henselizations with respect to finite maps (especially surjections), we refer the reader to [EGA, IV_4, 18.6–18.8]. In particular, since a semi-local ring which is integral over a local henselian ring is a finite product of henselian local rings [EGA, IV_4, 18.6.8], it follows by a direct limit argument that the normalization of a henselian *local* domain in a finite extension of its fraction field is again a henselian *local* domain. We use this without comment.

For any map of locally noetherian schemes $X \to S$, we say that X is a *regular* (resp. *normal*) *S-scheme* when X is intrinsically regular (resp. normal) as a scheme; that is, all of the noetherian local rings $\mathscr{O}_{X,x}$ for $x \in X$ are regular (resp. normal). In particular, this does not mean that the map $X \to S$ is a regular (resp. normal) morphism in the sense of [EGA, IV_2, 6.8.1] (i.e., the fibers of $X \to S$ do not have to be geometrically regular (resp. geometrically normal)).

For any field k, we define a *curve over k* to be a separated, finite type k-scheme with pure dimension 1. We do not require curves to be connected (this is purely for technical reasons, so we can avoid geometric connectivity assumptions and still use change of the base field).

A local map of local rings $A \to B$ is said to be *essentially étale* if B is a local ring on an étale A-algebra. For example, henselizations and strict henselizations of a local ring R are constructed as direct limits of essentially étale R-algebras. A map of schemes $g : V \to W$ is said to be *ind-étale* if, for every $v \in V$, the $\mathscr{O}_{W,g(v)}$-algebra $\mathscr{O}_{V,v}$ is a direct limit of essentially étale $\mathscr{O}_{W,g(v)}$-algebras. This property is preserved by base change.

Notation. When forming fiber products $X \times_Y Z$ with X, Y, or Z equal to an affine scheme $\mathrm{Spec}(A)$, we usually write A instead of $\mathrm{Spec}(A)$ in the fiber product notation (e.g., $X \times_Y A$ if $Z = \mathrm{Spec}(A)$).

For a point x on a scheme X, we write $k(x)$ for the residue field of the local ring $\mathscr{O}_{X,x}$. The same notation will be used without any risk of confusion in case X is a scheme of finite type over a base field which is also denoted k. However, in such cases, we will often write $k(X)$ for the product of the residue fields of X at its finitely many generic points.

A separable closure of a field K is denoted K_{sep}.

If R is a strictly henselian local ring and $e \in \mathbf{Z}$ is a unit in R, we often write μ_e, instead of $\mu_e(R)$, for the cyclic group of eth roots of unity in R. This should not cause confusion.

The symbol $\coprod$ is used to denote a disjoint union.

For a ring R, we denote by $R^\times$ the group of units in R. For a module M over R, we denote by $\mathrm{ann}_R(M)$ the ideal of elements $r \in R$ which annihilate M.

For elements $g_1, \dots, g_n$ in a group G, we denote by $\langle g_i \rangle$ the subgroup of G generated by the g_i's.

1. Galois Groups, Inertia Groups, and Tame Covers

Since we are concerned with Galois groups and inertia groups from a geometric point of view, we begin with a review of some standard geometric facts concerning inertia groups and Galois maps. All assertions in this section are explained in much greater detail in [SGA1, Exp. V, XIII].

Let G be a finite group. A faithfully flat, quasi-compact map $\pi : X \to Y$ between two schemes is said to be a (right) *G-torsor* if we are given a right action of G on X (over Y) such that the map of schemes $X \times G \to X \times_Y X$ given by "$(x, g) \mapsto (x, x.g)$" is an isomorphism. This implies that π is finite étale with constant degree equal to the order of G. When X and Y are connected and non-empty, a right action of G on X (over Y) makes π a right G-torsor if and only if π is étale with constant degree equal to the order of G and the map of groups $G \to \mathrm{Aut}(X/Y)^0$ is an isomorphism, where $(\cdot)^0$ denotes the "opposite group." In this case, we say that π is *Galois* and G is the *Galois group.* The property of being a G-torsor is preserved by base change, whereas the property of being Galois is not, due to the connectedness conditions. When passing to the fibers of a torsor, the following result will be used frequently:

Lemma 1.1. *Let G be a finite group, $Y = \mathrm{Spec}(k)$ for a field k, and $X = \mathrm{Spec}(k')$ for a non-zero, finite k-algebra k' which is equipped with a left G-action. The finite flat map $X \to Y$ is a* (*right*) *G-torsor if and only if X is reduced, $k(x)/k$ is Galois for all $x \in X$, G acts transitively on X, and the stabilizer group in G of each $x \in X$ maps isomorphically to* $\mathrm{Gal}(k(x)/k) = \mathrm{Aut}(\mathrm{Spec}(k(x))/\mathrm{Spec}(k))^0$.

Consider a finite, flat, generically étale map $\pi : X \to Y$ between normal noetherian schemes. The *branch scheme* $\mathscr{B}_{X/Y}$ is defined to be the closed subscheme of Y defined by the annihilator of $\pi_*(\Omega^1_{X/Y})$, so the complement of the branch scheme is the largest open in Y over which π is étale. Let $\mathrm{Spec}(k(Y))$ be the scheme of generic points of Y and let $\mathrm{Spec}(k(X))$ be the scheme of generic points of X. We say that π is a *generic G-torsor* if the map

$$X \times_Y \mathrm{Spec}(k(Y)) = \mathrm{Spec}(k(X)) \to \mathrm{Spec}(k(Y))$$

is a G-torsor. Since X is the normalization of Y in $k(X)$, every automorphism of $k(X)$ over $k(Y)$ uniquely extends to an automorphism of X over Y. Thus, by a normalization argument, we see that a *generic* G-torsor structure on π is 'the same' as a G-torsor structure on $\pi^{-1}(U) \to U$, where U is the complement of $\mathscr{B}_{X/Y}$ in Y.

Suppose that π is a generic G-torsor. From Lemma 1.1, it follows that for any point $y \in Y$ outside of $\mathscr{B}_{X/Y}$ and any $x \in \pi^{-1}(y)$, the extension $k(x)/k(y)$ is a finite Galois extension and the stablizer in G of x maps isomorphically to $\mathrm{Gal}(k(x)/k(y))$. Moreover, the action of G on $\pi^{-1}(y)$ is transitive, so the subgroups $\mathrm{Gal}(k(x)/k(y))$ are conjugate in G for $x \in \pi^{-1}(y)$. On the other hand, for any $y \in \mathscr{B}_{X/Y}$, it can be shown that G acts transitively on $\pi^{-1}(y)$ and for any $x \in \pi^{-1}(y)$ the extension $k(x)/k(y)$ is merely normal (perhaps inseparable), with the stabilizer of x in G surjecting onto $\mathrm{Aut}(k(x)/k(y))$. The kernel of this surjection is defined to be the *inertia group* $I(x|y)$ of x over y. These inertia groups are conjugate in G for all x over a fixed $y \in Y$, and we call any $I(x|y)$ an *inertia group over y.*

Now drop the hypothesis that π is a generic G-torsor, but assume that Y is regular. Let $\{a_i\}$ be the generic points of $\mathscr{B}_{X/Y}$. Since Y is regular, by the Zariski-Nagata theorem on purity of the branch locus [SGA1, Exp X, Thm 3.1], the points $a_i \in Y$ are all codimension 1 points. We say that π is *tamely ramified* if, for all i and all $x \in \pi^{-1}(a_i)$, the natural map of discrete valuation rings $\mathscr{O}_{Y,a_i} \to \mathscr{O}_{X,x}$ is tamely ramified in the usual sense; i.e., the residue field extension $k(x)/k(a_i)$ is separable and the ramification degree of $\mathscr{O}_{X,x}$ over $\mathscr{O}_{Y,a_i}$ is prime to the characteristic of $k(a_i)$. Of course, when π is a tamely ramified generic G-torsor, the inertia groups in G over each a_i are cyclic with order relatively prime to the characteristic of $k(a_i)$. It is well-known that if a generic G-torsor π is tamely ramified over a_i and $x \in \pi^{-1}(a_i)$, then there is a canonical isomorphism

$$I(x|a_i) \simeq \mu_{e(x|a_i)}(k(a_i)_{\mathrm{sep}}),$$

where $e(x|a_i)$ is the ramification degree of π at x.

In the special case where X and Y are of finite type over a field k, with π a generic G-torsor and a tamely ramified k-morphism, the inertia groups of G over a_i are 'geometric' in nature, in the sense that they behave well with respect to a separable extension of the base field. More precisely, consider base change by a separable extension k' of k. Each a_i decomposes into a finite set of (reduced) points $a_{ij} \in Y \times_k k'$. The a_{ij}'s are the generic points of the branch scheme of $\pi \times_k k'$. The following lemma is not difficult to prove and will be used later on, when we compute inertia groups after making a separable change of the base field.

Lemma 1.2. *For any j, the set of inertia groups in G over a_{ij} is equal to the set of inertia groups in G over a_i.*

We will be particularly interested in the case of tamely ramified π for which the branch scheme $\mathscr{B}_{X/Y}$ is a normal crossings divisor (which is automatic in the case of curves). Recall that an effective Cartier divisor D on a regular scheme Y is said to be a *strictly normal crossings divisor* if D is Zariski-locally defined by a product of part of a regular sequence of parameters. In more geometric terms, D is reduced and, Zariski-locally on Y, is set-theoretically a union of regular hypersurfaces, arbitrary intersections of which are again regular. Slightly more generally, if we relax 'Zariski local' to 'étale local', we get the notion of a *normal crossings divisor* D on a regular scheme Y.

The relative version of this concept goes as follows. If Y is a smooth scheme over some base S, then a *normal crossings divisor on Y relative to S* is an effective relative Cartier divisor D over Y over S which is étale-locally (on Y) isomorphic to the crossing of several coordinate hyperplanes in affine space (over S). Of course, when S is regular, this relative notion is a special case of the non-relative notion defined above.

For a rather degenerate example of normal crossings divisors, consider an effective divisor D on a regular curve C over a field k. The divisor D is a normal crossings divisor precisely when the corresponding closed subscheme of C is reduced, and D is a normal crossings divisor relative to k precisely when, in addition, $k(x)/k$ is separable for each of the finitely many closed points x in the support of D. Thus, in the case of curves over a field, the notion of a normal crossings divisor is not interesting. However, this concept will clarify what is really going on in our later considerations.

As a convenient reference for later on, we now mention some basic properties of normal crossings divisors.

Lemma 1.3. *An effective Cartier divisor D on an excellent, regular scheme Y is a (strictly) normal crossings divisor in a Zariski open neighborhood of $y \in Y$ if and only if the induced divisor D_y on the local scheme $\mathrm{Spec}(\mathscr{O}_{Y,y})$ is a (strictly) normal crossings divisor.*

Let D be an effective Cartier divisor on a quasi-compact, quasi-separated smooth S-scheme Y, with D a normal crossings divisor relative to S. If $\{S_i\}$ is a (filtered) inverse system of quasi-compact, quasi-separated schemes with affine transition maps and inverse limit S, then the pair (D, Y) over S is the base change of an analogous pair (D_i, Y_i) over some S_i, with D_i a normal crossings divisor on Y_i relative to S_i.

Proof. The essential content of the proof consists of the many tedious results in [EGA, IV$_3$, §8–§12] on inverse limits of schemes and the behavior of all 'reasonable' properties of schemes with respect to such limits. The idea is this. If $S = \mathrm{Spec}(A)$ and $A = \varinjlim A_i$ is a direct limit of rings, then any finitely presented A-scheme is defined in terms of finitely many equations, all of whose coefficients come from some A_i, and so any such A-scheme should be a base change of a finitely presented A_i-scheme; likewise with finitely presented quasi-coherent sheaves or maps between such schemes or sheaves. The theory in [EGA, IV$_3$, §8–§12] verfies that not only is this true, but more importantly all reasonable properties of schemes, sheaves, and morphisms, including flatness of maps and exactness of suitable complexes of quasi-coherent sheaves (which are not visibly 'defined by finitely many equations'), also 'descend' through such limits. For example, if (D, Y) is as in the second part of the lemma, with D defined by some sequence of functions $\{f_1, \dots, f_n\}$ on Y which form a regular sequence relative to S, then if we desend the data of the f_j's to some S_i-scheme Y_i, we want this to still be a regular sequence relative to S_i, or perhaps relative to $S_{i'}$ after base change to $S_{i'}$ for some $i' \geq i$. Intuitively, since the desired property holds after base change all the way up to S, it should hold after base change to some $S_{i'}$. This follows from [EGA, IV$_3$, 11.3.9]. Using the theory of limits in this way, we get the second part of the lemma.

For the first part of the lemma, we use this theory of limits (applied in the elementary case of a local ring viewed as a limit of 'basic affine opens') and the fact that the regular locus on an excellent scheme is always open [EGA, IV_2, 7.8.3(*iv*)]. The theory of limits takes care of regularity of sequences and étale neighborhoods, and the openness of the regular locus is relevant because we need to know that if a closed subscheme Z of Y through y (such as one defined by several of the equations cutting out D) has regular local ring $\mathscr{O}_{Z,y}$, then Z has an open neighborhood of $y \in Z$ which is regular; this is exactly the meaning of 'openness of the regular locus' on the (excellent) scheme Z. ■

For our purposes, the importance of the notion of a normal crossings divisor is its role in:

Lemma 1.4. (Abhyankar's Lemma) *Let Y be a regular noetherian scheme, X a normal noetherian scheme, and $f : X \to Y$ a finite, flat, generically étale map which is tamely ramified. If the support of the branch scheme of f coincides with the support of a normal crossings divisor D on Y, then*

- *X is regular,*
- *$\mathscr{B}_{X/Y} = D$ as closed subschemes of Y, so $\mathscr{B}_{X/Y}$ is a normal crossings divisor on Y,*
- *for each $y \in \mathscr{B}_{X/Y}$ and $x \in f^{-1}(y)$, there is an isomorphism of $\mathscr{O}_{Y,y}^{\mathrm{sh}}$-algebras*

$$\mathscr{O}_{X,x}^{\mathrm{sh}} \simeq \mathscr{O}_{Y,y}^{\mathrm{sh}}[T_1, \ldots, T_r]/(T_1^{e_1} - f_1, \ldots, T_r^{e_r} - f_r), \tag{1.1}$$

where $f_1, \ldots, f_r$ define the normal crossings divisor D in an étale neighborhood of y and $e_1, \ldots, e_r \geq 1$ are relatively prime to the characteristic of $k(y)$.

Proof. The essential content is [SGA1, XIII, Prop 5.2, Cor 5.3], but at the suggestion of the referee we explain why. For all $y \in Y$, we have

$$X \times_Y \mathscr{O}_{Y,y}^{\mathrm{sh}} \simeq \prod_{x \in f^{-1}(y)} \mathscr{O}_{X,x}^{\mathrm{sh}}.$$

Since the hypotheses are preserved by the base change $\mathrm{Spec}(\mathscr{O}_{Y,y}^{\mathrm{sh}}) \to Y$ (e.g., the normality and tameness assumptions are not harmed by ind-étale base change) and it suffices to check the conclusions after all such (flat) base changes (e.g., $\mathscr{O}_{X,x}^{\mathrm{sh}}$ is regular if and only if $\mathscr{O}_{X,x}$ is regular), we are reduced to the case where Y is local and strictly henselian with closed point y, so $X = \coprod X_i$ is a finite disjoint union of local and strictly henselian schemes X_i. If we can prove the theorem for each $X_i \to Y$, then

$$\mathscr{B}_{X/Y} = \bigcap_i \mathscr{B}_{X_i/Y} = \bigcap_i D = D.$$

Thus, we may assume X is connected.

Recall that the tame fundamental group $\pi_1^t(Y, D)$ of Y relative to D classifies *connected* normal finite Y-schemes which are étale over $Y - D$ and tamely ramified over the generic points of D (e.g., $X \to Y$). Since Y is regular, local, and strictly henselian, [SGA1, XIII, Cor 5.3] gives the determination of the tame fundamental group $\pi_1^t(Y, D)$ for Y relative to the divisor D. This group is *abelian*, so X is generically Galois over Y. Let D be defined by f_i's, so these form a regular sequence cutting out regular subschemes of Y. In particular, the f_i's are relatively prime in the UFD $\mathscr{O}_{Y,y}$. Once we know that (1.1) holds, it is clear by the *definition* of normal crossings divisor and the regularity of Y that X is regular. Likewise, since the exponents in (1.1) are invertible on the regular local Y, a direct computation gives the equality of ideal sheaves

$$\mathrm{ann}(f_* \Omega_{X/Y}^1) = \bigcap (f_i \mathscr{O}_Y) = (\prod f_i) \mathscr{O}_Y,$$

so $\mathscr{B}_{X/Y} = D$ as closed subschemes of Y.

Thus, we just have to verify (1.1). Since $X \to Y$ is generically Galois, the (tame) ramification degrees of $X \to Y$ at all points over the generic point of (f_i) are *equal*. Calling this common number e_i, it follows from [SGA1, XIII, Prop 5.2] that *all* of the e_i's are actually invertible on all of Y and for the *regular* scheme

$$Y' = \mathrm{Spec}\, \mathscr{O}_{Y,y}[T_1, \ldots, T_r]/(T^{e_1} - f_1, \ldots, T^{e_r} - f_r),$$

the normalization of the reduced Y'-scheme $X \times_Y Y'$ is finite étale over Y'. But Y' is local and strictly henselian, so the normalization of $X \times_Y Y'$ is a finite disjoint union of copies of Y'. Choosing any one of these components, we claim that the natural finite map $Y' \to X$ is an isomorphism. Obviously

$$\mathrm{Aut}(Y'/Y) \simeq \mu_{e_1} \times \cdots \times \mu_{e_r},$$

and the intermediate X corresponds to some subgroup G in $\mathrm{Aut}(Y'/Y)$. If we can show that $G = 1$, then $Y' = X$ by connectedness/normality and we are done. To see that $G = 1$, it suffices to check that G projects to 1 in each μ_{e_i}. But the quotient μ_{e_i} of $\mathrm{Aut}(Y'/Y)$ corresponds to the inertia group for Y' over the codimension 1 generic point ξ_i of (f_i), and $\mathrm{Aut}(X/Y)$ compatibly projects onto the same μ_{e_i} (rather than a proper quotient of it) by the very definition of e_i; what we are using here is that Y' and X have the *same* (tame) ramification degrees over the ξ_i's. The kernel G of $\mathrm{Aut}(Y'/Y) \twoheadrightarrow \mathrm{Aut}(X/Y)$ therefore does project to 1 in each μ_{e_i}, as desired. ■

When the conditions in Abhyankar's Lemma hold, we say that f is *tamely ramified along a normal crossings divisor*. As we noted in the proof above, the map of fraction fields corresponding to (1.1) is Galois, with Galois group canonically isomorphic to $\mu_{e_1} \times \cdots \times \mu_{e_r}$, where $(\zeta_1, \ldots, \zeta_r)$ sends T_j to $\zeta_j T_j$.

2. Integral Models

For the rest of this paper, we fix a finite, generically étale, surjective map $f_K : X_K \to Y_K$ between proper, normal schemes over a field K, with Y_K regular and X_K, Y_K of pure dimension $d \geq 1$. Assume also that f_K is tamely ramified along a normal crossings divisor. We want to relate ramification in the map f_K with 'ramification of codimension 1 points' in the closed fibers of f_K.

In order to make sense of 'ramification of codimension 1 points' in the fibers $f_K^{-1}(y)$ for closed points $y \in Y_K$, we assume that K is the function field of a connected, normal, noetherian scheme S. The cases of most interest below will be when S has infinitely many codimension 1 points (i.e., the generic point is not open in S). Fix a choice of S. For example, when K is finitely generated over its prime field (resp. over a field k), we can choose S to be a finite type scheme over $\mathbf{Z}$ (resp. over k). For technical reasons, we need to set things up for more general S, including the case $S = \mathrm{Spec}(R)$ for a discrete valuation ring R. The case $S = \mathrm{Spec}(K)$ is uninteresting.

Consider an arbitrary finite extension K' of K and let $S' \to S$ denote the normalization of S in K' [EGA, II, 6.3.6]. A *codimension* 1 *point* s' *in* K' is defined to be a codimension 1 point $s' \in S'$. This concept depends on the choice of S, but it generalizes the usual notion of 'prime (or non-archimedean place) in a global field'. We define the local ring $\mathscr{O}_{s'} = \mathscr{O}_{S',s'} \subseteq K'$.

Although S' might not be noetherian (e.g., if K'/K is not separable and S is not Japanese), by the Krull-Akizuki Theorem [M, Thm 11.7 and Corollary] we know that the normalization $\widetilde{\mathscr{O}_s}$ of $\mathscr{O}_s$ in K' is a semi-local Dedekind domain whose maximal ideals have residue fields which are of finite degree over $k(s)$. Thus, $\mathscr{O}_{s'}$ is a discrete valuation ring with residue field finite over $k(s)$ and there are only finitely many codimension 1 points s' in K' which lie over a given codimension 1 point $s \in S$ (corresponding to the finitely many maximal ideals in $\widetilde{\mathscr{O}_s}$). Terminology from classical valuation theory (e.g., unramified, tamely ramified, inertia groups) will be used when discussing these codimension 1 points.

In order to study ramification of codimension 1 points in the closed fibers of f_K, we will need to use certain models of f_K over open subschemes of S. The construction and basic properties of the models we need are straightfoward, and are intended to extend properties of f_K and X_K, Y_K over non-empty opens in S.

We define a *normal integral model* of f_K to be a triple (U, f_U, i) where

- $U \subseteq S$ is a non-empty open subscheme,
- $f_U : X_U \to Y_U$ is a finite flat map between proper, flat U-schemes, with X_U *normal* and Y_U regular,
- i is an identification of the K-fiber map $f_U \times_U K$ with f_K,
- all fibers of $X_U \to U$ and $Y_U \to U$ are of pure dimension d,
- the branch scheme $\mathscr{B}_{X_U/Y_U} \hookrightarrow Y_U$ is the scheme-theoretic closure of its generic fiber $\mathscr{B}_{X_K/Y_K}$,
- the branch scheme $\mathscr{B}_{X_U/Y_U}$ is a normal crossings divisor in Y_U.

When X_K and Y_K are K-smooth and $\mathscr{B}_{X_K/Y_K}$ is a normal crossings divisor relative to K, we define a *smooth integral model* of f_K to be a triple (U, f_U, i) as above, except that we require X_U and Y_U to be U-smooth (rather than normal and regular, respectively) and we require $\mathscr{B}_{X_U/Y_U}$ to be a normal crossings divisor in Y_U relative to U. When S is regular (e.g., the spectrum of a discrete valuation ring), smooth integral models are normal integral models.

The terminology *integral model* refers to either a smooth or normal integral model, with the understanding that X_K, Y_K are K-smooth and $\mathscr{B}_{X_K/Y_K}$ is a normal crossings divisor in Y_K relative to K whenever we speak of smooth integral models.

Lemma 2.1. *Let S, K, and f_K be as above.*

(1) *When S is excellent, a normal integral model f_U of f_K exists over some non-empty open U in S. If X_K and Y_K are K-smooth and $\mathscr{B}_{X_K/Y_K}$ is a normal crossings divisor in Y_K relative to K, then a smooth integral model f_U of f_K exists over some non-empty open U in S, without any excellence hypotheses.*

(2) *If f_U, f_V are integral models for f_K over non-empty opens $U, V \subseteq S$, then for a suitable non-empty open $W \subseteq U \cap V$, the integral models $f_U \times_U W$ and $f_V \times_V W$ are isomorphic (in a necessarily unique way).*

(3) *Let f_U be an integral model of f_K and U' a connected, normal, noetherian scheme with function field K'. If $U' \to U$ is an ind-étale map (so K' is separable algebraic over K), then $f_U \times_U U'$ is an integral model of $f_K \times_K K'$.*

Proof. The fact that integral models can be isomorphic in at most one way follows from flatness over the integral scheme S. Since the properties of integral models are analogues of properties which are satisfied by f_K and X_K, Y_K over the generic point $\mathrm{Spec}(K)$ of S, the existence and uniqueness of integral models (aside from regularity and normality properties) follows from Lemma 1.3 and various direct limit and constructibility results in [EGA, IV_3, §8–§12]. To illustrate the basic idea, once we construct a map $f_U : X_U \to Y_U$ between finite type U-schemes for some open U which induces f_K over $\mathrm{Spec}(K)$, we want to know that, if we shrink U a little, then f_U should be finite flat and the fibers of $X_U \to U$ should be pure dimension d. Since $f_K = f_U \times_U K$ is finite flat, by [EGA, IV_3, 9.6.1(*vi*), 11.2.6.1(*ii*)] it follows that f_U is finite flat for small U (we view the local scheme $\mathrm{Spec}(K)$ as the limit of its open affine neighborhoods). Now consider the question of fiber dimensions. Define Z to be the set Z of points $u \in U$ for which X_u is pure d-dimensional, so Z contains the generic point of U. We want Z to contain an open neighborhood of this generic point, so it suffices to show that Z is constructible, or equivalently that its complement is constructible. Since the image of a constructible set under f_U is again constructible, it suffices to show the constructibility of the set of points $x \in X_U$ for which $(X_U)_{f_U(x)}$ does *not* have dimension d at x, which is equivalent to the constructibility of its complement in X_U: the set of $x \in X_U$ at which $(X_U)_{f_U(x)}$ has dimension d. The constructibility of this latter set follows from [EGA, IV_3, 9.9.1] (and the equivalence of constructibility and local constructibility on noetherian schemes). The other properties (properness, etc.) follow by a similar kind of technique, via the theorems in [EGA, IV_3, §8–§12]

In order to get the regularity and normality conditions when S is excellent, it suffices to show more generally that if $Z \to U$ is a proper scheme with $Z \times_U K$ regular (resp. normal), then $Z \times_U V$ is regular (resp. normal) for some non-empty open $V \subseteq U$. This is immediate from properness considerations and the openness of the regular (resp. normal) locus in an excellent scheme [EGA, IV_2, 7.8.3(*iv*)].

For the last part of the lemma, we note that formation of the branch scheme commutes with flat base change. Since ind-étale maps are flat,

$$\mathscr{B}_{X_U/Y_U} \times_U U' \simeq \mathscr{B}_{X_{U'}/Y_{U'}},$$

where $X_{U'} = X_U \times_U U'$, $Y_{U'} = Y_U \times_U U'$. It remains to check that if a finite type U-scheme Z is regular, then so is the finite type U'-scheme $Z \times_U U'$. Since $Z \times_U U' \to Z$ is ind-étale, we just need to check that if $A \to B$ is a local ind-étale map of local noetherian rings, then A is regular if and only if B is. The natural map $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$ between strict henselizations is an isomorphism, so it suffices to treat the case of the ind-étale map $A \to A^{\mathrm{sh}}$. This is handled in [EGA, IV_4, 18.8.13]. ∎

The role of integral models is that they allow us to define certain intersection numbers $(Z'_K, Z_K)_{s'}$ as needed in Theorem 2.4 below (or Theorem 0.2 in the Introduction). The data used in the definition of these intersection numbers is a choice of integral model $f_U : X_U \to Y_U$ of f_K, a pair of disjoint closed subschemes Z'_K and Z_K on Y_K with respective dimensions 0 and $d-1$, and a choice of codimension 1 point s' in the field $K(y)$ for some closed point $y \in Y_K$.

Fix a choice of f_U and choose Z'_K, Z_K, and y. Let Z'_U and Z_U be the respective scheme-theoretic closures of Z'_K and Z_K in Y_U. By the valuative criterion for properness, the map

$$y : \mathrm{Spec}(K(y)) \to Y_K \tag{2.1}$$

over K extends uniquely to a map

$$y_{s'} : \mathrm{Spec}(\mathscr{O}_{s'}) \to Y_U \tag{2.2}$$

over U. The pullback of $Z_U \cap Z'_U$ under $y_{s'}$ is a closed subscheme of $\mathrm{Spec}(\mathscr{O}_{s'})$ with empty generic fiber, so it is artinian. We define the *intersection number* $(Z'_K, Z_K)_{s'} \geq 0$ to be the length of the corresponding local artinian quotient of $\mathscr{O}_{s'}$:

$$(Z'_K, Z_K)_{s'} = \mathrm{length}(y^*_{s'}(Z_U \cap Z'_U)). \tag{2.3}$$

This vanishes for all but the finitely many s' in $K(y)$ lying over the finitely many codimension 1 points in the (closed) image of $Z'_U \cap Z_U$ in U.

As an example, suppose $U = \mathrm{Spec}(\mathscr{O})$ for a Dedekind domain $\mathscr{O}$, $Y_U = \mathbf{P}^1_{\mathscr{O}}$, $K(y) = K$, and $Z'_K = \{a'\}$, $Z_K = \{a\}$ for closed points $a', a \in \mathrm{Spec}(K[t]) = \mathbf{A}^1_K \subseteq \mathbf{P}^1_K$ with $K(a') = K$. We have $K(a) \simeq K[t]/(q)$ for a unique irreducible, monic polynomial $q \in K[t]$. If $y = a'$, $\mathfrak{p}$ is the maximal ideal in $\mathscr{O}$ corresponding to s', and $q \in \mathscr{O}_{s'}[t] \subseteq K[t]$, then

$$(Z'_K, Z_K)_{s'} = \mathrm{ord}_{\mathfrak{p}}(q(a')).$$

As another example, in the special case where $y \in Z'_K$ (which is what we will use later), so $\mathrm{Spec}(\mathscr{O}_{s'}) \to Y_U$ factors through $Z'_U \subseteq Y_U$, we have

$$(Z'_K, Z_K)_{s'} = \mathrm{length}(y^*_{s'}(Z_U)). \tag{2.4}$$

For a fixed Z'_K, Z_K, and $y \in Y_K$, it is obvious that $(Z'_K, Z_K)_{s'} = 0$ for all but the finitely many codimension 1 points s' of $K(y)$ which lie over the image of $Z'_U \cap Z_U$ in U. Although (for fixed S) these intersection numbers depend heavily on the choice of integral model, by Lemma 2.1 we see that any two integral models define the same numbers $(Z'_K, Z_K)_{s'}$ for all but those s' lying over a finite set of codimension 1 points on S (depending only on the integral models being considered). Thus, the choice of integral model of f_K will be unimportant for our purposes.

Later calculations of these intersection numbers will only be possible after replacing U by its strict henselization at a codimension 1 point, due to the role of strict henselizations in Abhyankar's Lemma. Thus, we need to briefly discuss base change to strict henselizations. The following lemma (which is a variant on [EGA, IV_4, 18.8.11]) is useful for this purpose.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a discrete valuation ring with fraction field K and let R' denote the normalization of R in a finite extension K'/K, so R' is a semi-local Dedekind domain with $[R'/\mathfrak{m}' : R/\mathfrak{m}] < \infty$ for all $\mathfrak{m}' \in \mathrm{Max}(R')$. Choose a separable closure $(R/\mathfrak{m})_{\mathrm{sep}}$ of $R/\mathfrak{m}$ and let R^{sh} denote the corresponding strict henselization. The natural map*

$$R' \otimes_R R^{\mathrm{sh}} \to \prod_{\mathfrak{m}' \in \mathrm{Max}(R')} \quad \prod_{x \in \mathrm{Spec}(R'/\mathfrak{m}' \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})_{\mathrm{sep}})} R'^{\mathrm{sh}}_{\mathfrak{m}', i_x} \tag{2.5}$$

is an isomorphism, where $i_x : R'/\mathfrak{m}' \to k(x)$ is the separable closure of $R'/\mathfrak{m}'$ associated to a point

$$x \in \mathrm{Spec}(R'/\mathfrak{m}' \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})_{\mathrm{sep}}).$$

In particular, $R' \otimes_R R^{\mathrm{sh}}$ is noetherian and is the normalization of R^{sh} in $K' \otimes_R R^{\mathrm{sh}}$.

In this lemma, we do *not* assume K'/K is separable, so it may in fact happen that R' is not finite over R. But this does not cause any problems, because the Krull-Akizuki Theorem [M, 11.7] ensures that R' is nevertheless semi-local Dedekind and the residue field extensions are all finite. This is what we need.

Proof. The ring R' is a semi-local Dedekind domain which is integral over the discrete valuation ring R. Also, for each maximal ideal $\mathfrak{m}'$ of R', the residue field $R'/\mathfrak{m}'$ is finite over $R/\mathfrak{m}$. Thus, for a sufficiently large finite R-subalgebra $R_\alpha \subseteq R'$, the map $\mathrm{Spec}(R') \to \mathrm{Spec}(R_\alpha)$ is a bijection and if a maximal ideal $\mathfrak{m}'$ of R' contracts to the maximal ideal $\mathfrak{m}'_\alpha$ of R_α, then $\mathfrak{m}'_\alpha R' = \mathfrak{m}'$ and $R_\alpha/\mathfrak{m}'_\alpha = R'/\mathfrak{m}'$. In particular,

$$\mathrm{Spec}(R'/\mathfrak{m}' \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})_{\mathrm{sep}}) \to \mathrm{Spec}(R_\alpha/\mathfrak{m}'_\alpha \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})_{\mathrm{sep}})$$

is an isomorphism. Since R' is the direct limit of the R_α's and we can view a separable closure of $R'/\mathfrak{m}'$ as a separably closed extension of $R_\alpha/\mathfrak{m}'_\alpha$, it follows that there is a natural map

$$\varinjlim (R_\alpha)^{\mathrm{sh}}_{\mathfrak{m}'_\alpha} \to {R'}^{\mathrm{sh}}_{\mathfrak{m}'}$$

and this map is an isomorphism by [EGA, IV_4, 18.8.18].

Thus, in order to prove that (2.5) is an isomorphism, it suffices to prove the analogous assertion with R replaced by an arbitrary local ring and R' replaced by an arbitrary *finite* R-algebra. In fact, by using

$$R' \otimes_R R^{\mathrm{sh}} \simeq (R' \otimes_R R^{\mathrm{h}}) \otimes_{R^{\mathrm{h}}} R^{\mathrm{sh}},$$

it suffices to prove:

- for a finite algebra R' over a local ring R, the map

$$R' \otimes_R R^{\mathrm{h}} \to \prod_{\mathfrak{m}' \in \mathrm{Max}(R')} {R'}^{\mathrm{h}}_{\mathfrak{m}'} \tag{2.6}$$

  is an isomorphism,
- when $(R, \mathfrak{m})$ is a *henselian* local ring and $(R', \mathfrak{m}')$ is a finite local R-algebra, then the natural map

$$R' \otimes_R R^{\mathrm{sh}} \to \prod_{x \in \mathrm{Spec}(R'/\mathfrak{m}' \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})_{\mathrm{sep}})} {R'}^{\mathrm{sh}}_{i_x} \tag{2.7}$$

  is an isomorphism.

For a proof that (2.6) is an isomorphism, see [EGA, IV_4, 18.6.8]. In order to analyze (2.7), note that the ring $R' \otimes_R R^{\mathrm{sh}}$ is finite over R^{sh}, so it is a finite product of strictly henselian local rings. These local factor rings must be the localizations of $R' \otimes_R R^{\mathrm{sh}}$ at its maximal ideals, which are naturally indexed by the points of $\mathrm{Spec}(R'/\mathfrak{m}' \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})_{\mathrm{sep}})$. It remains to check each localization of $R' \otimes_R R^{\mathrm{sh}}$ at a maximal ideal is a strict henselization of R'. This follows from the proof of [EGA, IV_4, 18.8.10]. ∎

Here is how Lemma 2.2 reduces the calculation of $(Z'_K, Z_K)_{s'}$ to the case of the strictly henselian base $\mathrm{Spec}(\mathscr{O}_s^{\mathrm{sh}})$, where $s \in U$ is the image of s'. Let K_s^{sh} be the fraction field of $\mathscr{O}_s^{\mathrm{sh}}$. By Lemma 2.2, we can identify a strict henselization $\mathscr{O}_{s'}^{\mathrm{sh}}$ with the normalization of $\mathscr{O}_s^{\mathrm{sh}}$ in one of the factor fields of $K(y) \otimes_K K_s^{\mathrm{sh}}$. Such a choice of factor field corresponds to a choice of $y_{s'}^{\mathrm{sh}} \in Y_K \times_K K_s^{\mathrm{sh}}$ lying over $y \in Y_K$ under the canonical projection

$$\pi_s^{\mathrm{sh}} : Y_K \times_K K_s^{\mathrm{sh}} \to Y_K.$$

Note that the maximal ideal of $\mathscr{O}_{s'}^{\mathrm{sh}}$ is the unique codimension 1 point $\overline{s'}$ of $K_s^{\mathrm{sh}}(y_{s'}^{\mathrm{sh}})$ relative to the base $\mathrm{Spec}(\mathscr{O}_s^{\mathrm{sh}})$. In terms of such choices, one easily finds:

Lemma 2.3. *With the above notation, we have an equality*

$$(Z'_K, Z_K)_{s'} = (Z'_{K_s^{\mathrm{sh}}}, Z_{K_s^{\mathrm{sh}}})_{\overline{s'}},$$

where $Z_{K_s^{\mathrm{sh}}} = Z_K \times_K K_s^{\mathrm{sh}}$ *and* $Z'_{K_s^{\mathrm{sh}}} = Z'_K \times_K K_s^{\mathrm{sh}}$.

Proof. Going back to the definitions, the equality amounts to the statement that for a local ring A and a finite-length A-module M, the A^{sh}-length of $M \otimes_A A^{\mathrm{sh}}$ is equal to the A-length of M. This follows from the fact that $\mathrm{Spec}(A^{\mathrm{sh}}) \to \mathrm{Spec}(A)$ is flat and the fiber over the closed point is the spectrum of a field. ∎

We are now ready to state the main result. First, we recall the running notation. S is a normal, connected, noetherian scheme with function field K, $f_K : X_K \to Y_K$ is a map between proper, normal K-schemes, with Y_K regular and X_K, Y_K of pure dimension $d \geq 1$. The map f_K is tamely ramified along a normal crossings divisor relative to K, with branch scheme $\mathscr{B}_{X_K/Y_K}$ having generic points $\{a_i\}$. We assume moreover that either S is excellent or that X_K, Y_K are K-smooth, so (by Lemma 2.1) we may choose an integral model $f_U : X_U \to Y_U$ of f_K. All intersection numbers $(\cdot, \cdot)_{s'}$ will be computed in terms of this model. The finitely many codimension 1 points of S outside of U play a role analogous to the finite set Σ_f in Theorem 0.2. The following result, a more general version of Theorem 0.2 in the Introduction, is our goal:

Theorem 2.4. *With the above notation and hypotheses, let G be a finite group and suppose that our map f_K is a generic G-torsor. Choose a closed point $y \in Y_K$ outside of $\mathscr{B}_{X_K/Y_K}$ and pick some $x \in f_K^{-1}(y)$, so $K(x)/K(y)$ is a finite Galois extension with* $\mathrm{Gal}(K(x)/K(y)) \subseteq G$. *Let s' be a codimension 1 point of $K(y)$ lying over U.*

(1) *We have*

$$(y, \mathscr{B}_{X_K/Y_K})_{s'} = \sum_i (y, a_i)_{s'} \tag{2.8}$$

and s' is tamely ramified in $K(x)$.

(2) *If e_i is the ramification degree of f_K over a_i and $n_i = (y, a_i)_{s'}$, then the inertia groups over s' in* $\mathrm{Gal}(K(x)/K(y))$ *are abstractly isomorphic to the group generated by the $\mu_{e_i}^{n_i}$'s inside of $K_{\mathrm{sep}}^{\times}$, where $\mu_{e_i}^{n_i}$ denotes the subgroup of n_ith powers in μ_{e_i}. In particular, the ramification degree of s' in $K(x)$ is equal to the order of the subgroup $\langle n_i/e_i \rangle \subseteq \mathbf{Q}/\mathbf{Z}$ generated by the fractions n_i/e_i.*

(3) *There exists a choice of inertia group $I_i(y) \simeq \mu_{e_i}$ of f_K over a_i so that*
- *the $I_i(y)$'s commute in G,*
- *the canonical map of groups*

$$I_1(y) \times \cdots \times I_m(y) \to G$$

is injective,
- *the inertia groups over s' in* $\mathrm{Gal}(K(x)/K(y)) \subseteq G$ *are conjugate (in G) to*

$$\left\{(\zeta_1, \ldots, \zeta_m) \in \mu_{e_1}^{n_1} \times \cdots \times \mu_{e_m}^{n_m} \subseteq I_1(y) \times \cdots \times I_m(y) \subseteq G \mid \prod \zeta_j^{a_j} = 1 \text{ whenever } \sum a_j n_j/e_j \in \mathbf{Z}\right\}. \tag{2.9}$$

In order to prove Theorem 2.4, we first reduce to the case of a strictly henselian S and then will interpret everything geometrically in terms of 'specializations'. Let $s \in U$ be the image of s' and let K_s^{sh} denote the fraction field of a strict henselization of $\mathscr{O}_s^{\mathrm{sh}}$. By Lemma 2.2 and [BLR, 2.3/11], we know that for a discrete valuation ring $(A, \mathfrak{n})$ with fraction field F and integral closure A' in a finite Galois extension F'/F, the inertia group of A' over A at a maximal ideal $\mathfrak{n}' \in \mathrm{Spec}(A')$ is exactly the automorphism group of ${A'}_{\mathfrak{n}'}^{\mathrm{sh}}$ over $A_{\mathfrak{n}}^{\mathrm{sh}}$. Thus, by Lemma 1.2, (2.4), and considerations as in Lemma 2.3, we can reduce to analyzing the situation after base change by the ind-étale map $\mathrm{Spec}(\mathscr{O}_s^{\mathrm{sh}}) \to U$ and replacing y by a suitable point $y_{s'}^{\mathrm{sh}} \in Y \times_K K_s^{\mathrm{sh}}$ over y.

3. Specializations

We may now assume $U = S = \mathrm{Spec}(R)$ for a (strictly) henselian discrete valuation ring R. In particular, $\mathscr{O}_{s'}$ is the full integral closure of R in $K(y)$ and is a strictly henselian discrete valuation ring, so $\mathrm{Gal}(K(x)/K(y))$ is the full inertia group at s' in $K(x)$. For simplicity, we denote $\mathscr{O}_{s'}$ by R'. Also, we write $f : X \to Y$ for our integral model of $f_K : X_K \to Y_K$ over $\mathrm{Spec}(R)$ and we write $(\cdot, \cdot)$ instead of $(\cdot, \cdot)_{s'}$, since $y \in Y_K$ is fixed and the integral closure R' of R in $K(y)$ has only one height 1 prime. The main reason for making the base R a (strictly) henselian discrete valuation ring is that it allows us to work with 'specializations.' We need to precisely define what specializations are so that we may use them in order to prove Theorem 2.4.

If $z \in Y_K$ is a closed point and $R(z)$ is the integral closure of R in the finite extension $K(z)/K$, then $R(z)$ is a discrete valuation ring which is integral over R, so the unique map $\mathrm{Spec}(R(z)) \to Y$ extending

$z : \mathrm{Spec}(K(z)) \to Y_K$ has a closed image in Y which has the form $\{z, z_0\}$ for some closed point z_0 in the closed fiber of $Y \to \mathrm{Spec}(R)$. We call z_0 the *specialization* of $z \in Y_K$. The scheme-theoretic closure $\overline{\{z\}}$ in Y has underlying set $\{z, z_0\}$ with the topological structure of the spectrum of a discrete valuation ring. In particular, z_0 is not an open point in $\overline{\{z\}}$. If we carry out specialization on X_K as well, then the specializations of the points in $f_K^{-1}(z)$ obviously lie inside of $f^{-1}(z_0)$. An important fact is:

Lemma 3.1. *For any closed point $z \in Y_K$ with specialization $z_0 \in Y$, the specialization map of sets*

$$f_K^{-1}(z) \to f^{-1}(z_0)$$

is surjective.

Proof. The finite map $f^{-1}(\overline{\{z\}}) \to \overline{\{z\}}$ is flat, hence open [EGA, IV_1, 1.10.4]. Since the closed point $z_0 \in \overline{\{z\}}$ is not open, there can be no non-empty open subset of $f^{-1}(\overline{\{z\}})$ lying over z_0.

However, by the very definition of specialization, the closure of $f_K^{-1}(z)$ in $f^{-1}(\overline{\{z\}})$ is exactly the union of $f_K^{-1}(z)$ and the image of the specialization map. Therefore, the complement of the image of the specialization map in $f^{-1}(z_0)$ is open in $f^{-1}(\overline{\{z\}})$ and lies over z_0. This forces the complement to be empty. ■

Let y_0 be the specialization of $y \in Y_K$. Note that X and Y are normal, flat R-schemes and the local rings at all closed points in the closed fibers of X, Y over $\mathrm{Spec}(R)$ are $(d+1)$-dimensional normal local rings. The local rings on Y are even regular. Since X is a G-torsor over the generic points of Y, so the G-action on connected components of X over a fixed component of Y is transitive, it is easy to reduce to the case where X and Y are also connected (so X is generically Galois over Y). This step causes G to be replaced by a subgroup, but that is harmless.

With connectedness, $A = \mathscr{O}_{Y,y_0}$ is a $(d+1)$-dimensional regular local ring with fraction field $K(Y)$, so the integral closure B of A in $K(X)$ is a semi-local normal domain with fraction field $K(X)$ and

$$\mathrm{Spec}(B) = X \times_Y \mathrm{Spec}(A). \tag{3.1}$$

In particular, there is a natural identification of sets

$$\mathrm{Max}(B) = f^{-1}(y_0) \tag{3.2}$$

which we will use often. Recall that $\{a_i\}$ denotes the set of generic points of the branch locus of f. Each a_i with $(y, a_i) > 0$ gives rise to a height 1 prime $\mathfrak{p}_i$ in A, with $A/\mathfrak{p}_i$ the local ring of $\overline{\{a_i\}}$ at y_0. We want to use these $\mathfrak{p}_i$'s to explicitly describe X and Y in an étale neighborhood of y_0. This is going to be done via Abhyankar's Lemma, but we must first check the following conditions.

Lemma 3.2. *The height* 1 *primes in A which ramify in B (i.e., over which B is not étale) are exactly the $\mathfrak{p}_i$'s. Moreover,*

1. *$A/\mathfrak{p}_i$ is a regular local ring (with dimension d),*
2. *$\mathfrak{p}_i = (t_i)$ for elements $t_i \in A$ which form part of a regular system of parameters for A,*
3. *each $\mathfrak{p}_i$ is tamely ramified in B with inertia groups in G equal to those of f_K over a_i.*

Proof. Since A is a regular local ring, it is catenary and is a unique factorization domain [M, 17.8, 17.9, 20.3]. Thus, all height 1 primes $\mathfrak{p}$ of A are principal [M, 20.1] and

$$\dim A/\mathfrak{p} = \dim A - \dim A_{\mathfrak{p}} = d.$$

Since f_U is an integral model of f_K, $\mathscr{B}_{X/Y}$ is a normal crossings divisor and therefore its irreducible (reduced) components $\overline{\{a_j\}}$ are regular. Since $y_0 \in \overline{\{a_j\}}$ if and only if $\mathrm{Spec}(A)$ meets $\overline{\{a_j\}}$, we obtain the first two parts of the lemma.

In order to analyze the precise ramification at $\mathfrak{p}_i$, we just have to look at the map $A_{\mathfrak{p}_i} \to B_{\mathfrak{p}_i}$. These localizations can be computed by first inverting a uniformizer of R, as such elements do not lie in $\mathfrak{p}_i$ (since the closure $\overline{\{a_i\}}$ of $\mathrm{Spec}(A/\mathfrak{p}_i)$ in Y contains a generic fiber point, $a_i \in Y_K$). This makes it clear that $A_{\mathfrak{p}_i} = \mathscr{O}_{Y_K,a_i}$, so $A_{\mathfrak{p}_i} \to B_{\mathfrak{p}_i}$ is the normalization map of $\mathscr{O}_{Y_K,a_i}$ in $K(X) = K(X_K)$. But the finite map $f_K : X_K \to Y_K$ is tamely ramified over a_i, so we are done. ■

Since y will be fixed for the rest of this section, we only need to consider those a_i for which $(y, a_i) > 0$. Label these as $a_1, \ldots, a_r$ (and we may assume $r > 0$ or there is nothing to prove). This simplifies the exposition, since we will not have to repeatedly use the phrase "where i runs through those indices for which $(y, a_i) > 0$." Since A is a *regular* local ring, Lemma 3.2 provides us with all of the conditions required to apply Abhyankar's Lemma (Lemma 1.4). We conclude that

- B is regular
- the ramification degree e_i of $\mathfrak{p}_i = (t_i)$ in B is a *unit* in A, so since $R \to A$ is a *local* map,

$$e_i \in R^\times, \tag{3.3}$$

- for each maximal ideal $\mathfrak{m}$ of B there is an A^{sh}-algebra isomorphism

$$B_{\mathfrak{m}}^{\mathrm{sh}} \simeq A^{\mathrm{sh}}[T_{1,\overline{x}}, \ldots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - t_j), \tag{3.4}$$

  where $\overline{x} \in f^{-1}(y_0)$ corresponds to $\mathfrak{m} \in \mathrm{Max}(B)$ under (3.2).

Since A and $B_{\mathfrak{m}}$ have separably closed residue fields, we can rewrite (3.4) in the more convenient form

$$B_{\mathfrak{m}}^{\mathrm{h}} \simeq A^{\mathrm{h}}[T_{1,\overline{x}}, \ldots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - t_j). \tag{3.5}$$

As we noted after the statement of Lemma 1.4, the stabilizer $G_{\overline{x}}$ in G of $\overline{x}$ is

$$G_{\overline{x}} = I(\overline{x}|y_0) = \mu_{e_1} \times \cdots \times \mu_{e_r}. \tag{3.6}$$

Our goal is to calculate the inertia groups $I(x|y)$ of f_K at points $x \in f_K^{-1}(y)$. This calculation will require working with the regular local rings $R_i = A/\mathfrak{p}_i$ (whose fraction field is $K(a_i)$) and the integral closure R' of R in $K(y)$. Since y specializes to $y_0 \in Y$, we have a canonical map $\varphi' : A = \mathscr{O}_{Y,y_0} \to R'$ corresponding to the unique map $\mathrm{Spec}(R') \to Y$ over $\mathrm{Spec}(R)$ extending $y : \mathrm{Spec}(K(y)) \to Y_K$. The map φ' uniquely factors through the (strict) henselization A^{h} of A, since R' is (strictly) henselian.

Combining (3.5) with the fact [EGA, IV_4, 18.6.8] that

$$B \otimes_A A^{\mathrm{h}} \simeq \prod_{\mathfrak{m}} B_{\mathfrak{m}}^{\mathrm{h}}, \tag{3.7}$$

we get

$$B \otimes_A R' \simeq (B \otimes_A A^{\mathrm{h}}) \otimes_{A^{\mathrm{h}}} R' \simeq \prod_{\overline{x} \in f^{-1}(y_0)} R'[T_{1,\overline{x}}, \ldots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - \varphi'(t_j)), \tag{3.8}$$

or, more geometrically (by (3.1)),

$$X \times_Y R' = \mathrm{Spec}(B \otimes_A R') = \coprod_{\overline{x} \in f^{-1}(y_0)} \mathrm{Spec}(R'[T_{1,\overline{x}}, \ldots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - \varphi'(t_j))). \tag{3.9}$$

Since $f_K^{-1}(y) = (X \times_Y R') \times_{R'} K(y)$, we deduce the following result from (3.3) and (3.9):

Lemma 3.3. *The part of $f_K^{-1}(y)$ which specializes to a point $\overline{x} \in f^{-1}(y_0)$ is the generic fiber of*

$$\mathrm{Spec}(R'[T_{1,\overline{x}}, \ldots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - \varphi'(t_j))) \to \mathrm{Spec}(R'). \tag{3.10}$$

In particular, $K(x)/K(y)$ is tamely ramified for all $x \in f_K^{-1}(y)$.

Similar fiber calculations allow us to establish a useful uniqueness result:

Lemma 3.4. *Choose $x_0 \in f^{-1}(y_0)$. There is a unique point $x_i \in f_K^{-1}(a_i)$ whose closure $\overline{\{x_i\}}$ in X contains the point x_0, and*

$$[K(x_i) : K(a_i)] = \prod_{j \neq i} e_j. \tag{3.11}$$

In terms of the calculation (3.6) of $G_{x_0} = I(x_0|y_0)$, the inertia subgroup $I(x_i|a_i)$ of f_K at x_i is the ith factor subgroup μ_{e_i} of $G_{x_0} = \mu_{e_1} \times \cdots \times \mu_{e_r}$.

Proof. Let $K_i^{\rm h}$ denote the fraction field of $R_i^{\rm h} \simeq A^{\rm h}/t_i$ and let $\varphi_i : A^{\rm h} \to R_i^{\rm h}$ be the canonical map. Using (3.1), (3.5), and (3.7), we see that

$$\begin{aligned} X \times_Y R_i^{\rm h} &= \operatorname{Spec}(B \otimes_A R_i^{\rm h}) \\ &= \operatorname{Spec}((B \otimes_A A^{\rm h}) \otimes_{A^{\rm h}} A^{\rm h}/t_i) \\ &= \coprod_{\overline{x} \in f^{-1}(y_0)} \operatorname{Spec}((A^{\rm h}/t_i)[T_{1,\overline{x}}, \dots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - \varphi_i(t_j))) \\ &= \coprod_{\overline{x} \in f^{-1}(y_0)} \operatorname{Spec}(R_i^{\rm h}[T_{1,\overline{x}}, \dots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - \varphi_i(t_j))), \end{aligned}$$

where the factor indexed by $\overline{x} \in f^{-1}(y_0)$ has generic fiber over $\operatorname{Spec}(R_i^{\rm h})$ consisting of those points in $f_K^{-1}(a_i) \times_K K_i^{\rm h}$ whose closure in $f^{-1}(\overline{\{a_i\}}) \times_{R_i} R_i^{\rm h}$ contains $\overline{x}$.

Obviously $\varphi_i(t_i) = 0$ and the $\varphi_i(t_j)$'s for $j \neq i$ are part of a regular system of parameters in $R_i^{\rm h}$. Thus, for any $\overline{x} \in f^{-1}(y_0)$, the generic fiber of

$$\operatorname{Spec}(R_i^{\rm h}[T_{1,\overline{x}}, \dots, T_{r,\overline{x}}]/(T_{j,\overline{x}}^{e_j} - \varphi_i(t_j))) \to \operatorname{Spec}(R_i^{\rm h})$$

is

$$\operatorname{Spec}(F_{i,\overline{x}}[T_{i,\overline{x}}]/(T_{i,\overline{x}}^{e_i})),$$

where

$$F_{i,\overline{x}} = K_i^{\rm h}[T_{j,\overline{x}}; j \neq i]/(T_{j,\overline{x}}^{e_j} - \varphi_i(t_j)).$$

Since $R_i^{\rm h}$ is regular and the $\varphi_i(t_j)$'s for $j \neq i$ are part of a regular system of parameters, the local ring $R_i^{\rm h}[T_{j,\overline{x}}; j \neq i]/(T_{j,\overline{x}}^{e_j} - \varphi_i(t_j))$ is regular, hence a domain, and is finite over $R_i^{\rm h}$. Thus, $F_{i,\overline{x}}$ is a *field* and clearly has degree $\prod_{j \neq i} e_j$ over $K_i^{\rm h}$. This proves the uniqueness of the point $x_i \in f_K^{-1}(a_i)$ with closure $\overline{\{x_i\}}$ containing a chosen point $x_0 \in f^{-1}(y_0)$, and that (3.11) holds. In fact, we have proven the stronger result that this assertion holds after the separable algebraic base change $K(a_i) \to K_i^{\rm h}$.

It is obvious from the description via Abhyankar's Lemma that $I(x_i|a_i) = \mu_{e_i}$ inside of G_{x_0}. ■

It remains to compute the intersection numbers (y, a_i) and relate them to the group $\operatorname{Gal}(K(x)/K(y))$ for any $x \in f_K^{-1}(y)$. Choose $x_0 \in f^{-1}(y_0)$, and consider only those $x \in f_K^{-1}(y)$ which specialize to x_0.

By (2.4), the number (y, a_i) is equal to the length of the R'-module $(A/t_i) \otimes_A R'$. Using the factorization of $A \to R'$ through $A \to A^{\rm h}$, we have an R'-module isomorphism

$$(A/t_i) \otimes_A R' \simeq (A^{\rm h}/t_i) \otimes_{A^{\rm h}} R'. \tag{3.12}$$

Combining (3.3), (3.5), (3.7), and the second part of Lemma 3.2, it is easy to calculate that

$$\operatorname{ann}_{A^{\rm h}}(\Omega^1_{B \otimes_A A^{\rm h}/A^{\rm h}}) = \bigcap (t_i A^{\rm h}) = \left(\prod t_i\right) A^{\rm h},$$

so $(y, \mathscr{B}_{X_K/Y_K})$ is the sum of the lengths of the artin local rings

$$R'/\varphi'(t_i) \simeq (A/t_i) \otimes_A R'.$$

Thus, (2.8) holds. We remind the reader that $\varphi'(t_i) \in R' \subseteq K(y)$ is non-zero for all i because $y \notin \mathscr{B}_{X/Y}$.

Let $n_i = \operatorname{ord}_{R'}(\varphi'(t_i)) = (y, a_i) > 0$. We want to compute $G_x = \operatorname{Gal}(K(x)/K(y)) \subseteq G$ in terms of the n_i's and the groups $I(x_i|a_i)$ from Lemma 3.4. By Lemma 3.3 and Lemma 3.4, this amounts to determining the subgroup of $G_{x_0} = \mu_{e_1} \times \cdots \times \mu_{e_r}$ which fixes a choice of point x on the generic fiber of

$$\operatorname{Spec}(R'[T_{1,x_0}, \dots, T_{r,x_0}]/(T_{j,x_0}^{e_j} - \varphi'(t_j))) \to \operatorname{Spec}(R'). \tag{3.13}$$

The generic fiber in (3.13) is the G_{x_0}-torsor

$$\operatorname{Spec}(K(y)[T_{1,x_0}, \dots, T_{r,x_0}]/(T_{j,x_0}^{e_j} - \varphi'(t_j))) \to \operatorname{Spec}(K(y)).$$

Since G_{x_0} is abelian, we see that for all $x \in f_K^{-1}(y)$ specializing to x_0,

$$\operatorname{Gal}(K(x)/K(y)) \subseteq G_{x_0} \subseteq G$$

is independent of x. By (3.3) and the fact that R' is strictly henselian, we can write

$$\varphi'(t_j) = u_j^{e_j}\pi^{n_j},$$

where $u_j \in R'^{\times}$ is a unit and $\pi \in R'$ is a uniformizer.

Choose an e_jth root π^{1/e_j} of π in $K(y)_{\mathrm{sep}}$, so it suffices to consider the point

$$x : K(y)[T_{1,x_0}, \dots, T_{r,x_0}]/(T_{j,x_0}^{e_j} - \varphi'(t_j)) \to K(y)_{\mathrm{sep}}$$

which sends T_{j,x_0} to $u_j(\pi^{1/e_j})^{n_j}$ for all j. The extension field $K(x)/K(y)$ is identified with the subfield of $K(y)_{\mathrm{sep}}$ generated by the elements $(\pi^{1/e_j})^{n_j}$, so by Kummer theory $G_x \simeq \mathrm{Gal}(K(x)/K(y))$ is naturally identified with the subgroup of $R^{\times}$ generated by the $\mu_{e_j}^{n_j}$'s. Since the stabilizer G_x in G of x lies inside of $G_{x_0} = \mu_{e_1} \times \cdots \times \mu_{e_r}$, if we recall how the projections $G_{x_0} \to \mu_{e_j}$ are *defined*, then the inclusion $G_x \subseteq G_{x_0}$ corresponds to an injection

$$G_x \hookrightarrow \mu_{e_1} \times \cdots \times \mu_{e_r}$$

in which the image of $g \in G_x$ under projection to μ_{e_j} is the element in $\mu_{e_j}^{n_j}$ giving the action of g on $(\pi^{1/e_j})^{n_j}$. This gives an inclusion

$$G_x \hookrightarrow \mu_{e_1}^{n_1} \times \cdots \times \mu_{e_r}^{n_r}$$

and we need to check that the image is exactly the subgroup defined (2.9). The necessary and sufficient conditions for

$$(\zeta_1, \dots, \zeta_r) \in \mu_{e_1}^{n_1} \times \cdots \times \mu_{e_r}^{n_r}$$

to lie in G_x are exactly that there be a *well-defined* automorphism of $K(x)$ which sends $(\pi^{1/e_j})^{n_j} \mapsto \zeta_j(\pi^{1/e_j})^{n_j}$ for all j. Clearly it is necessary that

$$\prod \zeta_j^{a_j} = 1 \text{ whenever } \sum a_j n_j/e_j \in \mathbf{Z}. \tag{3.14}$$

For sufficiency we just have to check that the subgroup $H_1 \subseteq \mu_{e_1}^{n_1} \times \cdots \times \mu_{e_r}^{n_r}$ defined by (3.14) already has the same cardinality as the subgroup H_2 of $R^{\times}$ generated by the $\mu_{e_j}^{n_j}$'s (which we have seen has the same size as G_x).

If we non-canonically choose a primitive $(e_1 \cdot \ldots \cdot e_m)$th root of unity, then H_2 is identified with the subgroup $\langle n_j/e_j \rangle \subseteq \mathbf{Q}/\mathbf{Z}$, while H_1 is identified with the group

$$\left\{ \left(\frac{b_1}{e_1}, \dots, \frac{b_m}{e_m}\right) \in \left(\frac{1}{e_1}\mathbf{Z}/\mathbf{Z}\right) \times \cdots \times \left(\frac{1}{e_m}\mathbf{Z}/\mathbf{Z}\right) \mid \sum \frac{a_j b_j}{e_j} = 0 \text{ whenever } \sum \frac{a_j n_j}{e_j} = 0 \right\}.$$

Using the perfect pairing between $\prod \frac{1}{e_j}\mathbf{Z}/\mathbf{Z}$ and $\prod \mathbf{Z}/e_j\mathbf{Z}$ and the fact that the annihilator of the annihilator of a subgroup under this pairing is the subgroup itself, we see that H_1 is identified with the subgroup generated by the single element $(n_1/e_1, \dots, n_m/e_m)$ in $\prod \frac{1}{e_j}\mathbf{Z}/\mathbf{Z}$. We want the size of this subgroup to coincide with the size of the subgroup $\langle n_j/e_j \rangle \subseteq \mathbf{Q}/\mathbf{Z}$. Looking at p-primary components for all primes p, this is clear.

This completes the proof of Theorem 2.4.

4. The Non-Galois Case

In this last section, we explain the analogue of Theorem 2.4 when we remove the generic torsor condition (still assuming tame ramification). Also, we make the set Σ_f in Theorem 0.2 completely explicit.

When $f_K : X_K \to Y_K$ is generically étale but not necessarily a generic torsor, one can still ask how the ramification degrees in the closed fibers of f_K relate to the ramification degrees in the map f_K. We again assume that f_K is tamely ramified along a normal crossings divisor. These ramification degrees may now vary as we run through the points x_{ij} lying over a fixed generic point a_i of $\mathscr{B}_{X_K/Y_K}$. An analogue of Theorem 2.4 would be a formula for the ramification degrees of a codimension 1 point s' of $K(y)$ in the fibers $f_K^{-1}(y)$, for $y \in Y_K$ a closed point outside of $\mathscr{B}_{X_K/Y_K}$, in terms of the ramification degrees $e(x_{ij}|a_i)$ and the intersection numbers $(y, a_i)_{s'}$.

This question was considered by Beckmann in [B, §5] for K a number field, $Y_K = \mathbf{P}_K^1$, and X_K geometrically connected over K. However, Beckmann's formula is given in terms of a topological description (via fundamental groups) of the Galois closure of $X_K \times_K \mathbf{C}$ over $\mathbf{P}_{\mathbf{C}}^1$ (for a choice of embedding $K \hookrightarrow \mathbf{C}$).

Thanks to the more general geometric framework we set up above, this question can be answered in a purely algebraic manner with greater generality.

Theorem 4.1. *Let K, S, f_K be as above. Assume that S is excellent or that X_K, Y_K are K-smooth and f_K is tamely ramified along a normal crossings divisor relative to K. Let f_U be an integral model of f_K.*

(1) *For $y \in Y_K$ outside of $\mathscr{B}_{X_K/Y_K}$, any $x \in f_K^{-1}(y)$, and any codimension 1 point s' in $K(y)$ lying over U,*
  - $(y, \mathscr{B}_{X_K/Y_K})_{s'} = \sum (y, a_i)_{s'}$,
  - *s' is tamely ramified in $K(x)$, and for each codimension 1 point s'' in $K(x)$ over s', the ramification degree $e(s''|s')$ is equal to the order of the subgroup*

$$\left\langle \frac{(y, a_i)_{s'}}{e(x_{i,s''}|a_i)} \right\rangle \subseteq \mathbf{Q}/\mathbf{Z} \tag{4.1}$$

  for suitable $x_{i,s''} \in f_K^{-1}(a_i)$.

(2) *The ramification degrees for s' in the fiber $f_K^{-1}(y)$ are the orders of groups*

$$\left\langle \frac{(y, a_i)_{s'}}{e(x_i|a_i)} \right\rangle \subseteq \mathbf{Q}/\mathbf{Z}, \tag{4.2}$$

with

$$(x_i) \in \prod_{(y,a_i)_{s'} \neq 0} f_K^{-1}(a_i)$$

running through elements such that the $\overline{\{x_i\}}$'s contain a common point over s'.

Proof. As in the proof of Theorem 2.4, we can reduce to the case where $U = S = \mathrm{Spec}(R)$ and X_K, Y_K are connected. We let $f : X \to Y$ denote our integral model of f_K over R and let R' be the integral closure of R in $K(y)$. The ring $A = \mathscr{O}_{Y,y_0}$ is a $(d+1)$-dimensional regular local ring, its normalization B in $K(X)$ is a $(d+1)$-dimensional, normal, semi-local domain. Also, (3.1) and Lemma 3.2 still hold, except for the inertia group claim at the end of Lemma 3.2 (as this has no global analogue when $X_K \to Y_K$ is not generically Galois).

As before, let $a_1, \ldots, a_r$ be the a_i's with $(y, a_i) > 0$ and let $\mathfrak{p}_i = (t_i)$ be the height 1 prime in A corresponding to $\overline{\{a_i\}}$. It may now occur that the different height 1 primes in B over $\mathfrak{p}_i$ may have different ramification indices over $\mathfrak{p}_i$ (e.g., some may be unramified over $\mathfrak{p}_i$). By Abhyankar's Lemma, we conclude that B is regular and (as in the proof of Theorem 2.4) that

- the ramification degrees of $\mathfrak{p}_i$ in B lie in $R^\times$,
- for each $\mathfrak{m} \in \mathrm{Max}(B)$, there is an A^{h}-algebra isomorphism

$$B_{\mathfrak{m}}^{\mathrm{h}} \simeq A^{\mathrm{h}}[T_{1,\mathfrak{m}}, \ldots, T_{r,\mathfrak{m}}]/(T_{j,\mathfrak{m}'}^{e_j(\mathfrak{m})} - t_j). \tag{4.3}$$

  for suitable positive integers $e_j(\mathfrak{m})$ which are units in R.

Fix $\mathfrak{m}$, corresponding to a choice of $x_0 \in f^{-1}(y_0)$. If we now run through generic fiber calculations as near the end of the proof of Theorem 2.4, we see (as in Lemma 3.4) that there is a unique point $x_{i,\mathfrak{m}} \in f_K^{-1}(a_i)$ whose closure in X contains x_0. Moreover,

- $e_i(\mathfrak{m}) = e(x_{i,\mathfrak{m}}|a_i)$ for all i,
- for all $x \in f_K^{-1}(y)$ specializing to x_0, $e(x|y)$ is the order of the subgroup

$$\left\langle \frac{(y, a_i)}{e(x_{i,\mathfrak{m}}|a_i)} \right\rangle \subseteq \mathbf{Q}/\mathbf{Z}.$$

The second assertion in the theorem follows from Lemma 3.4 and some straightfoward base change considerations. ■

When X_K, Y_K are curves and S is an excellent Dedekind scheme, one would like to know some explicit finite set of closed points $\{s_1, \ldots, s_n\}$ which has to be removed from S so that a tamely ramified map $f_K : X_K \to Y_K$ as in Theorem 4.1 admits a normal integral model over the complement $U \subseteq S$ of the s_j's.

We give an explicit description of a set of s_j's which is adequate for this purpose. First, we recall a basic fact:

Lemma 4.2. *Let S be Dedekind and $X \to S$ a proper flat map whose generic fiber has pure dimension d. Then*

- *for every closed point $x \in X$, $\dim \mathscr{O}_{X,x} = d+1$,*
- *the closed fibers of $X \to S$ have pure dimension d.*

Proof. By the dimension formula for flat maps, the first assertion follows from the second. The second assertion is a consequence of the openness of the proper, flat map $X \to S$ and [EGA, IV_3, 14.2.5]. ■

Returning to the situation with our *curves* X_K and Y_K, there is *some* proper, flat, regular S-scheme Y whose generic fiber is Y_K and whose other fibers over S are curves (we call such a Y a *regular integral model* of Y_K over S). Indeed, since the curve Y_K is projective over K, by taking a suitable closure in some $\mathbf{P}^N_S$ and normalizing we obtain a proper, flat, normal S-scheme Y' whose generic fiber is Y_K. By Lemma 4.2 (with $d = 1$) and Lipman's resolution of singularities for normal, noetherian, excellent surfaces [L], we get a regular integral model Y of Y_K over S. Of course, in many cases there is an explicitly known Y; e.g., if $Y_K = \mathbf{P}^1_K$ we just take $Y = \mathbf{P}^1_S$.

Choose a regular integral model Y of Y_K over S and let X be the normalization of Y in $K(X_K)$. Since X is S-flat with generic fiber X_K and the map $f : X \to Y$ is finite, it follows Lemma 4.2 that the fibers of $X \to S$ are curves and all local rings at closed points on X are 2-dimensional and normal. In particular, by applying [M, 11.5(i)] at the closed points, we see that the local rings on X are Cohen-Macaulay. Thus, by [M, 23.1], the finite normalization map $f : X \to Y$ is flat at all points (only at the closed points is flatness not obvious).

Since S is Dedekind and f_K is tamely ramified, by Abhyankar's Lemma we see that $(\mathscr{B}_{X/Y})_{\mathrm{red}}$ is a normal crossings divisor and is S-flat if and only if $\mathscr{B}_{X/Y}$ is a normal crossings divisor and is the scheme-theoretic closure of its generic fiber. Thus, the only obstruction to f being a normal integral model of f_K over S is that $(\mathscr{B}_{X/Y})_{\mathrm{red}}$ might not be a normal crossings divisor or S-flat. Since the fibers of $Y \to S$ are curves, by purity of the branch locus we see that in order to get S-flatness for $(\mathscr{B}_{X/Y})_{\mathrm{red}}$ it is necessary and sufficient to remove those $s \in S$ for which the finite flat fiber map $f_s : X_s \to Y_s$ is not generically étale. With these preparations, we have essentially proven:

Corollary 4.3. *Let $f_K : X_K \to Y_K$ be a generically étale, tamely ramified map between curves as above, with S excellent and Dedekind. Let Y be a regular integral model of Y_K over S and let X be the normalization of Y in $K(X_K)$. Let $\{s_1, \dots, s_n\}$ be the finitely many closed points $s \in S$ such that at least one of the following holds:*

- *three of the closures $\overline{\{a_i\}} \subseteq Y$ meet over s, or two of the closures meet with non-reduced intersection over s,*
- *some closure $\overline{\{a_i\}}$ is not normal over a neighborhood of s,*
- *$X_s \to Y_s$ is not generically étale.*

The following assertions hold:

1. *The restriction of the finite flat map $f : X \to Y$ to $U = S - \{s_1, \dots, s_n\}$ is a normal integral model of f_K and U is the largest open subscheme of S with this property.*
2. *If $K(a_i)/K$ is separable for all i, then $U = S - \{s_1, \dots, s_n\}$ contains the complement V of the finite set of closed points $s \in S$ over which at least one of the following holds:*
   - *three of the closures $\overline{\{a_i\}}$ meet over s, or two of the closures meet with non-reduced intersection over s,*
   - *s is ramified in some $K(a_i)$,*
   - *the fiber map $X_s \to Y_s$ is not generically étale.*

   (*the point here is just that the second condition in this list is implied by the second of the three conditions originally defining the $\{s_i\}$'s*).

Proof. The maximality of U is because the s_i's are exactly the points over which $(\mathscr{B}_{X/Y})_{\rm red}$ is not a normal crossings divisor or is not S-flat, as we explained above. We now turn to the second part of the corollary. We just need to check that the condition of s being unramified in all $K(a_i)$ forces all $\overline{\{a_i\}}$ to be normal over s. Working over the local ring at s and passing to the strict henselization without loss of generality (as it suffices to check after such ind-étale base change), we may suppose $V = S = \mathrm{Spec}(R)$ for a strictly henselian discrete valuation ring R. Since $V = R$, we have $K(a_i) = K$ for all i due to the unramifiedness hypothesis. It is then obvious from the valuative criterion for properness that $\overline{\{a_i\}} \simeq \mathrm{Spec}(R)$, which is normal. ∎

As a special case, note that if all a_i are K-rational and Y is smooth over S with connected fibers, then $\{s_1, \dots, s_n\}$ is equal to the union of the following two finite sets:

- the set of $s \in S$ for which $K(Y_K) \hookrightarrow K(X_K)$ is ramified at the discrete valuation on $K(Y_K)$ corresponding to the generic point of Y_s,
- the set of $s \in S$ over which three of the sections $a_i \in Y_K(K) = Y(S)$ meet, or over which two of the sections a_i meet with non-reduced intersection.

The set of such s can be determined explicitly by working with $K(X_K)$ and a regular integral model Y of Y_K over S; one does not need to compute X.

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