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Limit Theory for Taboo-Regenerative Processes

PETER W. GLYNN * glynn@leland.Stanford.edu
Department of Engineering-Economic Systems and Operations Research, Stanford University, Stanford, CA 94305-4023, USA

HERMANN THORISSON hermann@hi.is
Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland

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Abstract. The paper considers the two-sided taboo limit process that arises when a regenerative process X is conditioned on staying out of a specified set of states (taboo set) over a long period of time. The taboo limit process after time 0 is a version of X, and the time-reversal of the taboo limit process before time 0 is regenerative with tabooed cycles having exponentially biased lengths. The cycle straddling zero has that same bias up to time 0, and is unbiased after time 0. This extends to processes regenerative in the wide sense, and to processes that only regenerate as long as they have not entered the taboo set.

Keywords: regenerative process, quasi-stationarity, renewal theory

1. Introduction

In this paper, we study the dynamics of a regenerative process X conditioned on its not having visited some specified subset (*taboo set*) over a long period of time t. For t large, it is desirable to approximate the conditioned process via a two-sided "taboo limit" in which the pre-t and post-t behavior of X are approximated by the limit process prior to time zero and subsequent to zero, respectively.

We establish conditions under which such a two-sided process describes the limiting behavior of the original conditioned process as t goes to infinity, and investigate the regenerative structure of the taboo limit process. As might be expected, the limit process has regenerative cycles of three different types. The first type of cycle describes the regenerative cycles completed prior to time t, under which X is prohibited from visiting the specified subset. It turns out that this type of cycle is not just a typical regenerative cycle on which the process is conditioned to not enter the taboo set. Rather, it is a cycle which has not only the taboo-conditioning property but is also exponentially twisted (or biased) so as to be longer than a tabooed cycle. The second type of cycle describes the cycle that traps time t. On the first part of the cycle, the process X is again prohibited from visiting the taboo set (and exponentially biased). However, over the latter segment

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of the cycle, such a visit is permissible. Finally, the third type of cycle is a standard (unconditioned) cycle of X. This describes the post-t cycles that are initiated after t and for which the conditioning enforces no restriction on the probabilistic behaviour of the process.

This paper is a companion paper to [6,7]. The former deals with the Markov case and the latter with general processes.

In [6], the taboo limit process is investigated in the Markov process setting using very different techniques (for example, by exploiting the eigenstructure of certain non-negative operators), and a perfect simulation algorithm for simulating the taboo limit is proposed. In contrast to the current paper, the taboo limit is characterized via the standard Markov descriptions of transition dynamics, namely the one-step transition kernel (in discrete time) and the infinitesimal generator (in continuous time). On the other hand, in the current paper, our goal is to fully expose the regenerative structure of such taboo limits. In particular, it turns out that the theory developed in this paper can be applied to more general stochastic objects than classically regenerative processes with i.i.d. cycles. Our limit theorems also extend beyond what is known in the Markov setting by exploiting regenerative analysis to obtain theorems that describe the uniform convergence that typically occurs in this setting; see, for example, theorem 2, propositions 8 and 9. In addition, we provide a central limit theorem and large deviations result for additive functionals of regenerative processes conditioned on not hitting the taboo set over a long period of time t.

Glynn and Thorisson [7] introduces the concept of "taboo-stationarity": a general stochastic process in two-sided time is defined to be taboo-stationary if its global distribution does not change by shifting the origin to an arbitrary time in the future *under taboo*, that is, conditionally on some taboo-event not having occurred up to the new time-origin. It is shown that taboo-stationarity is the characterizing property of a "taboo-limit" process in the same way as stationarity is the characterizing property of an ordinary limit process. The main result is the following basic structural characterization: a process is taboo-stationary *if and only if* it can be represented as a stochastic process with origin shifted backward in time by an independent exponential random variable.

The present paper provides a regenerative perspective on a body of literature that has a long tradition in the Markov process context. This includes the work on rarity and exponentiality by Keilson [8] and the substantial body of literature on R-recurrence for non-negative kernels and the associated quasi-stationary distribution theory; see, for example, [11–14,16]. Thorisson [15] presents a coupling approach to taboo limits.

The plan of the paper is as follows. After preparations in section 2, we develope the taboo limit theory in section 3 and consider the structure of the taboo-limit process in section 4. Section 5 presents some additional limit theory.

2. Preliminaries

Let $X = (X(t)\colon t \geqslant 0)$ be a stochastic process taking values in a complete separable metric space S; we assume that the paths of X are right continuous with left limits. Let

D denote the set of such paths and $\mathcal{D}$ denote the σ-algebra generated by the Skorohod topology. For a given nondecreasing sequence $(T(n)\colon\ n \geqslant 0)$ of random times satisfying $0 = T(-1) \leqslant T(0) < T(1) < \cdots$, take $\Delta \notin S$ and define the sequence $(W_n\colon\ n \geqslant 0)$ of "cycle processes" via

$$W_n(t) = \begin{cases} X\big(T(n-1)+t\big); & 0 \leqslant t < \tau_n, \\ \Delta; & t \geqslant \tau_n, \end{cases}$$

where $\tau_n = T(n) - T(n-1)$ for $n \geqslant 0$. Call W_0 the delay cycle and $X^\circ = X(T(-1) + t\colon\ t \geqslant 0)$ the non-delayed version of X.

Definition 1. We say that X is *classically regenerative* (with respect to $(T(n)\colon\ n \geqslant 0)$) if

(i) $(W_j\colon\ j \geqslant 0)$ is a sequence of independent random elements;

(ii) $(W_j\colon\ j \geqslant 1)$ is a sequence of identically distributed random elements.

Our goal is to study the behavior of a regenerative process when one conditions the process on not exiting a pre-specified set $A \subseteq S$ over a long period of time. For our pre-specified set A, let

$$\Gamma = \inf\big\{t \geqslant 0\colon\ X(t) \in A^c\big\}$$

be the exit time of the set A. Put $\widetilde{S} = S \cup \{\Delta\}$, and let $D_{\widetilde{S}}[0,\infty)$ be the corresponding function space of right-continuous $\widetilde{S}$-valued functions $x : [0,\infty) \to \widetilde{S}$ with left limits. For $x \in D_{\widetilde{S}}[0,\infty)$, let $\gamma(x) = \inf\{t \geqslant 0\colon\ x(t) \in A^c\}$, and put $i(x) = 0$ if $\gamma(x) < \infty$ and 1 otherwise. Assume that γ is measurable. Put $I_j = i(W_j)$ and note that $I_j = 1$ if and only if X remains within A over the j'th cycle. Note that

$$\Gamma = \gamma(X).$$

Put

$$\Gamma^\circ = \gamma\big(X^\circ\big),$$

the exit time of A for the non-delayed version of X. Put

$$\gamma_1 = \gamma(W_1),$$

the exit time of A for the cycle process W_1.

There are various ways of requiring that the process X not exit A over a long period of time. One obvious approach is to consider $\mathbf{P}(X \in \cdot|\Gamma > T(n))$ for n large.

Proposition 1. Suppose X is classically regenerative. Then, for $n, m \geqslant 0$,

$$\begin{aligned} &\mathbf{P}\big(W_j \in B_j\colon\ 0 \leqslant j \leqslant n+m \mid \Gamma > T(n)\big) \\ &\quad = \mathbf{P}(W_0 \in B_0 \mid I_0 = 1) \cdot \prod_{j=1}^{n} \mathbf{P}(W_1 \in B_j \mid I_1 = 1) \cdot \prod_{j=n+1}^{n+m} \mathbf{P}(W_1 \in B_j). \end{aligned}$$

Proof. Note that the independent cycle structure of X guarantees that

$$\begin{aligned}
&\mathbf{P}\big(W_j \in B_j\colon 0 \leqslant j \leqslant n+m \mid \Gamma > T(n)\big) \\
&\quad = \frac{\mathbf{E}\prod_{j=0}^{n} I(W_j \in B_j) I_j \prod_{j=n+1}^{n+m} I(W_j \in B_j)}{\mathbf{E}\prod_{j=0}^{n} I_j} \\
&\quad = \frac{\prod_{j=0}^{n} \mathbf{E} I(W_j \in B_j) I_j \prod_{j=n+1}^{n+m} \mathbf{E} I(W_j \in B_j)}{\prod_{j=0}^{n} \mathbf{E} I_j} \\
&\quad = \prod_{j=0}^{n} \mathbf{P}(W_j \in B_j \mid I_j = 1) \prod_{j=n+1}^{n+m} \mathbf{P}(W_j \in B_j) \\
&\quad = \mathbf{P}(W_0 \in B_0 \mid I_0 = 1) \cdot \prod_{j=1}^{n} \mathbf{P}(W_j \in B_j \mid I_j = 1) \cdot \prod_{j=n+1}^{n+m} \mathbf{P}(W_1 \in B_j);
\end{aligned}$$

property (ii) of classical regeneration was used in the final step. □

Thus, when conditioning on the random-cycle-based time scale, the conditional law of X just involves forcing the “taboo-cycles” to remain within A. Perhaps surprisingly, we will find in section 3 that the conditional law of X is quite different when conditioning X on $\Gamma > t$ for t large.

Remark 1. Let $X = (X_n\colon n \geqslant 0)$ be a discrete-time Markov chain (DTMC) living on a discrete state space S. The transition function of X within a cycle that is conditioned on $I_j = 1$ can easily be computed. Specifically, let $P = (P(x, y)\colon x, y \in S)$ be the transition matrix of X. Suppose the $T(j)$'s are the consecutive hitting times of a given state $z \in A$. For $x \in S$, put $\tilde{u}(x) = \mathbf{P}_x(T_z < \Gamma)$, where $T_z = \inf\{n \geqslant 1\colon X_n = z\}$ and $\mathbf{P}_x(\cdot) = \mathbf{P}(\cdot \mid X_0 = x)$. Then, note that

$$\begin{aligned}
&\mathbf{P}_z(X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, T_z = n \mid \Gamma > T_z) \\
&\quad = P(z, x_1) P(x_1, x_2) \cdots \frac{P(x_{n-1}, z)}{\tilde{u}(z)} \\
&\quad = \frac{1}{\tilde{u}(z)} \left[\widetilde{P}(z, x_1) \frac{\tilde{u}(z)}{\tilde{u}(x_1)} \cdots \widetilde{P}(x_{n-2}, x_{n-1}) \frac{\tilde{u}(x_{n-2})}{\tilde{u}(x_{n-1})} \widetilde{P}(x_{n-1}, z) \tilde{u}(x_{n-1}) \right] \\
&\quad = \widetilde{P}(z, x_1) \widetilde{P}(x_1, x_2) \cdots \widetilde{P}(x_{n-1}, z),
\end{aligned}$$

where $\widetilde{P}(x, y) = P(x, y)\tilde{u}(y)/\tilde{u}(x)$ for $x \in A$, $z \neq y \in A$, and $\widetilde{P}(x, z) = P(x, z)/\tilde{u}(z)$ for $x \in A$. It is easily verified that $(\widetilde{P}(x, y)\colon x, y \in A)$ is a stochastic matrix, as a consequence of the fact that $(\tilde{u}(x)\colon x \in A)$ is the minimal non-negative solution to the linear system

$$u(x) = P(x, z) + \sum_{y \neq z, y \in A} P(x, y) u(y), \qquad x \in A.$$

Hence, conditioned on staying within A within a z-cycle, X evolves as a DTMC with transition matrix $(\widetilde{P}(x, y)\colon\ x, y \in A)$.

Our principal emphasis in this paper is on the limiting "taboo" behavior of X, when X is conditioned on $\Gamma > t$ (with t large). Our analysis will require only that X be "taboo-regenerative". As in the setting of classically regenerative processes, the definition of taboo-regeneration requires the existence of a sequence $(T(j)\colon\ j \geqslant 0)$ of random times and corresponding cycles $(W_j\colon\ j \geqslant 0)$. In the taboo case we assume, however, that $T(n) = \infty$ on $\{\Gamma < T(n)\}$, $n \geqslant 0$.

Definition 2. We say that X is *taboo-regenerative in the wide sense* (with respect to $(T(j)\colon\ j \geqslant 0)$) if

$$\begin{aligned}&\mathbf{P}\big((W_{n+1}, W_{n+2}, \ldots) \in \cdot \mid T(0), T(1), \ldots, T(n);\ \Gamma \geqslant T(n)\big)\\ &\quad = \mathbf{P}\big((W_1, W_2, \ldots) \in \cdot \mid \Gamma \geqslant T(0)\big)\end{aligned}$$

for $n \geqslant 0$.

Remark 2. Any classically regenerative process is automatically taboo-regenerative in the wide sense (regardless of the choice of A).

Remark 3. The generalization of classical regeneration to taboo-regeneration in the wide sense is a valuable extension, particularly in the Markov chain context. Specifically, consider an S-valued Markov chain $X = (X_n\colon\ n \geqslant 0)$. For $n \geqslant 1$, $A \subseteq S$, let

$$B^n(x, \mathrm{d}y) = \mathbf{P}(X_n \in \mathrm{d}y,\ \Gamma > n \mid X_0 = x)$$

for $x, y \in A$. Suppose that there exists a set $C \subseteq A$ for which $\mathbf{P}_x(T_C < \infty, \Gamma > T_C) > 0$ for $x \in A$, where $T_C = \inf\{n \geqslant 0\colon\ X_n \in C\}$. We further demand that there exist $m \geqslant 1$, $\lambda > 0$, and a probability φ such that

$$B^m(x, \cdot) \geqslant \lambda\varphi(\cdot)$$

for $x \in C$. Let $\beta(1) = T_C$, $\beta(n+1) = \inf\{j \geqslant \beta(n) + m\colon\ X_j \in C\}$ be the consecutive hitting times of C (that are separated by at least m time units). At each time $\beta(n)$, distribute $X_{\beta(n)+m}$ according to φ with probability λ, according to $(B^m(x, \cdot) - \lambda\varphi(\cdot))/(B^m(x, A) - \lambda)$ with probability $B^m(x, A) - \lambda$, and according to $(\mathbf{P}(X_m \in \cdot \mid X_0 = x) - B^m(x, \cdot))/(1 - B^m(x, A))$ with probability $1 - B^m(x, A)$. In the first two cases "condition in" $X_{\beta(n)+1}, \ldots, X_{\beta(n)+m-1}$, based on $X_{\beta(n)} = x$ and $X_{\beta(n)+m} = y$, according to $\mathbf{P}_x(X_1 \in \cdot, \ldots, X_{m-1} \in \cdot \mid \Gamma > m, X_m = y)$ and in the third case according to $\mathbf{P}_x(X_1 \in \cdot, \ldots, X_{m-1} \in \cdot \mid \Gamma \leqslant m, X_m = y)$. To implement this randomization at time $\beta(n)$ we use a *r.v.* χ_n taking values 1, 2 and 3 to indicate the three cases. Note that whenever $X_{\beta(n)+m}$ is distributed according to φ, $X_{\beta(n)+i} \in A$ for $1 \leqslant i \leqslant m-1$. Let $\beta(\kappa(n)) + m, n \geqslant 1$, be the subsequence of times, at which X

has distribution φ. Put $T(n) = \beta(\kappa(n)) + m$ when $\beta(\kappa(n)) + m \leqslant \Gamma$ and $T(n) = \infty$ otherwise. If $\mathbf{P}_\varphi$ is the probability under which X has initial distribution φ, then

$$\begin{aligned}
&\mathbf{P}_\varphi\big(X_{T(n)} \in \cdot \mid T(0), \ldots, T(n); \Gamma \geqslant T(n)\big) \\
&\quad = \mathbf{E}_\varphi \mathbf{P}_\varphi\big(X_{T(n)} \in \cdot \mid X_0, \ldots, X_{\beta(\kappa(n))}, \chi_1, \ldots, \chi_{\kappa(n)};\ \Gamma \geqslant T(n)\big) \\
&\quad = \mathbf{E}_\varphi \mathbf{P}_\varphi\big(X_{\beta(\kappa(n))+m} \in \cdot \mid X_{\beta(\kappa(n))}; \chi_{\kappa(n)} = 1,\ \Gamma \geqslant \beta\big(\kappa(n)\big)\big) \\
&\quad = \varphi(\cdot),
\end{aligned}$$

proving that X can be viewed as taboo-regenerative in the wide sense. Note that $X_{T(n)-1}, \ldots, X_{T(n)-m+1}$ are typically correlated with both $X_{T(n)-m}$ and $X_{T(n)}$. Thus, X is not classically regenerative with respect to the $T(n)$'s.

It should be noted that the construction of the $T(n)$'s above is a taboo version of the "splitting" construction used by Athreya and Ney [4] and Nummelin [10] to produce regeneration times in the Harris chain context.

Remark 4. It should be obvious from definition 2 that the definition can be extended to random times Γ that are not defined as exit times. For example, Γ could be the first time that X visits A^c three times in a regenerative cycle. We will proceed, throughout the rest of this paper, under the assumption that Γ is the first exit time from A (for clarity of exposition), despite the fact that virtually all the theory presented here extends to a somewhat more general class of random times.

Returning now to the question of how X behaves when conditioned on $\Gamma > t$, a first step is understanding the asymptotics of $\mathbf{P}(\Gamma > t)$ as $t \to \infty$. Suppose $T(0) = 0$, so that we have a "non-delayed" taboo-regenerative process in the wide sense. Then,

$$\mathbf{P}(\Gamma > t) = \mathbf{P}(\Gamma \wedge \tau_1 > t) + \int_{[0,t]} \mathbf{P}(\Gamma > t - s)\mathbf{P}(\tau_1 \in \mathrm{d}s,\ \Gamma \geqslant \tau_1).$$

Consequently, $a(\cdot) \stackrel{\triangle}{=} \mathbf{P}(\Gamma > \cdot)$ satisfies a renewal equation. An assumption that is frequently imposed in order to transform such a defective renewal equation into a proper renewal equation is the following:

A1. There exists $\alpha \geqslant 0$ such that

$$\mathbf{E}\big[\mathrm{e}^{\alpha\tau_1} I_1\big] = 1.$$

With α at our disposal, we can re-write the above defective renewal equation as

$$\tilde{a}(\cdot) = \tilde{b}(\cdot) + \big(\widetilde{G} * \tilde{a}\big)(\cdot), \tag{2.1}$$

where $*$ denotes convolution, $\tilde{a}(t) = \mathrm{e}^{\alpha t} a(t)$, $\widetilde{G}(\mathrm{d}t) = \mathrm{e}^{\alpha t}\mathbf{E} I(\tau_1 \in \mathrm{d}t) I_1$, and $\tilde{b}(t) = \mathrm{e}^{\alpha t}\mathbf{P}(\gamma_1 \wedge \tau_1 > t)$, with $\gamma_1 = \gamma(W_1)$. (Note that when $T(0) = 0$, $\gamma_1 \wedge \tau_1 = \Gamma \wedge \tau_1$.)

To proceed further, we shall apply the renewal theorem. In particular, we shall invoke the hypotheses necessary to apply Smith's version of the renewal theorem:

A2. The distribution G is spread-out, where $G(\mathrm{d}t) \triangleq \mathbf{P}(\tau_1 \in \mathrm{d}t, \gamma_1 > \tau_1)$;

A3. $\mathbf{E}[\tau_1 \exp(\alpha\tau_1) I_1] < \infty$;

A4. $\mathrm{e}^{\alpha t}\mathbf{P}(\gamma_1 \wedge \tau_1 > t)$ is dominated by a non-increasing integrable function h.

Proposition 2. Suppose that X is taboo-regenerative in the wide sense and satisfies A1–A4. Then, if $T(0) = 0$,

$$\mathbf{P}(\Gamma > t) \sim c_1 \exp(-\alpha t)$$

as $t \to \infty$, where

$$c_1 = \frac{\mathbf{E}[\exp(\alpha(\gamma_1 \wedge \tau_1)) - 1]}{\alpha \mathbf{E}[\tau_1 \exp(\alpha\tau_1) I_1]}.$$

Proof. Given that

$$\begin{aligned}\int_0^\infty \tilde{b}(s)\,\mathrm{d}s &= \mathbf{E}\int_0^\infty \mathrm{e}^{\alpha s} I(\gamma_1 \wedge \tau_1 > s)\,\mathrm{d}s \\ &= \mathbf{E}\int_0^{\gamma_1\wedge\tau_1} \mathrm{e}^{\alpha s}\,\mathrm{d}s \\ &= \frac{\mathbf{E}[\exp(\alpha(\gamma_1 \wedge \tau_1)) - 1]}{\alpha}\end{aligned}$$

and

$$\begin{aligned}\int_{[0,\infty)} t\widetilde{G}(\mathrm{d}t) &= \int_{[0,\infty)} t\mathrm{e}^{\alpha t}\mathbf{P}(\tau_1 \in \mathrm{d}t,\ \gamma_1 > \tau_1) \\ &= \mathbf{E}\big[\tau_1 \exp(\alpha\tau_1) I_1\big],\end{aligned}$$

the result follows immediately from Smith's version of the renewal theorem; see, for example, [2]. □

Remark 5. Note that

$$\begin{aligned}\mathbf{E}\big[\exp\big(\alpha(\gamma_1 \wedge \tau_1)\big) - 1\big] &= \mathbf{E}\big[\exp(\alpha\gamma_1); \gamma_1 < \tau_1\big] + \mathbf{E}\big[\exp(\alpha\tau_1); \tau_1 < \gamma_1\big] - 1 \\ &= \mathbf{E}\big[\exp(\alpha\gamma_1); \gamma_1 < \tau_1\big],\end{aligned}$$

so c_1 can be alternatively expressed as $c_1 = \mathbf{E}[\exp(\alpha\gamma_1); \gamma_1 < \tau_1]/\alpha\mathbf{E}[\tau_1 \exp(\alpha\tau_1) I_1]$.

We now generalize proposition 2 to the case in which X has a nonzero delay.

A5. $\mathbf{E}[\exp(\alpha\tau_0); \Gamma > \tau_0] < \infty$;

A6. $\mathbf{P}(\tau_0 \wedge \Gamma > t)\,\mathrm{e}^{\alpha t} \to 0$ as $t \to \infty$.

Proposition 3. Under A1–A6,

$$\mathbf{P}(\Gamma > t) \sim c_2 \exp(-\alpha t)$$

as $t \to \infty$, where $c_2 = \mathbf{E}[\exp(\alpha\tau_0);\ \Gamma > \tau_0]c_1$.

Proof. Observe that

$$\begin{aligned} \mathrm{e}^{\alpha t}\mathbf{P}(\Gamma > t) = {} & \mathbf{P}(\Gamma \wedge \tau_0 > t)\mathrm{e}^{\alpha t} \\ & + \int_{[0,t]} \mathrm{e}^{\alpha s}\mathbf{P}(\tau_0 \in \mathrm{d}s,\ \Gamma > \tau_0) \cdot \exp\bigl(\alpha(t-s)\bigr)\mathbf{P}\bigl(\Gamma^\circ > t-s\bigr). \end{aligned}$$

The first term on the right-hand side goes to zero by A6. On the other hand, proposition 2 ensures that $\exp(\alpha(t-s))\mathbf{P}(\Gamma^\circ > t-s)$ is a bounded function which converges pointwise to c_1 as $t \to \infty$ (for each fixed s). The result then follows from A5 and the Bounded Convergence Theorem. $\square$

We conclude this section with a discussion of the assumptions A1–A6. We start with A1. Let

$$\begin{aligned} \kappa_1 &= \sup\bigl\{\theta\colon\ \mathbf{E}\bigl[\exp(\theta\tau_1)I_1\bigr] \leqslant 1\bigr\}, \\ \kappa_2 &= \sup\bigl\{\theta\colon\ \mathbf{E}\bigl[\exp(\theta\tau_1)I_1\bigr] < \infty\bigr\}. \end{aligned}$$

Note that if $\mathbf{P}(0 < \tau_1 < \gamma_1) > 0$, it follows that $\mathbf{E}[\exp(\theta\tau_1)I_1] \to +\infty$ as $\theta \to \infty$, and hence $\kappa_1 < \infty$. If $\kappa_2 > \kappa_1$, the continuity and strict monotonicity of $\mathbf{E}[\exp(\theta\tau_1)I_1]$ in θ on $[0, \kappa_2)$ implies the existence of a unique α in A1. Consequently, A1 is guaranteed if $\kappa_2 > \kappa_1$. Let

$$\begin{aligned} \kappa_3 &= \sup\bigl\{\theta\colon\ \mathbf{E}\exp\bigl(\theta\Gamma^\circ\bigr) < \infty\bigr\}, \\ \kappa_4 &= \sup\bigl\{\theta\colon\ \mathbf{E}\bigl[\exp\bigl(\theta\Gamma^\circ\bigr);\ \Gamma^\circ < \tau_1\bigr] < \infty\bigr\}, \\ \kappa_5 &= \sup\bigl\{\theta\colon\ \mathbf{E}\exp\bigl(\theta\bigl(\Gamma^\circ \wedge \tau_1\bigr)\bigr) < \infty\bigr\}. \end{aligned}$$

Clearly, $\kappa_5 = \min(\kappa_2, \kappa_4)$. Furthermore, $\kappa_3 = \min(\kappa_1, \kappa_4)$ because (due to taboo-regeneration)

$$\mathbf{E}\exp\bigl(\theta\Gamma^\circ\bigr) = \mathbf{E}\bigl[\exp\bigl(\theta\Gamma^\circ\bigr);\ \Gamma^\circ < \tau_1\bigr] + \mathbf{E}\bigl[\exp(\theta\tau_1);\ \tau_1 < \Gamma^\circ\bigr]\mathbf{E}\exp\bigl(\theta\Gamma^\circ\bigr).$$

We conclude that a sufficient condition for the existence of α is that $\kappa_3 < \kappa_5$.

Suppose that there exists $h > 0$ and $c < 1$ such that

$$\mathbf{P}\bigl(\Gamma^\circ \wedge \tau_1 > (n+1)h \mid \Gamma^\circ \wedge \tau_1 > nh\bigr) \leqslant c\mathbf{P}\bigl(\Gamma^\circ > (n+1)h \mid \Gamma^\circ > nh\bigr) \tag{2.2}$$

for n sufficiently large, say $n \geqslant n_0$. Then, for $n \geqslant n_0$,

$$\begin{aligned} & \mathbf{P}\bigl(\Gamma^\circ \wedge \tau_1 > nh\bigr) \\ & \quad = \mathbf{P}\bigl(\Gamma^\circ \wedge \tau_1 > 0\bigr)\prod_{j=0}^{n-1}\mathbf{P}\bigl(\Gamma^\circ \wedge \tau_1 > (j+1)h \mid \Gamma^\circ \wedge \tau_1 > jh\bigr) \end{aligned}$$

$$\leqslant \frac{\mathbf{P}(\Gamma^\circ \wedge \tau_1 > 0)}{\mathbf{P}(\Gamma^\circ > 0)} \prod_{j=0}^{n_0-1} \frac{\mathbf{P}(\Gamma^\circ \wedge \tau_1 > (j+1)h \mid \Gamma^\circ \wedge \tau_1 > jh)}{\mathbf{P}(\Gamma^\circ > (j+1)h \mid \Gamma^\circ > jh)}$$
$$\times c^{n-n_0}\mathbf{P}\big(\Gamma^\circ > nh\big).$$

It follows that $\kappa_5 \geqslant \kappa_3 - (\log c)/h$. Because $\kappa_3 \leqslant \kappa_1 < \infty$, we may summarize our discussion as follows.

Proposition 4. If $\mathbf{P}(0 < \tau_1 < \Gamma^\circ) > 0$, then A1 is implied either by $\kappa_3 < \kappa_5$ or (2.2).

Remark 6. In the Markov case, verifying (2.2) can be substantially simplified. Using the same notation as in remark 3, suppose there exists $a, b, m > 1$, and φ such that

$$a\varphi(\cdot) \leqslant B^m(x, \cdot) \leqslant b\varphi(\cdot) \tag{2.3}$$

for $x \in A$. Then, by utilizing the "splitting" idea as in remark 3, we can construct τ_1 so that

$$\begin{aligned}
&\mathbf{P}\big(\Gamma^\circ \wedge \tau_1 > (n+1)m \mid \Gamma^\circ \wedge \tau_1 > nm\big)\\
&\quad = B^m(x, A) - a\\
&\quad \leqslant B^m(x, A) - a\left(\frac{1}{b}B^m(x, A)\right)\\
&\quad = \left(1 - \frac{a}{b}\right)B^m(x, A)\\
&\quad = \left(1 - \frac{a}{b}\right)\mathbf{P}\big(\Gamma^\circ > (n+1)m \mid \Gamma^\circ > nm\big).
\end{aligned}$$

So, (2.3) ensures the validity of A1; this sufficient condition has been previously identified by Ney and Nummelin [9] in the Markov context.

We turn next to studying A3 and A4.

Proposition 5. If A1 holds and $\mathbf{E}[(\tau_1 \wedge \Gamma^\circ)\exp(x(\tau_1 \wedge \Gamma^\circ))] < \infty$, then A3 and A4 are satisfied.

Proof. Since A3 is trivial, we focus on A4. Note that

$$\exp(\alpha t)\mathbf{P}\big(\tau_1 \wedge \Gamma^\circ > t\big) \leqslant \mathbf{E}\big[\exp\big(\alpha\big(\Gamma^\circ \wedge \tau_1\big)\big); \Gamma^\circ \wedge \tau_1 > t\big].$$

The right-hand side is clearly non-increasing; it integrates to $\mathbf{E}[(\tau_1 \wedge \Gamma^\circ)\exp(\alpha(\tau_1 \wedge \Gamma^\circ))]$. □

We finally turn to consideration of A5 and A6.

Proposition 6. If $\mathbf{E}\exp(\alpha(\tau_0 \wedge \Gamma)) < \infty$, then A5 and A6 are satisfied.

Proof. Clearly, A5 is implied by our assumption. Also,

$$\mathbf{P}(\tau_0 \wedge \Gamma > t)\, \mathrm{e}^{\alpha t} \leqslant \mathbf{E}\big[\exp\big(\alpha(\tau_0 \wedge \Gamma)\big);\, \tau_0 \wedge \Gamma > t\big],$$

which converges to zero because $\mathbf{E}[\exp(\alpha(\tau_0 \wedge \Gamma))] < \infty$. □

Remark 7. In our companion paper [6] on taboo limit theory in the Markov setting, our fundamental hypotheses involve the solution of an eigenvalue problem. To see how this is related to our current hypotheses, suppose (for simplicity) that A is discrete in remark 3. Adopting the notation of remark 1, A1 asserts that

$$\mathbf{E}_z\big[\mathrm{e}^{\alpha T_z};\, T_z < \Gamma\big] = 1, \tag{2.4}$$

where $\mathbf{E}_x(\cdot)$ is the expectation operator associated with $\mathbf{P}_x(\cdot)$. Set

$$r(x) = \mathbf{E}_x\big[\exp(\alpha T_z);\, T_z < \Gamma\big]$$

with the aid of (2.4), it is easy to establish that

$$r(x) = \mathrm{e}^{\alpha} \sum_{y \in A} P(x, y) r(y)$$

for $x \in A$. In other words, r is the eigenfunction of B associated with the eigenvalue $\mathrm{e}^{-\alpha}$. Hence, in the Markov context, A1 can be viewed as offering a regenerative characterization of the solution to the (Perron–Frobenius) eigenvalue problem associated with B.

Remark 8. Note that proposition 2 implies that for each fixed $s \geqslant 0$, $\mathbf{P}(\Gamma > t + s \mid \Gamma > t) \to \exp(-\alpha s)$ as $t \to \infty$. It is easy to see that if $\mathbf{P}(\Gamma > t + \cdot \mid \Gamma > t) \to \overline{G}(\cdot)$ as $t \to \infty$, then $\overline{G}$ must be the tail of an exponential r.v. In particular, $\mathbf{P}(\Gamma > t + s_1 + s_2 \mid \Gamma > t) = \mathbf{P}(\Gamma > t + s_1 + s_2 \mid \Gamma > t + s_2)\mathbf{P}(\Gamma > t + s_2 \mid \Gamma > t)$, so taking limits with respect to t establishes that $\overline{G}(s_1 + s_2) = \overline{G}(s_1)\overline{G}(s_2)$. So, $\overline{G}$ is memoryless, and therefore exponential.

3. Taboo limit theory for regenerative processes

In this section, we shall be concerned with developing limit theory that can be used to establish approximations for the process X, conditional on requiring that X remain in A over some long period of time $[0, t]$ (with t large). For $t \geqslant 0$, let $X_t = (X(t+s)\colon s \geqslant 0)$ be the post-t process. Our first result concerns the case in which X is non-delayed.

Theorem 1. Let X be taboo-regenerative in the wide sense, and suppose $T(0) = 0$. If X satisfies A1–A4, then,

$$\lim_{t\to\infty} \sup_{B \in \mathcal{D}} \left| \mathbf{P}(X_t \in B \mid \Gamma > t + s) - \frac{\mathbf{E} \int_0^{\tau_1} \mathrm{e}^{\alpha(u+s)} I(X_u \in B, \Gamma > u + s)\, \mathrm{d}u}{\mathbf{E} \int_0^{\tau_1 \wedge \Gamma} \mathrm{e}^{\alpha u}\, \mathrm{d}u} \right| = 0.$$

Proof. Note that for any measurable subset B and $s \geqslant 0$, $a_{B,s}(t) = \mathrm{e}^{\alpha t}\mathbf{P}(X_t \in B, \Gamma > t+s)$ satisfies the renewal equation

$$a_{B,s}(t) = b_{B,s}(t) + \big(\widetilde{G} * a_{B,s}\big)(t),$$

where

$$b_{B,s}(t) = \mathrm{e}^{\alpha t}\mathbf{P}(X_t \in B,\ \tau_1 > t, \Gamma > t+s).$$

But

$$\sup_{B,s}\big|b_{B,s}(t)\big| \leqslant \mathrm{e}^{\alpha t}\mathbf{P}(\tau_1 \wedge \Gamma > t),$$

and the right-hand side is bounded above by a non-increasing integrable function; see A4. It follows that

$$\sup_{B,s}\left|a_{B,s}(t) - \frac{\int_0^\infty b_{B,s}(u)\,\mathrm{d}u}{\int_0^\infty u\widetilde{G}(\mathrm{d}u)}\right| \to 0$$

as $t \to \infty$; see [1].

Proposition 2 therefore implies that

$$\sup_{B\in\mathcal{D}}\left|\mathbf{P}(X_t \in B \mid \Gamma > t+s) - \frac{\int_0^\infty \mathrm{e}^{\alpha s} b_{B,s}(u)\,\mathrm{d}u}{\mathbf{E}\int_0^{\tau_1\wedge\Gamma}\mathrm{e}^{\alpha u}\,\mathrm{d}u}\right| \to 0$$

as $t \to \infty$. Using Fubini's theorem, the above limit may be simplified to the form specified by theorem 1, thereby completing the proof. □

Let $X_u^\circ = (X(T(0)+u+s)\colon\ s \geqslant 0)$ for $u \geqslant 0$. The following "delayed" analog of theorem 1 is an easy consequence of the result.

Corollary 1. Let X be taboo-regenerative in the wide sense. If X satisfies A1–A6, then

$$\sup_{\substack{s\geqslant 0\\ B\in\mathcal{D}}}\left|\mathbf{P}(X_t \in B \mid \Gamma > t+s) - \frac{\mathbf{E}\int_0^{\tau_1}\mathrm{e}^{\alpha(u+s)}I(X_u^\circ \in B, \Gamma^\circ > u+s)\,\mathrm{d}u}{\mathbf{E}\int_0^{\tau_1\wedge\Gamma}\mathrm{e}^{\alpha u}\,\mathrm{d}u}\right|$$

converges to zero as $t \to \infty$.

Remark 9. Note that corollary 1 asserts that whenever A1–A6 are in force,

$$\mathbf{P}(X(t) \in \cdot \mid \Gamma > t) \stackrel{\text{t.v.}}{\to} \frac{\mathbf{E}\int_0^{\tau_1}\mathrm{e}^{\alpha u}I(X^\circ(u) \in \cdot, \Gamma^\circ > u)\,\mathrm{d}u}{\mathbf{E}\int_0^{\tau_1\wedge\Gamma^\circ}\mathrm{e}^{\alpha u}\,\mathrm{d}u}$$

as $t \to \infty$, where $\stackrel{\text{t.v.}}{\to}$ denotes convergence in the sense of total variation. The above limiting marginal distribution (on S) is the regenerative analog of the "quasi-stationary distribution" that appears in the corresponding theory for Markov processes. Corollary 1 shows that under A1–A6, this limit distribution is independent of the initial delay, just as in the Markov context.

Remark 10. If $(a_t\colon t \geqslant 0)$ is a deterministic function for which $a_t \to \infty$ as $t \to \infty$ with $a_t \leqslant t$, then corollary 1 asserts that, under the conditions stated there,

$$\sup_{B\in\mathcal{D}}\left|\mathbf{P}(X_{a_t} \in B \mid \Gamma > t) - \frac{\mathbf{E}\int_0^{\tau_1} \mathrm{e}^{\alpha(u+(t-a_t))} I(X_u \in B, \Gamma > u + (t - a_t))\,\mathrm{d}u}{\mathbf{E}\int_0^{\tau_1\wedge\Gamma} \mathrm{e}^{\alpha u}\,\mathrm{d}u}\right|$$

converges to zero as $t \to \infty$. So, conditional on $\Gamma > t$, corollary 1 provides an approximation to the tabooed process over almost the entire time parameter set of X, excepting a (relatively) small interval $[0, a_t)$ at the beginning.

In view of remark 10, we now develop a separate limit theorem that describes the asymptotic behavior of the initial segment of the tabooed process. For this purpose, we add the following assumption.

A7. $\kappa_2 > \kappa_1$.

As noted in section 2, this implies A1. Let $N(t) = \max\{n\colon T(n) \leqslant t\}$.

Proposition 7. Let X be classically regenerative, and suppose $T(0) = 0$. Assume A2–A4 and A7 and let $(a_t\colon t \geqslant 0)$ be a deterministic non-negative nondecreasing function for which $t - a_t \to \infty$ as $t \to \infty$. Then,

$$\begin{aligned}\sup_{B\in\mathcal{D}}\Big|&\mathbf{P}\big(\big(X(u)\colon 0 \leqslant u \leqslant a_t\big) \in B \mid \Gamma > t\big)\\ &-\mathbf{E}\big[I\big(\big(X(u) : 0 \leqslant u \leqslant a_t\big) \in B\big)\exp\big(\alpha T\big(N(a_t)+1\big)\big); \Gamma > T\big(N(a_t)+1\big)\big]\Big| \to 0\end{aligned}$$

as $t \to \infty$.

Proof. We start by observing that A7 implies that

$$\int_{[0,\infty)} \mathrm{e}^{\theta t}\widetilde{G}(\mathrm{d}t) = \mathbf{E}\big[\exp\big((\theta+\alpha)\tau_1\big)I_1\big] < \infty$$

for some $\theta > 0$. Consequently, it follows, in the proof of proposition 2, that

$$\tilde{a}(t) = c_1 + \mathrm{o}\big(\mathrm{e}^{-\varepsilon t}\big)$$

for some $\varepsilon > 0$; see, for example, [2]. Hence, we may conclude that for $0 \leqslant r \leqslant t$,

$$\begin{aligned}\frac{\mathbf{P}(\Gamma > t - r)}{\mathbf{P}(\Gamma > t)} &= \mathrm{e}^{\alpha r}\frac{\tilde{a}(t-r)}{\tilde{a}(t)}\\ &= \mathrm{e}^{\alpha r}\big(1 + \mathrm{o}\big(\mathrm{e}^{-\varepsilon(t-r)}\big)\big).\end{aligned} \tag{3.1}$$

With (3.1) in hand, note that

$$\begin{aligned}&\mathbf{P}\big(\big(X(u)\colon 0 \leqslant u \leqslant a_t\big) \in B \mid \Gamma > t\big)\\ &\quad= \int_{(a_t,t]}\sum_{j=0}^{\infty}\mathbf{P}\big(\big(X(u)\colon 0 \leqslant u \leqslant a_t\big) \in B,\ T(j) \leqslant a_t < T(j+1)\big),\end{aligned}$$

$$
\begin{aligned}
& T\big((j+1) \in \mathrm{d}r,\ \Gamma > r\big) \cdot \frac{\mathbf{P}(\Gamma > t-r)}{\mathbf{P}(\Gamma > t)} \\
&\quad + \frac{\mathbf{P}((X(u)\colon\ 0 \leqslant u \leqslant a_t) \in B, T(N(a_t)+1) > t, \Gamma > t)}{\mathbf{P}(\Gamma > t)} \\
&= \int_{(a_t,t]} \mathbf{P}\big((X(u)\colon\ 0 \leqslant u \leqslant a_t\big) \in B,\ T\big(N(a_t)+1\big) \in \mathrm{d}r, \Gamma > r\big) \\
&\quad \times \mathrm{e}^{\alpha r}\big(1 + \mathrm{o}\big(\mathrm{e}^{-\varepsilon(t-r)}\big)\big) \\
&\quad + \frac{\mathbf{P}((X(u)\colon\ 0 \leqslant u \leqslant a_t) \in B, T(N(a_t)+1) > t, \Gamma > t)}{\mathbf{P}(\Gamma > t)} \\
&= \mathbf{E}\big[I\big(\big(X(u)\colon\ 0 \leqslant u \leqslant a_t\big) \in B\big) \exp\big(\alpha\big(T\big(N(a_t)+1\big)\big)\big); \Gamma > T\big(N(a_t)+1\big)\big] \\
&\quad + \int_{(a_t,t]} \mathbf{P}\big(\big(X(u)\colon\ 0 \leqslant u \leqslant a_t\big) \in B,\ T\big(N(a_t)+1\big) \in \mathrm{d}r, \Gamma > r\big)\mathrm{e}^{\alpha r} \\
&\quad \times \mathrm{o}\big(\mathrm{e}^{-\varepsilon(t-r)}\big) \\
&\quad - \mathbf{E}\big[I\big(\big(X(u)\colon\ 0 \leqslant u \leqslant a_t\big) \in B\big) \exp\big(\alpha\big(T\big(N(a_t)+1\big)\big)\big);\ \Gamma > T\big(N(a_t)+1\big), \\
&\qquad T\big(N(a_t)+1\big) > t\big] \\
&\quad + \frac{\mathbf{P}((X(u)\colon\ 0 \leqslant u \leqslant a_t) \in B,\ T(N(a_t)+1) > t,\ \Gamma > t)}{\mathbf{P}(\Gamma > t)} \\
&\triangleq \mathbf{E}\big[I\big(\big(X(u)\colon\ 0 \leqslant u \leqslant a_t\big) \in B\big) \exp\big(\alpha\big(T\big(N(a_t)+1\big)\big)\big);\ \Gamma > T\big(N(a_t)+1\big)\big] \\
&\quad + r_1(t) - r_2(t) + r_3(t).
\end{aligned}
$$

Turning first to r_3, let $k(\cdot)$ be the dominating function of A4. For t large enough, $\mathbf{P}(\tau > t) \geqslant 2^{-1}c_1\mathrm{e}^{-\alpha t}$, so r_3 can be dominated (uniformly in B) by

$$
\begin{aligned}
& 2c_1^{-1}\mathrm{e}^{\alpha t}\mathbf{P}\big(T\big(N(a_t)+1\big) > t,\ \Gamma > t\big) \\
&= 2c_1^{-1} \sum_{j=0}^{\infty} \int_{[0,a_t]} \mathrm{e}^{\alpha u}\mathbf{P}\big(T(j) \in \mathrm{d}u, \Gamma > u\big) \cdot \mathrm{e}^{\alpha(t-u)}\mathbf{P}(\tau_1 > t-u,\ \Gamma > t-u) \\
&\leqslant 2c_1^{-1} \sum_{j=0}^{\infty} \int_{[0,a_t]} \widetilde{G}^{(j)}(\mathrm{d}u)k(t-u),
\end{aligned}
$$

where $\widetilde{G}^{(j)}$ denotes the j-fold convolution of $\widetilde{G}$. For $c > 0$, let $k_c(u) = k(c+u)$. For t large enough, $k(t-u) \leqslant k_c(a_t - u)$ for $0 \leqslant u \leqslant a_t$ (since k is non-increasing). So, we can bound the above by $2c_1^{-1}\sum_{j=0}^{\infty}(\widetilde{G}^{(j)} \star k_c)(a_t)$ which converges (by Smith's renewal theorem) to $\int_0^{\infty} k_c(u)\,\mathrm{d}u / \int_{[0,\infty)} u\widetilde{G}(u)\,\mathrm{d}u$. But

$$
\int_0^{\infty} k_c(u)\,\mathrm{d}u = \int_c^{\infty} k(u)\,\mathrm{d}u
$$

converges to zero as $c \to \infty$. Hence, by first letting $t \to \infty$ and then letting $c \to \infty$, we conclude that r_3 goes to zero uniforming in B.

To analyze r_1 and r_2, we introduce the probability $\mathbf{P}_*$ as defined by

$$\mathbf{P}_*\big((W_1, W_2, \dots, W_n) \in \cdot\big) = EI\big((W_1, W_2, \dots, W_n) \in \cdot\big) \cdot \exp\big(\alpha(\tau_1 + \cdots + \tau_n)\big) I\big(\Gamma > T(n)\big).$$

Then, r_2 is clearly dominated (uniformly in B) by $\mathbf{E}[\exp(\alpha(T(N(a_t) + 1)); \Gamma > T(N(a_t) + 1), T(N(a_t) + 1) > t]$. Because $N(a_t) + 1$ is a stopping time adapted to the W_ns, a simple calculation shows that this latter quantity equals $\mathbf{P}_*(T(N(a_t) + 1) > t)$. Assumption A3 guarantiees that the τ_is are a finite mean sequence of $r.v.$s under $\mathbf{P}_*$, so $(T(N(a_t) + 1) - a_t\colon t \geqslant 0)$ is tight under $\mathbf{P}_*$. Since $t - a_t \to \infty$ as $t \to \infty$, it follows that $\mathbf{P}_*(T(N(a_t) + 1)\cdot > t) \to 0$ as $t \to \infty$.

For r_1, note that it can be dominated (uniformly in B) by a multiple of $\mathbf{E}[\exp(-\varepsilon(t - T(N(a_t) + 1))) \exp(\alpha(T(N(a_t) + 1))); \Gamma > T(N(a_t) + 1)), T(N(a_t) +1) \leqslant t]$. But this can be re-expressed as $\mathbf{E}_*[\exp(-\varepsilon(t - T(N(a_t) + 1))); T(N(a_t) +1) \leqslant t]$, where $\mathbf{E}_*(\cdot)$ is the expectation under $\mathbf{P}_*$. Under our assumptions, the "residual life" process $T(N(a_t) + 1) - a_t$ is converging to a proper $r.v.$ under $\mathbf{P}_*$, so $\exp(-\varepsilon(t - T(N(a_t) + 1)))$ is converging in distribution to zero under $\mathbf{P}_*$ (on account of the fact that $t - a_t \to \infty$). Because of the presence of the event $\{T(N(a_t) + 1) \leqslant t\}$ in the expectation, the Bounded Convergence Theorem applies, and we may conclude that $\mathbf{E}_*[\exp(-\varepsilon(t - T(N(a_t) + 1))); T(N(a_t) + 1) \leqslant t] \to 0$ as $t \to \infty$. This concludes the proof. $\square$

4. Structure of the taboo-limit process

We turn now to studying the structure of the "taboo limit". Given that we are working with a regenerative process, it is natural to consider the process on the time scale of regenerative cycles. Fix an arbitrary $a > 0$. Note that for $t \geqslant a$, the behavior of $(X(t + s)\colon |s| \leqslant a)$ is a simple (deterministic) functional of the cycles $(W_{N(t)+j}\colon j \geqslant -N(t))$ (with $N(t) = \max\{n\colon T(n) \leqslant t\}$), and the location of time t within the $(N(t) +1)$'st cycle, as given by the "age" r.v. $\beta(t) = t - T(N(t))$. So, rather than studying $(X(t + s)\colon |s| \leqslant a)$, the structure of the limit process is more transparent if we instead focus on the behavior of $((W_{N(t)+j}\colon |j| \leqslant k), \beta(t))$, conditional on $\Gamma \geqslant t$, when t is large.

In order to obtain an elegant representation of the taboo limit, we shall express our taboo limit in terms of a two-sided process. Roughly speaking, we let the limit process to the left of the time origin correspond to those cycles completed prior to time t, and the limit process to the right correspond to those cycles completed subsequent to time t. So, we assume now that our original probability space supports a two-sided sequence $W = (W_n\colon -\infty < n < \infty)$ of $D_{\widetilde{S}}[0, \infty)$-valued cycles, and a real-valued r.v. β. (We will use β later as our limiting age r.v.) We extend the definition of the τ_j's, I_j's, and γ_j's to negative indices j in the obvious way. Since our limit process is "tabooed" to the left, we first define $\widetilde{\mathbf{P}}$ via

$$\widetilde{\mathbf{P}}\big((W_j\colon j \geqslant -n) \in \cdot\big) = \mathbf{P}\big((W_j\colon j \geqslant 1) \in \cdot \mid \Gamma^\circ \geqslant T(n)\big). \tag{4.1}$$

Relation (4.1) consistently defines a joint probability on $(W_n\colon\ -\infty < n < \infty)$, because definition 2 implies that

$$\mathbf{P}\big((W_j\colon\ j \geqslant 1) \in \cdot \mid \Gamma^\circ \geqslant T(n+1)\big) = \mathbf{P}\big((W_j\colon\ j \geqslant 2) \in \cdot \mid \Gamma^\circ \geqslant T(n+2)\big).$$

We next define the probability $\mathbf{P}^*$ on (W, β) via the change-of-measure formula

$$\mathbf{P}^*\big((W_j\colon\ j \geqslant -n) \in \cdot, \beta \in \mathrm{d}x\big) = \widetilde{\mathbf{E}} I\big((W_j\colon\ j \geqslant -n) \in \cdot\big) L(-n) \cdot F_{W_0}(\mathrm{d}x),$$

where

$$L(-n) = \exp\big(\alpha(\tau_{-n} + \cdots + \tau_{-1})\big) \cdot (\mathbf{E}I_1)^n \cdot \frac{\int_0^{\gamma_0 \wedge \tau_0} \mathrm{e}^{\alpha u}\,\mathrm{d}u}{\mathbf{E}\int_0^{\gamma_0 \wedge \tau_0} \mathrm{e}^{\alpha u}\,\mathrm{d}u}$$

and

$$F_{W_0}(\mathrm{d}x) = \frac{\mathrm{e}^{\alpha x} I(\gamma_0 \wedge \tau_0 > x)\,\mathrm{d}x}{\int_0^{\gamma_0 \wedge \tau_0} \mathrm{e}^{\alpha u}\,\mathrm{d}u}.$$

In words this means firstly that the process going backward in time from time zero does not enter the taboo set and its cycle lengths are exponentially biased, while the process going forward in time from time zero behaves without restriction and bias; and secondly that the position-from-the-right of time zero in the cycle straddling zero, β, is exponentially distributed conditional on staying in the cycle.

As for $\widetilde{\mathbf{P}}$, it is easily shown that $\mathbf{P}^*$ consistently defines a joint probability on (W, β). Here, the central argument hinges on the fact that A1 ensures that

$$\begin{aligned}
&\widetilde{\mathbf{E}} L(-n-1) I\big((W_j\colon\ j \geqslant -n) \in \cdot\big) \\
&\quad = \frac{\mathbf{E} L(-n-1) I((W_j\colon\ j \geqslant -n) \in \cdot) I_{-n-1} \cdots I_{-1}}{\mathbf{E}I_{-n-1} \cdots \mathbf{E}I_{-1}} \\
&\quad = \frac{\mathbf{E}\mathrm{e}^{\alpha\tau_{-n-1}} I_{-n-1} \cdot \mathbf{E}I_1 \mathbf{E} L(-n) I((W_j\colon\ j \geqslant -n) \in \cdot) I_{-n} \cdots I_{-1}}{\mathbf{E}I_1 \cdot (\mathbf{E}I_{-n} \cdots \mathbf{E}I_{-1})} \\
&\quad = \mathbf{E}\mathrm{e}^{\alpha\tau_{-n-1}} I_{-n-1} \widetilde{\mathbf{E}} L(-n) I\big((W_j\colon\ j \geqslant -n) \in \cdot\big) \\
&\quad = \widetilde{\mathbf{E}} L(-n) I\big((W_j\colon\ j \geqslant -n) \in \cdot\big).
\end{aligned}$$

Remark 11. Under $\mathbf{P}^*$, the r.v. β has a distribution, conditional on W_0, equal to $F_{W_0}(\cdot)$. Note that $F_{W_0}(\cdot)$ is just an exponential distribution, truncated at $\tau_0 \wedge \gamma_0$. Its unconditional distribution is given by

$$\mathbf{P}(\beta \in \mathrm{d}x) = \frac{\mathrm{e}^{\alpha x} \mathbf{P}(\tau_0 \wedge \gamma_0 > x)}{\mathbf{E}\int_0^{\tau_0 \wedge \gamma_0} \mathrm{e}^{\alpha u}\,\mathrm{d}u}\,\mathrm{d}x.$$

Remark 12. Suppose that X is classically regenerative. Then the cycles of the taboo-limit are independent and

$$\mathbf{P}^*(W_j \in B_j : -k \leqslant j \leqslant k) = \prod_{j=-k}^{-1} \mathbf{E}\big[e^{\alpha\tau_1} I(W_1 \in B_j,\ \gamma_1 \geqslant \tau_1)\big]$$
$$\times \frac{\mathbf{E}[\int_0^{\tau_1\wedge\gamma_1} e^{\alpha u}\,du \cdot I(w_1 \in B_0)]}{\mathbf{E}\int_0^{\tau_1\wedge\gamma_1} e^{\alpha r}\,dr} \cdot \prod_{j=1}^{k} \mathbf{P}(W_1 \in B_j).$$

Note that three types of cycles exist; fully tabooed cycles (completed before time zero), a specially biased cycle straddling the origin, and ordinary cycles (starting after time zero).

We now turn to a theorem that establishes $\mathbf{P}^*$ as the limit distribution that appears under conditioning on the event $\Gamma > t$, with t large. Put $\lambda^{-1} = \int_{[0,\infty)} t\widetilde{G}(dt) = \mathbf{E}^*\tau_{-1}$, where $\mathbf{E}^*(\cdot)$ is the expectation operator associated with $\mathbf{P}^*$. Note that λ is the rate at which the fully tabooed cycles are completed under $\mathbf{P}^*$.

Theorem 2. Let X be a process that is taboo-regenerative in the wide sense. If X satisfies A1–A7 and $(k_t\colon t \geqslant 0)$ is a deterministic non-negative nondecreasing integer-valued function for which $\overline{\lim}_{t\to\infty} k_t/t < \lambda$, then

$$\sup_{B\in\mathcal{D}} \Big|\mathbf{P}\Big(\big(\big(W_{N(t)+j}\colon |j| \leqslant k_t\big),\ \beta(t)\big) \in B,\ N(t) \geqslant k_t \mid \Gamma > t\Big)$$
$$- \mathbf{P}^*\Big(\big(\big(W_j\colon |j| \leqslant k_t\big), \beta\big) \in B\Big)\Big| \to 0$$

as $t \to \infty$.

Proof. We assume X is non-delayed, so that $T(0) = 0$. The extension to the delayed case follows the same pattern that we have used in earlier arguments.

For each suitably measurable B,

$$\begin{aligned}
&e^{\alpha t}\mathbf{P}\Big(\big(\big(W_{N(t)+j}\colon |j| \leqslant k_t\big),\ \beta(t)\big) \in B,\ N(t) \geqslant k_t,\ \Gamma > t\Big) \\
&\quad = \sum_{j=0}^{\infty} \int_{[0,t]} e^{\alpha u}\mathbf{P}\big(T(j) \in du,\ \Gamma > u\big) \cdot e^{\alpha(t-u)} \\
&\qquad \times \mathbf{P}\Big(\big(\big(W_{k_t+j+1}\colon |j| \leqslant k_t\big), t-u-T(k_t)\big) \in B, \\
&\qquad\qquad T(k_t) \leqslant t-u < T(k_t+1), \Gamma > t-u\Big) \\
&\quad = \sum_{j=0}^{\infty} \int_{[0,t]} \widetilde{G}^{(j)}(du) \cdot e^{\alpha(t-u)}\mathbf{P}\Big(\big(\big(W_{k_t+j+1}\colon |j| \leqslant k_t\big), t-u-T(k_t)\big) \in B, \\
&\qquad\qquad T(k_t) \leqslant t-u < T(k_t+1), \Gamma > t-u\Big).
\end{aligned} \tag{4.2}$$

Put

$$\tilde{b}_{B,\ell}(t) = \mathbf{P}\Big(\big(\big(W_{\ell+j+1}\colon |j| \leqslant \ell\big),\ t-T(\ell)\big) \in B,\ T(\ell) \leqslant t < T(\ell+1),\ \Gamma > t\Big),$$

and note that uniformly in B and ℓ,

$$\begin{aligned}\big|\tilde{b}_{B,\ell}(t)\big| &\leqslant \mathrm{e}^{\alpha t}\mathbf{P}\big(T(\ell) \leqslant t < T(\ell+1), \Gamma > t\big)\\ &= \int_{[0,t]} \widetilde{G}^{(\ell)}(\mathrm{d}u)\cdot \mathrm{e}^{\alpha(t-u)}\mathbf{P}\big(\Gamma^\circ \wedge \tau_1 > t-u\big)\\ &\leqslant \int_{[0,t]} \widetilde{G}^{(\ell)}(\mathrm{d}u)k(t-u)\\ &= \mathbf{E}^*\big[k\big(t-T(\ell)\big); T(\ell) \leqslant t\big],\end{aligned}$$

where $k(\cdot)$ is the dominating function that is guaranteed by A4. Hence, for ℓ satisfying $\lambda^{-1}\ell \leqslant (1-\eta)t$ with $\eta > 0$, we have

$$\begin{aligned}\big|\tilde{b}_{B,\ell}(t)\big| &= \mathbf{E}^*\left[k\big(t-T(\ell)\big),\ T(\ell) \leqslant t,\ T(\ell) \leqslant t\left(1-\frac{\eta}{2}\right)t\right]\\ &\quad + \mathbf{E}^*\left[k\big(t-T(\ell)\big), T(\ell) \leqslant \left(1-\frac{\eta}{2}\right)t\right]\\ &\leqslant k(0)\mathbf{P}^*\left(T(\ell) > \left(1-\frac{\eta}{2}\right)t\right) + k\left(\frac{\eta}{2}t\right).\end{aligned}$$

Recall that A7 ensures that $\mathbf{E}^*\exp(\varepsilon\tau_1) < \infty$ for ε in a neighborhood of the origin. Hence,

$$\begin{aligned}\mathbf{P}^*\left(T(\ell) > \left(1-\frac{\eta}{2}\right)t\right) &\leqslant \exp\left(-\varepsilon\left(1-\frac{\eta}{2}\right)t\right)\mathbf{E}^*\exp\big(\varepsilon T(\ell)\big)\\ &= \exp\left(-\varepsilon\left(1-\frac{\eta}{2}\right)t + \ell\psi(\varepsilon)\right),\end{aligned}$$

where $\psi(\varepsilon) = \log \mathbf{E}^*\exp(\varepsilon\tau_1)$. But $\psi(\varepsilon) = \varepsilon\lambda^{-1} + \mathrm{o}(\varepsilon^2)$ so

$$\begin{aligned}\mathbf{P}^*\left(T(\ell) > \left(1-\frac{\eta}{2}\right)t\right) &\leqslant \exp\left(-\varepsilon\left(\left(1-\frac{\eta}{2}\right)t - \lambda^{-1}\ell\right) + \ell\mathrm{o}\big(\varepsilon^2\big)\right)\\ &\leqslant \exp\left(-\frac{\varepsilon\eta}{2}t + \ell\mathrm{o}\big(\varepsilon^2\big)\right).\end{aligned}$$

We conclude that uniformly in R and ℓ satisfying $\lambda^{-1}\ell \leqslant (1-\eta)t$, we have

$$|\tilde{b}_{B,\ell}(t)| \leqslant k(0)\exp\left(-\frac{\varepsilon\eta t}{2} + t\mathrm{o}\big(\varepsilon^2\big)\right) + k\left(\frac{\eta t}{2}\right).$$

By choosing ε sufficiently small and positive, we obtain a uniform bound on $\tilde{b}_{B,\ell}$ which is non-increasing and integrable. Hence, the renewal theorem guarantees uniform convergence (over B, ℓ) of the convolution of $\tilde{b}_{B,\ell}$ with $\sum_0^\infty \widetilde{G}^{(\ell)}$ to

$$\frac{\int_0^\infty \tilde{b}_{B,\ell}(t)\,\mathrm{d}t}{\int_{[0,\infty)} t\widetilde{G}(\mathrm{d}t)}.$$

Since our growth condition on k_t ensures that $\lambda^{-1}k_t \leqslant (1-\eta)t$ for some $\eta > 0$ and t sufficiently large, it follows that the difference between (4.2) and

$$\frac{\int_0^\infty \tilde{b}_{B,k_t}(u)\,\mathrm{d}u}{\int_{[0,\infty)} u\widetilde{G}(\mathrm{d}u)}$$

goes to zero uniformly in B. Utilizing proposition 2 then yields the desired result. $\square$

Remark 13. The above theorems can be used to develop perfect simulation algorithms for sampling from the quasi-stationary distribution. See [6] for a related algorithm in the Markov setting and [3] for analogous algorithms for sampling from the steady-state of a regenerative process. To be specific, remark 9 establishes that if one can generate a variate R from the distribution having density

$$\frac{\mathrm{e}^{\alpha u}\mathbf{P}(\Gamma^\circ \wedge \tau_1 > u)}{\mathbf{E}\int_0^{\Gamma^\circ\wedge\tau_1} \mathrm{e}^{\alpha r}\,\mathrm{d}r},$$

then we can simply generate cycles $W_1, W_2, \ldots$ until we get a cycle with $\gamma_i \wedge \tau_i > R$. The r.v. $W_i(R)$ is then a sample from the quasi-stationary distribution.

A second perfect simulation algorithm assumes that one can compute a finite upper bound m on the r.v. $\int_0^{\gamma_1\wedge\tau_1} \mathrm{e}^{\alpha u}\,\mathrm{d}u$; this would be the case, for example, if α is known and τ_1 is bounded by some known constant. Again, we generate cycles $W_1, W_2, \ldots$. However, this time, we also generate an independent auxiliary sequence of *i.i.d.* uniform r.v.s $U_1, U_2, \ldots$. We continue sampling until the event B_i occures, where

$$B_i = \left\{U_i \leqslant \int_0^{\gamma_i\wedge\tau_i} \frac{\mathrm{e}^{\alpha u}\mathrm{d}u}{m}\right\}.$$

It is easily seen that

$$\mathbf{P}(W_i \in \cdot \mid B_i) = \mathbf{P}^*(W_1 \in \cdot).$$

One then uses the distribution $F_{W_i}(\cdot)$ to generate the r.v. β; $W_i(\beta)$ then has the required quasi-stationary distribution.

Both of these algorithms require a priori knowledge of α. It is worth noting that the Markov algorithm appearing in [6] requires no such knowledge.

Remark 14. It should be recognized that only A1 and A4 are required in order that $\mathbf{P}^*$ be unambiguously defined as a probability. In particular, the moment condition A3 is unnecessary. This raises the question of whether $\mathbf{P}^*$ can be interpreted as a limit distribution in the absence of A3; we leave this issue to future research.

Theorem 2 focuses attention on the cycle-structure of the tabooed process, excepting the first $N(t) - k_t$ cycles. As in the case of proposition 7, we can also study the asymptotic structure of the first few cycles, conditional on $\Gamma > t$ (with t large).

To describe the limit distribution, we extend $\mathbf{P}^*$ to a probability on the zero'th cycle, as follows: for $n \geqslant 0$,

$$\begin{aligned}&\mathbf{P}_*\big((W_0, W_1, \dots, W_n) \in \cdot\big)\\ &\quad = \frac{\mathbf{E}[\exp(\alpha T(n)) I((W_0, W_1, \dots, W_n) \in \cdot)_j \Gamma \geqslant T(n)]}{\mathbf{E}[\exp(\alpha T(0)); \Gamma \geqslant T(0)]}.\end{aligned}$$

On account of A1, this consistently defines a probability on $(W_n\colon\ n \geqslant 0)$. Note that the marginal distribution of the cycles having index 1 or higher matches the marginal distribution of the fully tabooed cycles under $\mathbf{P}^*$.

Proposition 8. Let X be a process that is classically regenerative. If X satisfies A1–A7 and $(m_t\colon\ t \geqslant 0)$ is a deterministic non-negative nondecreasing integer-valued function satisfying $\overline{\lim}_{t\to\infty} m_t/t < \lambda$, then

$$\sup_{B\in\mathcal{D}} \Big|\mathbf{P}\big((W_0, \dots, W_{m_t}) \in B \mid \Gamma > t\big) - \mathbf{P}_*\big((W_0, \dots, W_{m_t}) \in B\big)\Big| \to 0$$

as $t \to \infty$.

Proof. Note that if $\overline{F}(x) = \mathbf{P}(\Gamma^\circ > x)$, then

$$\begin{aligned}
&\frac{\mathbf{P}((W_0, \dots, W_{m_t}) \in B, \Gamma > t)}{\mathbf{P}(\Gamma > t)}\\
&\quad = \mathbf{E}\bigg[I((W_0, \dots, W_{m_t}) \in B) \mathrm{e}^{\alpha T(m_t)} I(T(m_t) \leqslant t,\ \Gamma \geqslant T(m_t))\\
&\qquad \times \frac{\exp(\alpha(t - T(m_t)))\overline{F}(t - T(m_t))}{\mathrm{e}^{\alpha t}\mathbf{P}(\Gamma > t)}\bigg]\\
&\qquad + \frac{\mathbf{P}((W_0, \dots, W_{m_t}) \in B, T(m_t) > t, \Gamma > t)}{\mathbf{P}(\Gamma > t)}\\
&\quad = \mathbf{P}_*\big(\big((W_0, \dots, W_{m_t}) \in B\big),\ T(m_t) \leqslant t, \big)\\
&\qquad + \mathbf{E}_*\bigg[I\big((W_0, \dots, W_{m_t}) \in B\big) I\big(T(m_t) \leqslant t\big)\\
&\qquad \times \bigg(\frac{\exp(\alpha(t - T(m_t)))\overline{F}(t - T(m_t))}{\mathrm{e}^{\alpha t}\mathbf{P}(\Gamma > t)} \mathbf{E}\big[\exp\big(\alpha\big(T(0)\big); \Gamma \geqslant T(0)\big)\big] - 1\bigg)\bigg]\\
&\qquad + \frac{\mathbf{P}((W_0, \dots, W_{m_t}) \in B, T(m_t) > t,\ \Gamma > t)}{\mathbf{P}(\Gamma > t)}\\
&\quad \stackrel{\triangle}{=} \mathbf{P}_*\big((W_0, \dots, W_{m_t}) \in B,\ T(m_t) \leqslant t\big) + r_4(t) + r_5(t).
\end{aligned}$$

To deal with $r_4(t)$, this can be bounded (uniformly in B) by

$$\mathbf{E}_*\bigg[\bigg(\frac{\exp(\alpha(t - T(m_t)))\overline{F}(t - T(m_t))}{\mathrm{e}^{\alpha t}\mathbf{P}(\Gamma > t)} \mathbf{E}\big[\exp\big(\alpha T(0)\big); \Gamma \geqslant T(0)\big] - 1\bigg); T(m_t) \leqslant t\bigg]. \tag{4.3}$$

The weak law $T(m_t)/m_t \Longrightarrow \lambda^{-1}$ as $t \to \infty$ holds under $\mathbf{P}_*$, where $\Longrightarrow$ denotes weak convergence. Under our growth condition on m_t, $t - T(m_t) \Longrightarrow +\infty$ as $t \to \infty$, so propositions 2 and 3 ensure that

$$\frac{\exp(\alpha(t - T(m_t)))\overline{F}(t - T(m_t))}{\mathrm{e}^{\alpha t}\mathbf{P}(\Gamma > t)}\mathbf{E}\big[\exp\big(\alpha T(0)\big); \Gamma \geqslant T(0)\big] \Longrightarrow 1$$

as $t \to \infty$. The Bounded Convergence Theorem therefore implies that (4.3) goes to zero as $t \to \infty$.

For $r_5(t)$, this can be bounded (uniformly in B) by a multiple of $\mathrm{e}^{\alpha t}\mathbf{P}(T(m_t) > t, \Gamma > t)$. But

$$\begin{aligned}
&\mathrm{e}^{\alpha t}\mathbf{P}\big(T(m_t) > t,\ \Gamma > t\big)\\
&\quad = \mathrm{e}^{\alpha t}\sum_{j=-1}^{m_t-1}\mathbf{P}\big(T(j) \leqslant t < T(j+1),\ T(m_t) > t, \Gamma > t\big)\\
&\quad \leqslant \mathrm{e}^{\alpha t}\mathbf{P}\big(T(0) > t, \Gamma > t\big) + \sum_{j=0}^{m_t-1}\mathbf{P}\big(T(j) \leqslant t < T(j+1), \Gamma > t\big)\cdot \mathrm{e}^{\alpha t}\\
&\quad \leqslant \mathrm{e}^{\alpha t}\mathbf{P}(\tau_0 \wedge \Gamma > t) + \sum_{j=0}^{m_t-1}\int_{[0,t]}\widetilde{G}^{(j)}(\mathrm{d}u)\mathbf{P}(\tau_1 > t-u, \Gamma^\circ > t-u)\mathrm{e}^{\alpha(t-u)}\\
&\quad \leqslant \mathrm{e}^{\alpha t}\mathbf{P}(\tau_0 \wedge \Gamma > t) + \sum_{j=0}^{m_t-1}\mathbf{E}_*\big[k\big(t - T(j)\big); T(j) \leqslant t\big],
\end{aligned}$$

where k is the dominating function of A4. The first term in the last inequality goes to zero by A6. The sum is dominated by

$$\begin{aligned}
&\sum_{j=0}^{m_t-1}\mathbf{E}_*\big[k\big(t - T(j)\big);\ T(m_t) \leqslant \big(\lambda^{-1} + \varepsilon\big)m_t\big]\\
&\qquad + \sum_{j=0}^{m_t-1}\mathbf{E}_*\big[k\big(t - T(j)\big);\ T(j) \leqslant t,\ T(m_t) > \big(\lambda^{-1} + \varepsilon\big)m_t\big]\\
&\quad \leqslant m_t\mathbf{E}_*\big[k\big(t - T(m_t)\big); T(m_t) \leqslant \big(\lambda^{-1} + \varepsilon\big)m_t\big] + k(0)m_t\mathbf{P}_*\big(T(m_t) > \big(\lambda^{-1} + \varepsilon\big)m_t\big)\\
&\quad \leqslant m_t k\big(t - \big(\lambda^{-1} + \varepsilon\big)m_t\big) + k(0)m_t\mathbf{P}_*\big(T(m_t) > \big(\lambda^{-1} + \varepsilon\big)m_t\big).
\end{aligned}$$

For ε sufficiently small and positive, $\mathbf{P}_*(T(m_t) > (\lambda^{-1} + \varepsilon)m_t)$ goes to zero exponentially fast in t; see the proof of theorem 2. So, the second term goes to zero. For the first, the growth condition on m_t ensures that $t - (\lambda^{-1} + \varepsilon)m_t \geqslant \delta t$ for $\delta > 0$ and t large enough. Hence, $k(t - (\lambda^{-1} + \varepsilon)m_t) \geqslant k(\delta t)$ for t large. But

$$\frac{\delta t}{2}k(\delta t) \leqslant \int_{\delta t/2}^{\delta t} k(u)\,\mathrm{d}u \leqslant \int_{\delta t/2}^{\infty} k(u)\,\mathrm{d}u,$$

so

$$tk(t) \leqslant \frac{2}{\delta} \int_{\delta t/2}^{\infty} k(u)\,\mathrm{d}u \to 0$$

as $t \to \infty$. Hence, the first term also goes to zero, proving the result. □

5. Additional taboo-limit theory

In this section, we offer several additional limit theorems that provide further insight into the structure of taboo-regenerative processes. We start by considering the behavior of a taboo-regenerative process that is conditioned on exitting A precisely at time t (with t large). Specifically, we shall impose regularity conditions on the process that ensure that Γ possesses a density.

A8. $\mathbf{P}(\Gamma^\circ \in \mathrm{d}t, \tau_1 > t)$ has a bounded density β such that $\exp(\alpha\cdot)\beta(\cdot)$ is dominated by a non-increasing integrable function.

Note that Γ° then has a density $\gamma(\cdot)$ given by

$$\gamma(t) = \sum_{j=0}^{\infty} \int_{[0,t]} \mathbf{P}\big(T(j) \in \mathrm{d}u,\ \Gamma \geqslant T(j)\big)\beta(t-u). \tag{5.1}$$

Multiplying through (5.1) by $\mathrm{e}^{\alpha t}$, letting $t \to \infty$, and applying the renewal theorem, we obtain the limit relation

$$\gamma(t) \sim \mathrm{e}^{-\alpha t} \frac{\mathbf{E}[\exp(\alpha\Gamma^\circ; \Gamma^\circ < \tau_1]}{\int_{[0,\infty)} u\widetilde{G}(\mathrm{d}u)} \tag{5.2}$$

as $t \to \infty$.

To study the regenerative structure of the taboo-regenerative limit, conditioned on $\Gamma = t$, note that because $D_{\widetilde{S}}[0,\infty)$ is Polish, there exist regular conditional distributions for the W_j's. This permits us to assert the existence of $\mathbf{P}((W_{k+j+1}\colon |j| \leqslant k) \in \cdot,\ T(k) \leqslant t < T(k+1) \mid \Gamma^\circ = t)$ for $t \geqslant 0$. Then, if $T(0) = 0$ and A1–A3 hold, we have

$$\begin{aligned}
&\mathbf{P}\big((W_{n(t)+j}\colon |j| \leqslant k\big) \in B,\ N(t) \geqslant k \mid \Gamma^\circ = t\big)\\
&\quad = \sum_{j=0}^{\infty} \int_{[0,t]} \widetilde{G}^{(j)}(\mathrm{d}u) \cdot \mathrm{e}^{\alpha(t-u)}\\
&\qquad \times \mathbf{P}\big((W_{k+j+1}\colon |j| \leqslant k\big) \in B,\ T(k) \leqslant t-u < T(k+1) \mid \Gamma^\circ = t-u\big) \frac{\gamma(t-u)}{\mathrm{e}^{\alpha t}\gamma(t)}\\
&\quad \to \int_0^{\infty} \frac{\mathrm{e}^{\alpha u}\mathbf{P}((W_{k+j+1}\colon |j| \leqslant k) \in B,\ T(k) \leqslant u < T(k+1), \Gamma^\circ \in \mathrm{d}u)}{\mathbf{E}[\exp(\alpha\Gamma^\circ);\ \Gamma^\circ < \tau_1]}\\
&\quad = \frac{\mathbf{E}[\mathrm{e}^{\alpha\Gamma^\circ} I((W_{k+j+1}\colon |j| \leqslant k) \in B;\ T(k) \leqslant \Gamma^\circ < T(k+1))]}{\mathbf{E}[\exp(\alpha\Gamma^\circ);\ \Gamma^\circ < \tau_1]}
\end{aligned}$$

as $t \to \infty$. Noting that the convergence is uniform in B, we may summarize the above discussion with following result.

Proposition 9. Under A1–A3 and A8, Γ° has a density $\gamma(\cdot)$ satisfying (5.2). If $T(0) = 0$, we further have

$$\mathbf{P}\big((W_{N(t)+j}\colon\ |j| \leqslant k\big) \in \cdot,\ N(t) \geqslant k \mid \Gamma^\circ = t\big)$$
$$\xrightarrow{\text{t.v.}} \frac{\mathbf{E}[\mathrm{e}^{\alpha\Gamma^\circ} I((W_{k+j+1}\colon\ |j| \leqslant k) \in B;\ T(k) \leqslant \Gamma^\circ < T(k+1))]}{\mathbf{E}[\exp(\alpha\Gamma^\circ);\ \Gamma^\circ < \tau_1]}$$

as $t \to \infty$.

So, proposition 9 describes the cycle structure relative to the location of Γ°, conditional on Γ° taking on a large value. By utilizing remark 5, it is straightforward to show that the conditional limit distribution can alternatively be obtained by considering the limit of $\mathbf{P}^*(\cdot \mid \Gamma^\circ \in (0, h))$ and letting $h \downarrow 0$. In section 6, this limit is discussed in the context of general taboo-stationary processes.

We conclude this section with a brief discussion of the behavior of additive functionals of X, over the tabooed time horizon $[0, t]$ (with t large). For Borel measurable $f : S \to \mathbb{R}$, let

$$C(t) = \int_0^t f\big(X(s)\big)\,\mathrm{d}s.$$

To study $C(\cdot)$ over the tabooed time horizon, we assume that:

A9. $\mathbf{E}[\exp(\theta_1 \int_0^{\tau_1} |f(X(T(0) + u))|\,\mathrm{d}u + \theta_2\tau_1); \Gamma^\circ \geqslant \tau_1] < \infty$ for (θ_1, θ_2) in a neighborhood of the origin.

Put $Y_1 = \int_0^{\tau_1} f(W_i(s))\,\mathrm{d}s$ for $i \geqslant 0$, and set

$$\varphi(\theta_1, \theta_2) = \mathbf{E}\big[\exp(\theta_1 Y_1 + \theta_2\tau_1);\ \Gamma^\circ \geqslant \tau_1\big].$$

Since $\varphi(\theta_1, \cdot)$ is strictly increasing, continuous in a neighborhood of the origin, and satisfies $\varphi(0, -\alpha) = 1$, it follows that for θ in a neighborhood of the origin, there exists $\kappa(\theta)$ for which

$$\mathbf{E}\big[\exp\big(\theta Y_1 - \kappa(\theta)\tau_1\big);\ \Gamma^\circ \geqslant \tau_1\big] = 1.$$

Furthermore, because φ is infinitely differentiable on the interior of its domain of finiteness and $\varphi(\theta_1, \cdot)$ is strictly increasing, the Implicit Function Theorem guarantees that κ is also infinitely differentiable in a neighborhood of the origin. In addition, $\varphi(0) = -\alpha$.

In preparation for our next theorem, let

$$\mu = \frac{\mathbf{E}[Y_1\mathrm{e}^{\alpha\tau_1};\ \Gamma^\circ \geqslant \tau_1]}{\mathbf{E}[\tau_1\mathrm{e}^{\alpha\tau_1};\ \Gamma^\circ \geqslant \tau_1]} \quad \left(= \frac{\mathbf{E}^*Y_{-1}}{\mathbf{E}^*\tau_{-1}}\right),$$
$$\sigma^2 = \frac{\mathbf{E}[(Y_1 - \mu\tau_1)^2\mathrm{e}^{\alpha\tau_1};\ \Gamma^\circ \geqslant \tau_1]}{\mathbf{E}[\tau_1\mathrm{e}^{\alpha\tau_1};\ \Gamma^\circ \geqslant \tau_1]}.$$

Theorem 3. Let X be classically regenerative and suppose $T(0) = 0$. Under A1–A2 and A9,

$$\mathbf{P}\big(t^{-1/2}\big(C(t) - \mu t\big) \leqslant \cdot \mid \Gamma^\circ > t\big) \Longrightarrow \mathbf{P}\big(\sigma N(0,1) \leqslant \cdot\big) \tag{5.3}$$

as $t \to \infty$, where $N(0, 1)$ denotes a standard normal r.v. Furthermore, for $x > \mu$ in a neighborhood of μ,

$$\frac{1}{t} \log \mathbf{P}\big(C(t) > xt \mid \Gamma^\circ > t\big) \to -\theta_x x + \varphi(\theta_x) \tag{5.4}$$

as $t \to \infty$, where $\varphi'(\theta_x) = x$. Similarly, for $x < \mu$ in a neighborhood of the origin,

$$\frac{1}{t} \log \mathbf{P}\big(C(t) < \mu t \mid \Gamma^\circ > t\big) \to -\theta_x + \varphi(\theta_x) \tag{5.5}$$

as $t \to \infty$.

Proof. Let

$$\varphi(\theta, t) = \mathbf{E}\big[\exp\big(\theta C(t)\big); \Gamma^\circ > t\big]$$

for $t \geqslant 0$. Then,

$$\varphi(\theta, \cdot) = b(\theta, \cdot) + \int_{[0,t]} G(\theta, \mathrm{d}s)\varphi(\theta, \cdot - s),$$

where

$$\begin{aligned} b(\theta, t) &= \mathbf{E}\big[\exp\big(\theta C(t)\big); \Gamma^\circ > t, \tau_1 > t\big], \\ G(\theta, \mathrm{d}t) &= \mathbf{E}\big[\exp(\theta Y_1); \tau_1 \in \mathrm{d}t, \Gamma^\circ \geqslant \tau_1\big]. \end{aligned}$$

This becomes a proper renewal equation if we multiply through by $\exp(-\varphi(\theta)t)$. We then obtain

$$\tilde{\varphi}(\theta, \cdot) = \tilde{b}(\theta, \cdot) + \int_{[0,t]} \widetilde{G}(\theta, \mathrm{d}s)\tilde{\varphi}(\theta, \cdot - s),$$

where $\tilde{\varphi}(\theta, t) = \mathbf{E}[\exp(\theta C(t) - \varphi(\theta)t); \Gamma^\circ > t]$, $\tilde{b}(\theta, t) = \mathbf{E}[\exp(\theta C(t) - \varphi(\theta)t); \Gamma^\circ \wedge \tau_1 > t]$, $\widetilde{G}(\theta, \mathrm{d}t) = \mathbf{E}[\exp(\theta Y_1 - \varphi(\theta)\tau_1); \tau_1 \in \mathrm{d}t, \Gamma^\circ \geqslant \tau_1]$. Using A9, it is straightforward to show that $\widetilde{G}(\theta, \cdot)$ has finite mean and $\tilde{b}(\theta, \cdot)$ is dominated by a non-increasing integrable function, in a neighborhood of the origin. Then, Smith's renewal theorem implies that

$$\frac{\tilde{\varphi}(\theta, t) \to \int_0^\infty \tilde{b}(\theta, u)\,\mathrm{d}u}{\int_{[0,\infty)} u\widetilde{G}(\theta, \mathrm{d}u)}$$

as $t \to \infty$ from which it follows that

$$\frac{1}{t} \log \mathbf{E}\big[\exp\big(\theta C(t)\big) \mid \Gamma^\circ > t\big] \to \varphi(\theta) + \alpha$$

as $t \to \infty$. The Gartner–Ellis theorem then yields (5.4) and (5.5); see, for example, [5]. The proof of the central limit theorem (5.3) follows via a direct argument based on use of moment generating functions. $\square$

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