Casselman's Basis of Iwahori vectors and the Bruhat Order

Daniel Bump and Maki Nakasuji

February 15, 2010

1 Introduction

Casselman [3] described an interesting basis of the vectors in a spherical representation of a reductive p-adic group that are fixed by the Iwahori subgroup. This basis is defined as being dual to the standard intertwining operators. He remarked (p.402) that it was an unsolved and apparently difficult problem to compute this basis explicitly. For his applications, which include the computation of the spherical function and, in Casselman and Shalika [4] the spherical Whittaker function, it is only necessary to compute one element of the basis explicitly. Despite this difficulty, we began to look at the Casselman basis and we obtained interesting partial results. These lead to some interesting combinatorial questions about the Bruhat order.

Let G be a split semisimple algebraic group over the nonarchimedean field F. Let $B(F)$ be the standard Borel subgroup of $G(F)$, K the standard maximal compact subgroup, and J the Iwahori subgroup of K. (See Section 2 for definitions of these.)

We write $B = TN$ where T is the maximal split torus and N its unipotent radical. If χ is a character of $T(F)$ then $V(\chi)$ will be the representation of $G(F)$ induced from χ. Its space consists of locally constant functions $f : G(F) \longrightarrow \mathbb{C}$ such that

$$f(bg) = (\delta^{1/2}\chi)(bg)$$

where $\delta : B(F) \longrightarrow \mathbb{C}$ is the modular quasicharacter and χ, δ are extended to B to be trivial on $N(F)$. The action of $G(F)$ is by right translation.

If χ is in general position then $V(\chi)$ is irreducible. If χ is unramified (which we assume) then the space $V(\chi)^J$ of J-fixed vectors has dimension equal to the order of the Weyl group W, and so it is natural to parametrize bases of $V(\chi)^J$ by W. There is one natural basis, namely $\{\phi_w | w \in W\}$ defined as follows. If $b \in B(F), u \in W$ and $k \in J$, define

$$\phi_w(buk) = \begin{cases} \delta^{1/2}\chi(b) & \text{if } k \in J,\ u = w, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is clear that this is a basis of $V(\chi)^J$.

If $w \in W$ then there is an intertwining integral $M_w : V(\chi) \longrightarrow V({}^w\chi)$. It is given by (5) below. These have the property that if $l(ww') = l(w) + l(w')$ then $M_{ww'} = M_w \circ M_{w'}$, where $l : W \longrightarrow \mathbb{Z}$ is the length function. The Casselman basis $\{f_w | w \in W\}$ is the basis defined by the condition that

$$(M_w f_v)(1) = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{if } w \neq v. \end{cases}$$

The question of Casselman mentioned above is to express the basis f_w in terms of the basis ϕ_w. However we found it better to try to express it in terms of the basis

$$\psi_u = \sum_{v \geqslant u} \phi_v,$$

where $\geqslant$ is the Bruhat order. By Verma [21] or Stembridge [20]

$$\phi_u = \sum_{v \geqslant u} (-1)^{l(v)-l(u)} \psi_v,$$

so expressing the f_w in terms of ψ_w is equivalent to Casselman's question.

This problem can be divided into two parts: first, to compute the values of $m(u,v) = (M_v \psi_u)(1)$, and second, to invert the matrix $m(u,v)_{u,v \in W}$. Indeed, if $\tilde{m}(u,v)_{u,v \in W}$ is the inverse matrix, so $\sum_v \tilde{m}(u,v) m(v,w) = \delta_{u,w}$ (Kronecker δ) then $\sum_u \tilde{m}(v,u)\psi_u$ will satisfy $M_w \left(\sum_u \tilde{m}(v,u)\psi_u\right)(1) = \delta_{v,w}$ and so $f_v = \sum_u \tilde{m}(v,u)\psi_u$ is the Casselman basis.

Let $\hat{\Phi}$ be the root system with respect to $\hat{T}$, the dual torus of T. This is a complex torus in the L-group ${}^LG$.

If $\alpha \in \hat{\Phi}^+$ let r_α be the reflection in the hyperplace perpendicular to α. Thus if α is simple, r_α is the simple reflection s_α. Then for any $u \leqslant y \leqslant v$ we have

$$\#\{\alpha \in \hat{\Phi}^+ | u \leqslant y.r_\alpha \leqslant v\} \geqslant l(v) - l(u).$$

This statement is known as *Deodhar's conjecture*. The condition is sometimes written $u \leqslant r_\alpha.y \leqslant v$ but this does not change the cardinality of the set since $y.r_\alpha = r_\beta.y$

for another positive root $\beta = \pm y(\alpha)$. This inequality was stated by Deodhar [7] who proved it in some cases; the general statement is a theorem of Dyer [9] and (independently) Polo [16] and Carrell and Peterson (Carrell [2]). In particular, taking $y = v$ or u gives

$$S(u,v) = \{\alpha \in \hat{\Phi}^+ | u \leqslant v r_\alpha < v\}, \qquad S'(u,v) = \{\alpha \in \hat{\Phi}^+ | u \leqslant u r_\alpha < v\}.$$

Then Deodhar's conjecture implies that $S(u,v)$ and $S'(u,v)$ have cardinality $\geqslant l(v) - l(u)$.

Proposition 1 *If the Kazhdan-Lusztig polynomial $P_{u,v} = 1$ then $|S'(u,v)| = l(v) - l(u)$. If the Kazhdan-Lusztig polynomial $P_{w_0 v, w_0 u} = 1$ then $|S(u,v)| = l(v) - l(u)$.*

Proof The first statement follows from Carrell [2], Theorem C. If $P_{w_0 v, w_0 u} = 1$ then it follows that $|S'(w_0 v, w_0 u)| = l(w_0 u) - l(w_0 v)$. Since $x \leqslant y$ if and only if $w_0 y \leqslant w_0 x$, this is equivalent to $|S(u,v)| = l(v) - l(u)$. □

We assume that $\hat{\Phi}$ is simply-laced, that is, of Cartan type A, D or E. In this case, we make the following conjectures. The unramified character $\chi = \chi_{\boldsymbol{z}}$ of $T(F)$ is parametrized by an element $\boldsymbol{z}$ of the complex torus $\hat{T}$ in the L-group ${}^L G$. (See Section 2.)

Conjecture 1 *Assume that $\hat{\Phi}$ is simply-laced. Suppose that $u \leqslant v$ in the Bruhat order. In this case $w_0 v \leqslant w_0 u$. Suppose that the $|S(u,v)| = l(v) - l(u)$. Then we conjecture that*

$$(M_v \psi_u)(1) = \prod_{\alpha \in S(u,v)} \frac{1 - q^{-1} \boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}. \tag{2}$$

Conjecture 2 *Assume that $\hat{\Phi}$ is simply-laced. Suppose that $u \leqslant v$ in the Bruhat order. Suppose that $|S'(u,v)| = l(v) - l(u)$. Then we conjecture that*

$$\tilde{m}(u,v) = (-1)^{|S'(u,v)|} \prod_{\alpha \in S'(u,v)} \frac{1 - q^{-1} \boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}. \tag{3}$$

We give an example to show that the assumption that $\hat{\Phi}$ is simply-laced is necessary. Let $\hat{\Phi}$ have Cartan type B_2, with α_1, α_2 being the long and short simple roots, respectively and $\sigma_1 = s_{\alpha_1}, \sigma_2 = s_{\alpha_2}$ being the simple reflections. Then we find that when $(u,v) = (\sigma_1, \sigma_1\sigma_2\sigma_1)$ or $(\sigma_1, \sigma_1\sigma_2\sigma_1\sigma_2)$ the conclusion of Conjecture 1 fails, though the Kazhdan-Lusztig polynomial $P_{w_0 v, w_0 u} = 1$. Nevertheless the conjecture is *often* true for type B_2, for these are the only failures. There are 33 pairs (u,v)

with $u \leqslant v$, and Conjecture 1 gives the correct value for $(M_v \psi_u)(1)$ in every case except for these two. Hence it becomes interesting to ask how the hypothesis in Conjectures 1 and 2 should be modified if when $\hat{\Phi}$ is not simply-laced.

We recall the formula of Gindikin and Karpelevich. Let $\phi^\circ = {}^\chi\phi^\circ$ be the standard spherical vector in $\mathrm{Ind}_B^G(\delta^{1/2}\chi)$ defined by $\phi^\circ(bk) = \delta^{1/2}\chi(b)$ when $b \in B(F)$ and $k \in K$. In this case

$$M(v)^\chi\phi^\circ = \left[\prod_{\substack{\alpha \in \hat{\Phi}^+ \\ v(\alpha) \in \hat{\Phi}^-}} \frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}\right] {}^{v\chi}\phi^\circ. \tag{4}$$

This well-known formula was proved by Langlands [14] after Gindikin and Karpelevich proved a similar statement for real groups. See Theorem 3.1 of Casselman [3] for a proof.

Theorem 1 *If $u = 1$ then Conjecture 1 is true.*

Proof We will deduce this from (4). In this case $\psi_1 = \phi^\circ$, so to prove Conjecture 1 we need to know that if $\alpha \in \hat{\Phi}^+$ then $\alpha \in S(1, v)$ if and only if $v(\alpha) \in \hat{\Phi}^-$. This follows from Proposition 4 with $w = v$. □

Thus Conjecture 1 generalizes the formula of Gindikin and Karpelevich. If $u \neq 1$ it resembles the formula of Gindikin and Karpelevich but there are some important differences, which we will now discuss.

We say that a subset S of $\hat{\Phi}$ is *convex* if $\alpha \in S$ implies $-\alpha \notin S$ and whenever $\alpha, \beta \in S$ and $\alpha + \beta \in \hat{\Phi}$ we have $\alpha + \beta \in S$. The set $S(1, v) = \{\alpha \in \hat{\Phi}^+ | v(\alpha) \in \hat{\Phi}^-\}$ is convex in this sense. Moreover it has the property that if it is nonempty then it contains simple roots; this follows from the fact that its complement in $\hat{\Phi}^+$ is $\{\alpha \in \hat{\Phi}^+ | v(\alpha) \in \hat{\Phi}^+\}$, which is also convex. These are special properties that $S(u, v)$ may not have in general.

Example 1 Suppose that $\hat{\Phi} = A_2$ with simple roots α_1 and α_2 and simple reflections $\sigma_i = s_{\alpha_i}$. Let $u = \sigma_1$, $v = w_0 = \sigma_1\sigma_2\sigma_1$. Then $S(u, v) = \{\alpha_1, \alpha_2\}$ is not convex.

Example 2 Suppose that $\hat{\Phi} = A_2$ and that $u = \sigma_2$, $v = \sigma_1\sigma_2$. Then $S(u, v) = \{\alpha_1 + \alpha_2\}$. Thus $S(u, v)$ contains no simple roots.

We see that $S(u, v)$ has two special properties in the case where $u = 1$, namely that it is convex and that its complement is convex, which implies that (if nonempty) it always contains simple roots. These properties fail for general u.

We turn now to an interesting combinatorial conjecture which implies Conjecture 1.

Let W be a Coxeter group with generators Σ, whose elements will be referred to as *simple reflections*. If $u, v \in W$ and $u \leqslant v$ with respect to the Bruhat order, then we will define the notion of a *good word* for v with respect to u. First, this is a reduced decomposition $v = s_1 \cdots s_n$ into a product of simple reflections, where n equals the length $l(v)$. It has the following property. Let S be the set of integers j such that

$$u \leqslant s_1 \cdots \widehat{s_j} \cdots s_n,$$

where the "hat" means that the factor s_j is omitted. Let $S = \{j_1, \cdots, j_d\}$, which we arrange in ascending order: $j_1 < \cdots < j_d$. Then we say that the decomposition $s_1 \cdots s_n$ is a *good word* for v with respect to u if

$$u = s_1 \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_d}} \cdots s_n.$$

Now d has an intrinsic characterization in terms of u and v independent of the decomposition $v = s_1 \cdots s_n$. It is the number of reflections r in W such that $u \leqslant vr < v$. Indeed, given any reflection r such that $u \leqslant vr < v$ there is a unique j such that

$$r = s_j s_{j+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_j$$

and so $vr = s_1 \cdots \widehat{s_j} \cdots s_n$. Thus $d = |S(u, v)|$ and by Deodhar's conjecture $d \geqslant l(v) - l(u)$. Therefore a good word can exist only if $d = l(v) - l(u)$.

Let us consider some examples. First consider the case where $W = A_2$, with generators $\sigma_1 = s_{\alpha_1}$ and $\sigma_2 = s_{\alpha_2}$ satisfying $\sigma_i^2 = 1$ and $(\sigma_1\sigma_2)^3 = 1$. Let $u = \sigma_1$ and $v = \sigma_1\sigma_2\sigma_1$. Then $\sigma_1\sigma_2\sigma_1$ is not a good word for v with respect to u, since

$$\sigma_1 \leqslant \widehat{\sigma}_1\sigma_2\sigma_1, \qquad \sigma_1 \leqslant \sigma_1\sigma_2\widehat{\sigma}_1, \qquad \text{but } \sigma_1 \neq \widehat{\sigma}_1\sigma_2\widehat{\sigma}_1.$$

But $v = \sigma_2\sigma_1\sigma_2$ by the braid relation, and this word is good. Indeed, we have

$$\sigma_1 \leqslant \widehat{\sigma}_2\sigma_1\sigma_2, \qquad \sigma_1 \leqslant \sigma_2\sigma_1\widehat{\sigma}_2, \qquad \sigma_1 = \widehat{\sigma}_2\sigma_1\widehat{\sigma}_2.$$

Conjecture 3 *If W is simply-laced and $d = l(v) - l(u)$ then v admits a good word with respect to u.*

Proposition 2 *Conjecture 3 is true for $W = A_4$ or D_4.*

Proof This was established by computer computation using Sage. □

If W is not simply-laced, then this fails: for example, let $W = B_2$, with generators σ_1 and σ_2 satisfying $\sigma_i^2 = 1$ and $(\sigma_1\sigma_2)^4 = 1$. Let $u, v = \sigma_1, \sigma_1\sigma_2\sigma_1$. Then there is no good word for v with respect to u. It is an interesting question to give other characterizations (for example in terms of Schubert varieties) of the pairs u, v such that v admits a good word for u when W is not simply-laced.

Our main theorem is the following result:

Theorem 2 *If v admits a good word for u then (2) is true.*

By Theorem 2, Conjecture 3 implies Conjecture 1. Theorem 2 is true whether or not $\hat{\Phi}$ is simply-laced. However, as we have mentioned, if $\hat{\Phi}$ is not simply-laced, there may not exist a good word even if $l(v) - l(u) = d$.

By Proposition 2 it follows that Conjecture 1 is true if $G = \mathrm{GL}_r$ with $r \leqslant 5$ or $G = \mathrm{SO}(8)$ (split).

We have investigated Conjecture 2 less than Conjecture 1 and have less evidence for it. Conjecture 2 also is related to a combinatorial conjecture which we will now state.

Conjecture 4 *Assume that $\hat{\Phi}$ is simply-laced. If $u < v$ and $P_{u,v} = 1$ then there exists $\beta \in \hat{\Phi}^+$ such that $u \leqslant t \leqslant v$ if and only if $u \leqslant r_\beta t \leqslant v$.*

It is shown in Proposition 3.7 of Deodhar [6] that the Bruhat interval $[u, v] = \{t | u \leqslant t \leqslant v\}$ has as many elements of even length as of odd length. Conjecture 4 (when applicable) gives a strengthening of this since $t \mapsto r_\beta t$ is a specific bijection of $[u, v]$ to itself that interchanges elements of odd and even length.

We have checked using a computer that Conjecture 4 is true for A_r when $r \leqslant 4$. For example if $\hat{\Phi} = A_3$ then there exists such a β for every pair $u \leqslant v$ except the pair $\sigma_2, \sigma_2\sigma_1\sigma_3\sigma_2$ and $\sigma_1\sigma_3$, $\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3$. For these pairs, we have $u \prec v$ (in the notation of Kazhdan and Lusztig [13]) but $l(v) > l(u) + 1$ and so $P_{u,v} \neq 1$. For A_4, there are pairs $u \leqslant v$ such that $u \prec v$ is not true but still the Bruhat interval $\{t | u \leqslant t \leqslant v\}$ is not stabilized for any simple reflection. However for these examples we have $P_{u,v} \neq 1$ and $P_{w_0 v, w_0 u} \neq 1$, and Conjecture 4 is still true.

We will prove in Theorem 5 that Conjecture 4 and Conjecture 1 together imply a weak form of Conjecture 2.

When this work was at an early stage we spoke with Thomas Lam, Anne Schilling, Mark Shimozono, Nicolas Thiéry and others and their remarks were helpful in correctly formulating Conjecture 1. We also thank Ben Brubaker, Gautam Chinta, Solomon Friedberg and Paul Gunnells for helpful conversations.

Sage mathematical software [19] was crucial in these investigations. (Versions 4.3.2 and later have support for Iwahori Hecke algebras and Bruhat order.)

This work was supported in part by the JSPS Research Fellowship for Young Scientists and by NSF grant DMS-0652817.

2 Preliminaries

Let $\mathfrak{g}_\mathbb{C}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\mathfrak{t}_\mathbb{C}$ be a split Cartan subalgebra of $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}_\mathbb{C}$ corresponding to $\mathfrak{t}$ and let W be the Weyl group, and let $\hat{\Phi}$ be the dual root system.

Let $H_\alpha \in \mathfrak{t}$ ($\alpha \in \Phi$) be the coroots. Thus the root α is the linear functional $x \longmapsto \frac{2\langle x,H_a\rangle}{\langle H_\alpha,H_\alpha\rangle}$ with respect to a fixed W-invariant inner product on $\mathfrak{t}$. Using Théorème 1 of Chevalley [5] we may choose a basis $\mathfrak{g}$ that consists of $X_\alpha, X_{-\alpha}$ where α runs through the set Φ^+ of positive roots and $H_\alpha \in \mathfrak{t}$ where α runs through the simple roots. These have the properties that $[X_\alpha, X_\beta] = \pm(p+1)X_{\alpha+\beta}$ when $\alpha, \beta, \alpha+\beta \in \Phi$ is a root, where p is the greatest integer such that $\beta - p\alpha \in \Phi$ and $[H_\alpha, X_\beta] = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}X_\beta$. Let $\mathfrak{g}_\mathbb{Z}$ be the lattice spanned by this Chevalley basis. It is a Lie algebra over $\mathbb{Z}$ such that $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes \mathfrak{g}_\mathbb{Z}$.

Now if F is a field let $\mathfrak{g}_F = F \otimes \mathfrak{g}_\mathbb{Z}$. We will take F to be a nonarchimedean local field. Let G be a split semisimple algebraic group defined over F with Lie algebra $\mathfrak{g}_F$. Let $\mathfrak{o}$ be the ring of integers in F, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$ and q the cardinality of the residue field.

If $\alpha \in \Phi^+$ then there exists a homomorphism $i_\alpha : \mathrm{SL}_2 \longrightarrow G$ such that under the differential $di_\alpha : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ we have

$$di_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha, \qquad di_\alpha \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = X_{-\alpha}, \qquad di_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H_\alpha.$$

Let $x_\alpha : F \longrightarrow G(F)$ be the one-parameter subgroup $x_\alpha(t) = \exp(tX_\alpha)$. The Borel subgroup $B(F) = N(F)T(F)$ where $T(F)$ is the split Cartan subgroup with $\mathrm{Lie}(T) = \mathfrak{t}$ and N is generated by the $x_\alpha(F)$ with $\alpha \in \Phi^+$. If $\mathfrak{a}$ is a fractional ideal we will also denote by $N(\mathfrak{a})$ the subgroup generated by $x_\alpha(\mathfrak{a})$ with $\alpha \in \Phi^+$. Similarly $N_-(F)$ and $N_-(\mathfrak{a})$ are generated by $x_{-\alpha}(F)$ or $x_{-\alpha}(\mathfrak{a})$ with $\alpha \in \Phi^+$, and $B_-(F) = N_-(F)T(F)$. Let w_0 be the long element of W. Let $a_\alpha = i_\alpha \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix}$ where p is a fixed generator of $\mathfrak{p}$.

Let K be the maximal compact subgroup of $G(F)$ that stabilizes $\mathfrak{g}_\mathfrak{o}$ in the adjoint representation. Then reduction modulo $\mathfrak{p}$ gives a homomorphism $K \longrightarrow G(\mathbb{F}_q)$. Let J be the preimage of $B(\mathbb{F}_q)$ under this homomorphism. This is the *Iwahori subgroup*.

By a result of Iwahori and Matsumoto [11] (Section 2), we have a generalized Tits system in $G(F)$ with respect to J and the normalizer N of the maximal torus T of G that has Lie algebra $\mathfrak{t}_F = F \otimes \mathfrak{t}$. See also Iwahori [12]. The subgroup denoted B in these papers and in Matsumoto [15] is actually $w_0 J w_0^{-1}$. This is a bornological (B, N)-pair in the sense of Matsumoto [15], and we may make use of his results. In particular we have the Iwasawa decomposition $G(F) = B(F)K$ and let $T(\mathfrak{o}) = T(F) \cap K$. The Iwahori subgroup J is the subgroup generated by $T(\mathfrak{o})$, $N(\mathfrak{o})$ and $N_-(\mathfrak{p})$.

We have the *Iwahori factorization*, which is the statement that the multiplication map $T(\mathfrak{o}) \times N_-(\mathfrak{p}) \times N(\mathfrak{o}) \longrightarrow J$ is a homeomorphism. The three factors for this may be taken in any order. See Matsumoto [15] Proposition 5.3.3.

Let χ be a quasicharacter of $T(F)$. We say χ is *unramified* if χ is trivial on $T(\mathfrak{o})$. Let $X^*(T(F)/T(\mathfrak{o}))$ be the group of unramified quasicharacters. It is isomorphic to $X^*(\mathbb{Z}^r) = \mathbb{C}^r$ where r is the rank of G. The (connected) L-group $\hat{G} = {}^L G^\circ$ defined by Langlands [14] is a complex analytic group with a maximal torus $\hat{T}$ such that the unramified quasicharacters of $T(F)$ are in bijection with the elements of $\hat{T}$. If $\boldsymbol{z} \in \hat{T}$ let $\chi_{\boldsymbol{z}}$ be the corresponding unramified quasicharacter.

The Weyl groups $N_G(T)/T$ and $N_{\hat{G}}(\hat{T})/\hat{T}$ are isomorphic and may be identified. If $\boldsymbol{z} \in \hat{T}$ and $w \in W$ then $\chi_{w(\boldsymbol{z})} =^w \chi_{\boldsymbol{z}}$ where ${}^w\chi(t) = \chi(w^{-1}tw)$. If $\chi = \chi_{\boldsymbol{z}}$ is an unramified quasicharacter let $V(\chi) = \mathrm{Ind}_B^G(\delta^{1/2}\chi)$ denote the space of locally constant functions f on $G(F)$ such that if $b \in B(F)$ then

$$f(bg) = (\delta^{1/2}\chi)(b)\, f(g)$$

where $\delta : B(F) \longrightarrow \mathbb{C}$ is the modular quasicharacter. This is a module for $G(F)$ under right translation, and if $\boldsymbol{z}$ is in general position it is irreducible. The standard intertwining operators $M_w : V(\chi) \longrightarrow V({}^w\chi)$ are defined by

$$(M_w f)(g) = \int_{N \cap wN_- w^{-1}} f(w^{-1}ng)\, dn = \int_{(N \cap wNw^{-1})\backslash N} f(w^{-1}ng)\, dn. \tag{5}$$

The integral is absolutely convergent if $|\chi(a_\alpha)| < 1$, and may be meromorphically continued to all χ.

We recall that ϕ_w defined by (1) are a basis of $V(\chi)^J$. By the Iwasawa decomposition, $G(F) = B(F)K$ and by the Bruhat decomposition for $G(\mathbb{F}_q)$ pulled back to K under the canonical map we have $K = \bigcup_{u \in W} JuJ = \bigcup_{u \in W} B(\mathfrak{o})uJ$. Therefore

$$G(F) = \bigcup_{u \in W} B(F)uJ \qquad \text{(disjoint)}.$$

Proposition 3 *Let $x \in W$ and let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition into simple reflections. Then*

$$\left\{\alpha \in \hat{\Phi}^+ | w(\alpha) \in \hat{\Phi}^-\right\} = \{\alpha_{i_k}, s_{i_k}(\alpha_{i_{k-1}}), s_{i_k}s_{i_{k-1}}(\alpha_{i_{k-2}}), \cdots, s_{i_k} \cdots s_{i_2}(\alpha_{i_1})\}. \tag{6}$$

The elements in this list are distinct, so $k = l(x)$ is the cardinality of this set.

Proof This is Corollary 2 to Proposition 17 in VI.1.6 of Bourbaki [1]. $\square$

Proposition 4 *Let $w \in W$. If $w(\alpha) \in \hat{\Phi}^-$ then $wr_\alpha < w$. If $w(\alpha) \in \hat{\Phi}^+$ then $w < wr_\alpha$.*

Proof Suppose that $w(\alpha) \in \hat{\Phi}^-$. Write $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ a reduced expression. Then by Proposition 3 $\alpha = s_{i_m}\cdots s_{i_{k+1}}(\alpha_{i_k})$ for some k. Then

$$wr_\alpha = s_{i_1}\cdots\widehat{s_{i_k}}\cdots s_{i_m} < w, \qquad r_\alpha = (s_{i_m}\cdots s_{i_{k+1}})s_{i_k}(s_{i_{k+1}}\cdots s_{i_m}).$$

where the caret denotes the omitted factor. This proves the first case.

In the second case, $w(\alpha) \in \hat{\Phi}^+$ implies $w_0w(\alpha) \in \hat{\Phi}^-$ so the first case is applicable and implies that $w_0wr_\alpha < w_0w$. Now $w_0x < w_0y$ is equivalent to $y < x$ and so $w < wr_\alpha$. $\square$

3 Upper triangularity of $m(u, v)$

The Iwahori subgroup J admits the *Iwahori factorization*

$$J = T(\mathfrak{o})\, N_-(\mathfrak{p})\, N(\mathfrak{o}).$$

The factors may be written in any order. This is a special case of the following.

Proposition 5 *If $w \in W$ then*

$$xJx^{-1} = T(\mathfrak{o})(xJx^{-1} \cap N)(xJx^{-1} \cap N_-).$$

Proof It follows from Matsumoto [15], Lemme 5.4.2 on page 154 that

$$J = T(\mathfrak{o})(J \cap wNw^{-1})(J \cap wN_-w^{-1}).$$

Taking $w = x^{-1}$ and conjugating gives the result. $\square$

Proposition 6 *If $b \in B$ and $x, y \in W$ and if $yb \in BxJ$ then $x \leqslant y$.*

Proof Using the Iwahori factorization of J we may write $yb = b''xn_-b'$ where $b'' \in B$, $n_- \in N_-(\mathfrak{p})$, and $b' \in B(\mathfrak{o})$. Then $yb(b')^{-1} = b''xn_- \in BxB_-$ where B_- is the opposite Borel subgroup to B, so $ByB \cap BxB_- \neq \varnothing$. By Corollary 1.2 in Deodhar [8] it follows that $x \leqslant y$. $\square$

Proposition 7 *Suppose that $n = n_1 n_2$ with $n_1, n_2 \in N$, and that $xn \in BxJ$, $xn_1x^{-1} \in N$ and $xn_2x^{-1} \in N_-$. Then $n_2 \in N(\mathfrak{o})$.*

Proof We write $xn = bxk$ with $k \in J$, so $xn_1x^{-1} \cdot xn_2x^{-1} = bxkx^{-1}$. Then by Proposition 5 we write $xkx^{-1} = an_+n_-$ with $a \in T(\mathfrak{o})$, $n_+ \in xJx^{-1} \cap N$ and $n_- \in xJx^{-1} \cap N_-$. So

$$xn_1^{-1}x^{-1}ban_+ = xn_2x^{-1}n_-^{-1}.$$

Here the left-hand side is in B and the right hand side is in N_-, so both sides are 1. Thus $n_2 = x^{-1}n_-x \in N(\mathfrak{o})$. □

Proposition 8 *If $n \in N$ and $x \in W$, and $xnx^{-1} \in N_-$, and if $xn \in BxJ$ then $n \in N(\mathfrak{o})$.*

Proof This is the special case of the previous Proposition with $n_1 = 1$. □

Theorem 3 *If $(M_v\psi_u)(1) \neq 0$ then $u \leqslant v$. Moreover $(M_u\psi_u)(1) = 1$.*

Proof We may write

$$(M_v\psi_u)(1) = \int_{N \cap vN_-v^{-1}} \psi_u(v^{-1}n)\, dn.$$

If this is nonzero, then $\psi_u(v^{-1}n) \neq 0$ for some $n \in N$. Find $w \in W$ such that $v^{-1}n \in Bw^{-1}J$. Then by definition of ψ_u we have $w \geqslant u$. By Proposition 6 $w^{-1} \leqslant v^{-1}$ or $w \leqslant v$ and therefore $u \leqslant v$.

Now if $u = v$ then

$$(M_u\psi_u)(1) = \int_{N \cap uN_-u^{-1}} \psi_u(u^{-1}n)\, dn.$$

If $\psi_u(u^{-1}n) \neq 0$ then by definition of ψ_u we have $u^{-1}n \in Bw^{-1}J$ for some w such that $w \geqslant u$. By Proposition 6, $w \leqslant u$ and so $w = u$. Now by Proposition 8 $n \in N(\mathfrak{o})$. Thus the domain of integration can be taken to be $N(\mathfrak{o}) \cap wN_-(\mathfrak{o})w$. On this domain, the integrand is 1, and the measure is normalized so that the volume of $N(\mathfrak{o}) \cap wN_-(\mathfrak{o})w$ is 1. Hence $(M_u\psi_u)(1) = 1$. □

Proposition 9 *If $s = s_\alpha$ is a simple reflection then*

$$M_s\left({}^{\chi}\phi_1\right) = \frac{1}{q}\left({}^{s\chi}\phi_s\right) + \left(1 - \frac{1}{q}\right)\frac{\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}\left({}^{s\chi}\phi_1\right).$$

Proof See Casselman [3], Theorem 3.4. □

4 Hecke algebra

It is shown by Rogawski [18] that one may use the Iwahori Hecke algebra to express the intertwining operators. We will review this method. See also Reeder [17] and Haines, Kottwitz and Prasad [10].

We assume that the split semisimple group G is simply-connected. There is no loss of generality in assuming this for the purpose of computing the intertwining operators and Casselman basis.

There are two Weyl groups which we must consider. There is the affine Weyl group W_{aff} which is $N_G(T(F))/T(\mathfrak{o})$, and the ordinary Weyl group $N_G(T(F))/T(F)$. Following Iwahori and Matsumoto [11], [12], these Weyl groups and their Hecke algebras may be described as follows. Let $\sigma_1, \cdots, \sigma_r$ be the simple reflections. Then σ_i and σ_j commute unless i and j are adjacent nodes in the Dynkin diagram, in which case they satisfy the braid relation; assuming G is simply-laced, this has the form $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$. Then $\sigma_1, \cdots, \sigma_r$ generate W. Another generator σ_0 is needed for W_{aff}. Since we are assuming that G is simply-connected, then $\sigma_0, \cdots, \sigma_r$ generate $N(T(F))/T(\mathfrak{o})$, the affine Weyl group, with generators and relations as above except that one uses the extended Dynkin diagram to decide whether i and j are adjacent.

The Iwahori Hecke algebra is the convolution ring of compactly supported functions f on G such that $f(kgk') = f(g)$ when $k, k' \in J$. Its structure was determined by Iwahori and Matsumoto [11]. Normalizing the Haar measure so that J has volume 1, let t_w be the characteristic function of JwJ, and if $1 \leqslant i \leqslant r$ let t_i denote t_{σ_i}. The t_w with $w \in W_{\text{aff}}$ form a basis, and the t_i form a set of algebra generators The t_i satisfy the same braid relations as the s_i, but the relation $\sigma_i^2 = 1$ is replaced by $t_i^2 = (q-1)t_i + q$.

The subalgebra elements of H_{aff} consisting of functions that are supported in K is the finite Iwahori Hecke algebra H. Thus $\dim(H) = |W|$ but H_{aff} is infinite-dimensional The subalgebra H has generators $t_1, \cdots, t_r$ but omits t_0.

With notation as in the introduction, $V(\chi)^J$ is a module for H_{aff}. If $\phi \in H$ and $f \in V(\chi)$ then $\phi f(g) = \int_G \phi(h) f(gh)\, dh$.

We define a vector space isomorphism $\alpha = \alpha(\chi) : V(\chi)^J \longrightarrow H$ as follows. If $F \in V(\chi)^J$ then let $\alpha(F) = f$ where f is the function $f(g) = F(g^{-1})$ if $g \in K$, 0 if $g \notin K$. It may be checked using the Iwahori factorization that $\alpha(F) \in H$. Now $V(\chi)^J$ is a left-module for H (since $H \in H_{\text{aff}}$) and so is H. It is easy to check that α is a homomorphism of left H-modules. Now let $w \in W$ and define a map

$\mathcal{M}_w = \mathcal{M}_{w,\boldsymbol{z}} : H \longrightarrow H$ by requiring the diagram:

$$\begin{array}{ccc} V(\chi)^J & \xrightarrow{M_w} & V({}^w\chi)^J \\ \big\downarrow{\scriptstyle \alpha(\chi)} & & \big\downarrow{\scriptstyle \alpha({}^w\chi)} \\ H & \xrightarrow{\mathcal{M}_w} & H \end{array}$$

to be commutative. If $w \in W$ let us define $\mu_{\boldsymbol{z}}(w) = \mathcal{M}_w(1_H) \in H$, where 1_H is the unit element in the ring H. Note that $\alpha_\chi(\phi_1) = 1_H$, so

$$\mu_{\boldsymbol{z}}(w) = \alpha({}^w\chi)\phi_1.$$

Proposition 10 *We have*

$$\mathcal{M}_w(h) = h \cdot \mu_{\boldsymbol{z}}(w)$$

for all $h \in H$.

Proof $\mathcal{M}_w$ is a homomorphism of left H-modules, where H, being a ring, is a bimodule. Therefore $\mathcal{M}_w(h) = \mathcal{M}_w(h \cdot 1) = h\mathcal{M}_w(1) = h \cdot \mu_{\boldsymbol{z}}(w)$. □

Lemma 1 *If $l(w_1w_2) = l(w_1) + l(w_2)$ then*

$$\mu_{\boldsymbol{z}}(w_1w_2) = \mu_{\boldsymbol{z}}(w_2)\mu_{w_2\boldsymbol{z}}(w_1).$$

Proof We have $M_{w_1w_2} = M_{w_1} \circ M_{w_2}$. Therefore this follows from the commutativity of the diagram:

$$\begin{array}{ccccc} V(\chi)^J & \xrightarrow{M_{w_2}} & V({}^{w_2}\chi)^J & \xrightarrow{M_{w_1}} & V({}^{w_1w_2}\chi)^J \\ \big\downarrow{\scriptstyle \alpha(\chi)} & & \big\downarrow{\scriptstyle \alpha({}^{w_2}\chi)} & & \big\downarrow{\scriptstyle \alpha({}^{w_1w_2}\chi)} \\ H & \xrightarrow[\mathcal{M}_{w_2}]{} & H & \xrightarrow[\mathcal{M}_{w_1}]{} & H \end{array}$$

□

Lemma 2 *If $w = \sigma_i$ is a simple reflection, then $\mathcal{M}_w(1) = \frac{1}{q}t_i + (1 - \frac{1}{q})\frac{\boldsymbol{z}^{\alpha_i}}{1-\boldsymbol{z}^{\alpha_i}}$.*

Proof This follows from Proposition 9. □

We will denote $\alpha_\chi(\psi_u) = \psi(u)$. Note that this element of H is independent of χ: it is just the union of the characteristic functions of the double cosets JwJ with

$w \geqslant u$. If $f \in H$, let $\Lambda(f)$ denote the coefficient of 1 in the expansion of f in terms of the basis elements. Then

$$m_{\boldsymbol{z}}(u,v) = \Lambda(\psi(u)\mu_{\boldsymbol{z}}(v)).$$

Let us introduce the following notation. If $f, g \in H$ and $x \in W$, we will write $f \equiv g \bmod x$ if the only t_w ($w \in W$) that have nonzero coefficient in $f - g$ are those with $w \geqslant x$.

Proposition 11 *Let $x, y \in W$ let s be a simple reflection. Assume $x \leqslant y$.*
(a) Either $xs \leqslant y$ or $xs \leqslant ys$.
(b) Either $x \leqslant ys$ or $xs \leqslant ys$.

Proof Part (a) is proved in Humpfreys, *Reflection Groups and Coxeter Groups*, Proposition 5.9. For (b), since W is a finite Weyl group, it has a long element w_0 and $w_0 y < w_0 x$. Therefore by (a) either $w_0 ys < w_0 x$ or $w_0 ys < w_0 xy$, which implies (b). □

Proposition 12 *Let $u \in W$ and let s be a simple reflection.*
(a) Assume that $us > u$. Then for all $x \in W$ we have $x \geqslant u$ if and only if $xs \geqslant u$.
(b) Assume that $us < u$. Then for all $x \in W$ we have $x \leqslant u$ if and only if $xs \leqslant u$.

Proof For (a), if $x \geqslant u$ then either $xs \geqslant u$ or $xs \geqslant us$ by Proposition 11, but if $us > u$, both cases imply $xs \geqslant u$. Conversely if $xs \geqslant u$ the same argument shows $x \geqslant u$. This proves (a) and (b) is similar. □

If s is a simple reflection, let us denote

$$u \ominus s = \begin{cases} u & \text{if } u < us, \\ us & \text{if } us < u. \end{cases} \tag{7}$$

Proposition 13 *If $x \geqslant u$ and $y \leqslant s$ then $xy \geqslant u \ominus s$.*

Proof This follows from Proposition 12. □

Proposition 14 *Let s be a simple reflection, and let $u \in W$ such that $us > u$. Then $\psi(u)t_s = q\psi(u)$ and $\psi(u)\mu_{\boldsymbol{z}}(s) = \left(\frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha}\right)\psi(u)$.*

Proof The second conclusion follows from the first and Lemma 2, so we prove $\psi(u)t_s = q\psi(u)$. By Proposition 12 $\{x \in W | x \geqslant u\}$ is stable under right multiplication by s, so we may write $\psi(u)$ as a sum of terms of the form $t_x + t_{xs}$ with $xs > x$. But

$$(t_x + t_{xs})(t_s - q) = t_x(1 + t_s)(t_s - q) = 0$$

so $\psi(u)(t_s - q) = 0$. $\square$

Proposition 15 *Let s be a simple reflection, and let $u \in W$ such that $us > u$. Then*

$$\psi(us)t_s \equiv q\psi(u) \ mod \ us, \qquad \psi(us)\mu_{\mathbf{z}}(s) \equiv \psi(u) \ mod \ us. \tag{8}$$

Proof The first equation in (8) implies the second since by Lemma 2 $\mu_{\mathbf{z}}(s)$ differs from $\frac{1}{q}t_s$ by a scalar and $\psi(us) \equiv 0$ mod us. We prove the first equation.

Let us determine the coefficient of t_x in $\psi(us)t_s$ under the assumption that $x \not\geqslant us$. We will show that this coefficient equals q if $x \geqslant u$ and 0 otherwise. This will prove the Proposition since this is also the coefficient of t_x in $q\psi(u)$.

If $x \geqslant u$ then by Proposition 11 either $us \leqslant x$ or $us \leqslant xs$. Since we are assuming that $x \not\geqslant us$ it follows that $us \leqslant xs$. Hence $\psi(us)$ has a term t_{xs} but no term t_x. Therefore the only term in the sum

$$\psi(us)t_s = \sum_{z \geqslant us} t_z t_s \tag{9}$$

that can contribute to the coefficient of t_x is the term is $t_{xs}t_s$. Since $xs \geqslant us$ but $x \not\geqslant us$ we have $xs > x$. Thus $t_{xs} = t_x t_s$ and $t_{xs}t_s = t_x q$. Therefore the coefficient of t_x is q.

If $x \not\geqslant u$ then we claim that there is no contribution to t_x from any term in the sum (9). Indeed, the only z which could produce a contribution would $z = x$ or $z = xs$, but the condition $z \geqslant us$ is not satisfied for these. Indeed, $x \not\geqslant us$ since $x \not\geqslant u$. If $xs \geqslant us$ then by Proposition 11, either $x \geqslant us$ or $x \geqslant u$. Since $us > u$ we have $x \geqslant u$ in either case, contradicting our assumption. $\square$

5 Proof of Theorem 2

In this section we will not assume that $\hat{\Phi}$ is simply-laced.

Theorem 4 *Suppose that there exist reduced words*

$$\begin{aligned} v &= s_1 \dots s_n \\ u &= s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots\cdots \widehat{s_{i_m}} \cdots s_n, \end{aligned}$$

so that $l(v) = n$ and $l(u) = n - m$. Suppose moreover that $|S(u,v)| = l(v) - l(u)$ and that

$$S(u,v) = \{s_n s_{n-1} \cdots s_{i_k+1}(\alpha_{i_k}) | 1 \leqslant k \leqslant m\}.$$

Then

$$m(u,v) = \prod_{\alpha \in S(u,v)} \frac{1 - q^{-1}\boldsymbol{z}^{\alpha}}{1 - \boldsymbol{z}^{\alpha}}.$$

The reflections $r = r_\alpha$ with $\alpha \in S(u,v)$ such that $u \leqslant vr < v$ are precisely the $s_n s_{n-1} \cdots s_{i_k+1} s_{i_k} s_{i_k+1} \cdots s_n$ with $1 \leqslant k \leqslant m$, and so $u \leqslant s_1 \cdots \widehat{s_{i_k}} \cdots s_n < v$. Therefore the hypothesis that $u = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \cdots \widehat{s_{i_m}} \cdots s_n$ is equivalent to the assumption that $s_1 \cdots s_n$ is a good word for v with respect to u. It follows that this theorem is equivalent to Theorem 2.

Proof Here s_i is a simple reflection. Let α_i be the corresponding simple root. We will write $\mu(s_n) = \mu_{\boldsymbol{z}}(s_n)$, $\mu(s_{n-1}) = \mu_{s_n(\boldsymbol{z})}(s_{n-1})$, ... , suppressing the dependence on the spectral parameters. We have $m(u,v) = \Lambda(\psi(u)\mu_{\boldsymbol{z}}(v))$ where we may write $\psi(u)\mu_{\boldsymbol{z}}(v)$ as a sum of terms

$$\begin{aligned}
[\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots s_n)\mu(s_n) - \psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots s_{n-1})]\mu(s_{n-1}) \cdots \mu(s_1) \quad &+ \\
[\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots s_{n-1})\mu(s_{n-1}) - \psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots s_{n-2})]\mu(s_{n-2}) \cdots \mu(s_1) \quad &+ \\
\vdots \quad & \\
[\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}})\mu(s_{i_m}) - C(m)\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}})]\mu(s_{i_m-1}) \cdots \mu(s_1) \quad &+ \\
C(m)[\psi(s_1 \cdots \widehat{s_{i_1}} \cdots s_{i_m-1})\mu(s_{i_m-1}) - \psi(s_1 \cdots \widehat{s_{i_1}} \cdots s_{i_m-2})]\mu(s_{i_m-2}) \cdots \mu(s_1) \quad &+ \\
\vdots \quad & \\
C(m) \cdots C(1)[\psi(s_1)\mu(s_1) - \psi(1)] \quad &+ \\
C(m) \cdots C(1)\psi(1), \quad &
\end{aligned}$$

where

$$C(k) = \frac{1 - q^{-1}\boldsymbol{z}^{\gamma_k}}{1 - \boldsymbol{z}^{\gamma_k}}, \qquad \gamma_k = s_n s_{n-1} \cdots s_{i_k+1}(\alpha_{i_k}).$$

The summation telescopes with the terms cancelling in pairs. We will show that Λ annihilates every term except the last, so that $m(u,v) = C(m) \cdots C(1)$, as required.

We note that the terms of the form

$$\prod_{j>k} C(j)[\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_k}})\mu(s_{i_k}) - C(k)\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_k}})]\mu(s_{i_k+1}) \cdots \mu(s_1)$$

are equal to zero by Proposition 14. The spectral parameter for $\mu(s_{i_k})$ is $s_{i_{k+1}} \cdots s_{i_n}(\boldsymbol{z})$ so

$$C(k) = \frac{1 - q^{-1}\boldsymbol{z}^{\gamma_k}}{1 - \boldsymbol{z}^{\gamma_k}} = \frac{1 - q^{-1}(s_{i_{k+1}} \cdots s_{i_n}(\boldsymbol{z}))^{\alpha_{i_k}}}{1 - (s_{i_{k+1}} \cdots s_{i_n}(\boldsymbol{z}))^{\alpha_{i_k}}},$$

as in Proposition 14.

Each remaining terms is a constant times

$$[\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j)\mu(s_j) - \psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_{j-1})]\mu(s_{j-1}) \cdots \mu(s_1).$$

By Proposition 14 we have

$$\psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j)\mu(s_j) - \psi(s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_{j-1}) \geqslant s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j,$$

and so by Proposition 13 this term is $\geqslant s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j \ominus s_{j-1} \ominus s_{j-2} \ominus \cdots \ominus s_1$ in the notation (7). Thus this term is annihilated by Λ unless

$$s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j \ominus s_{j-1} \ominus s_{j-2} \ominus \cdots \ominus s_1 = 1. \tag{10}$$

We will assume this and deduce a contradiction. If this is true, then we may write

$$s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j = s_1 \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_k}} \cdots s_{j-1}, \tag{11}$$

where $j_k, j_{k-1}, \cdots\ j_1$ are the locations in (10) where the $\ominus$ is *not* a descent. In other words, the left hand side of (10) is of the form $s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots s_j s_{j-1} \cdots \widehat{s_{j_k}} \cdots \widehat{s_{j_1}} \cdots s_1$, and we have moved terms to the other side to obtain (11).

Now using (11) we may write

$$u = s_1 \cdots \widehat{s_{i_1}} \cdots s_n = s_1 \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_k}} \cdots s_{j-1} \widehat{s_j} s_{j+1} \cdots \widehat{s_{i_m}} \cdots s_n, \tag{12}$$

for we have substituted the right-hand side of (11) for an initial segment in the word representing u. Now let $\delta = s_n s_{n-1} \cdots s_{j+1}(\alpha_j)$. Then

$$v\, r_\delta = s_1 \cdots \widehat{s_j} \cdots s_n,$$

with only one omitted entry s_j. Clearly $vr_\delta < v$ and by (12) we have $u \leqslant vr_\delta$. Thus $\delta \in S(u, v)$. This is a contradiction, however, because the list

$$s_n \cdots s_{k+1}(\alpha_k)$$

of positive roots α such that $v(\alpha) \in \hat{\Phi}^-$ has no repetitions by Proposition 3. But j is not among the set $\{j_1, \cdots, j_m\}$. $\square$

6 Towards Conjecture 2

Proposition 16 *If Conjecture 4 is true and if* $u < v$ *and* $P_{w_0v,w_0u} = 1$ *then there exists* $\beta \in \hat{\Phi}^+$ *such that* $u \leqslant t \leqslant v$ *if and only if* $u \leqslant r_\beta t \leqslant v$.

Proof The conjecture implies that there exists γ such that $w_0v \leqslant t \leqslant w_0u$ if and only if $w_0v \leqslant r_\gamma t \leqslant w_0u$. Then we may take $\beta = -w_0(\gamma)$ so that $r_\beta = w_0 r_\gamma w_0$ and

$$\begin{aligned} u \leqslant t \leqslant v \quad \Leftrightarrow \quad w_0v \leqslant w_0t \leqslant w_0u \quad & \Leftrightarrow \quad w_0v \leqslant r_\gamma w_0 t \leqslant w_0u \\ & \Leftrightarrow \quad u \leqslant w_0 r_\gamma w_0 t \leqslant v. \end{aligned}$$

□

Although we do not see how to deduce Conjecture 2 from Conjecture 1, we have the following special case.

Theorem 5 *Conjectures 1 and 4 imply that if* $P_{u,v} = P_{w_0v,w_0u} = 1$ *then (3) is satisfied.*

Proof With notation as in Conjecture 4 we note that the map $t \longmapsto t' = r_\beta t$ is a bijection of the set $\{t|u \leqslant t \leqslant v\}$ to itself such that $l(t) \not\equiv l(t')$ mod 2 and such that

$$\{\alpha|u \leqslant tr_\alpha \leqslant v\} = \{\alpha|u \leqslant t'r_\alpha \leqslant v\}. \tag{13}$$

Indeed the property that r_β has, applied to tr_α instead of t, implies that $u \leqslant tr_\alpha \leqslant v$ if and only if $u \leqslant r_\beta tr_\alpha \leqslant v$.

Let M and $\tilde{M}$ be the matrices with coefficients $m(u,v)$ and $\tilde{m}(u,v)$. We know that these matrices are upper triangular with respect to the Bruhat order, that is, $m(u,v) = \tilde{m}(u,v) = 0$ unless $u \leqslant v$. Assuming Conjecture 1 if $P_{w_0v,w_0u} = 1$ then $m(u,v) = m'(u,v)$ where

$$m'(u,v) = \prod_{\substack{\alpha \in \hat{\Phi}^+ \\ u \leqslant vr_\alpha < v}} R(\alpha), \qquad R(\alpha) = \frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}. \tag{14}$$

According to Conjecture 2 if $P_{u,v} = 1$ then we should have $\tilde{m}(u,v) = \tilde{m}'(u,v)$ where

$$\tilde{m}'(u,v) = (-1)^{l(v)-l(u)} \prod_{\substack{\alpha \in \hat{\Phi}^+ \\ u \leqslant ur_\alpha < v}} R(\alpha). \tag{15}$$

By induction, we may assume that the counterexample minimizes $l(v) - l(u)$. If $u < t \leqslant v$ then $P_{t,v} = 1$ and $P_{w_0t,w_0u} = 1$. Therefore $m(u,t) = m'(u,t)$ and $\tilde{m}(t,v) = \tilde{m}'(t,v)$. Now since M and $\tilde{M}$ are inverse matrices, we have

$$\sum_{u \leqslant t \leqslant v} m(u,t)\tilde{m}(t,v) = 0,$$

and in this relation $m(u,t) = m'(u,t)$ is assumed for all t and $\tilde{m}(t,v) = \tilde{m}'(t,v)$ is proved for all t except when $t = u$. Therefore $\tilde{m}(u,v) = \tilde{m}'(u,v)$ will follow if we prove

$$\sum_{u \leqslant t \leqslant v} m'(u,t)\tilde{m}'(t,v) = 0.$$

We have

$$m'(u,t)\tilde{m}'(t,v) = (-1)^{l(v)-l(t)} \prod_{u \leqslant tr_\alpha \leqslant t} R(\alpha) \prod_{t \leqslant tr_\alpha \leqslant v} R(\alpha) = (-1)^{l(v)-l(t)} \prod_{u \leqslant tr_\alpha \leqslant v} R(\alpha)$$

because by Proposition 4 we always have either $tr_\alpha < t$ or $t < tr_\alpha$. Using Conjecture 4 and Proposition 16 there is a bijection $t \mapsto r_\beta t$ for some reflection r_β that stabilizes the Bruhat interval $u \leqslant t \leqslant v$. With $u \leqslant t \leqslant v$ this means that $u \leqslant tr_\alpha \leqslant v$ if and only if $u \leqslant r_\beta tr_\alpha \leqslant v$ and

$$(-1)^{l(v)-l(t)} \prod_{u \leqslant tr_\alpha \leqslant v} R(\alpha) = -(-1)^{l(v)-l(r_\beta t)} \prod_{u \leqslant r_\beta tr_\alpha \leqslant v} R(\alpha),$$

so the terms corresponding to t and $r_\beta t$ cancel. □

References

[1] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[2] James B. Carrell. The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties. In *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 53–61. Amer. Math. Soc., Providence, RI, 1994.

[3] W. Casselman. The unramified principal series of p-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[4] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[5] C. Chevalley. Sur certains groupes simples. *Tôhoku Math. J. (2)*, 7:14–66, 1955.

[6] Vinay V. Deodhar. Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. *Invent. Math.*, 39(2):187–198, 1977.

[7] Vinay V. Deodhar. Local Poincaré duality and nonsingularity of Schubert varieties. *Comm. Algebra*, 13(6):1379–1388, 1985.

[8] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985.

[9] M. J. Dyer. The nil Hecke ring and Deodhar's conjecture on Bruhat intervals. *Invent. Math.*, 111(3):571–574, 1993.

[10] Thomas J. Haines, Robert E. Kottwitz, and Amritanshu Prasad. Iwahori-Hecke algebras. *http://arxiv.org/abs/math/0309168*, 2003.

[11] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):5–48, 1965.

[12] Nagayoshi Iwahori. Generalized Tits system (Bruhat decompostition) on p-adic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 71–83. Amer. Math. Soc., Providence, R.I., 1966.

[13] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[14] Robert P. Langlands. *Euler products.* Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1.

[15] Hideya Matsumoto. *Analyse harmonique dans les systèmes de Tits bornologiques de type affine.* Lecture Notes in Mathematics, Vol. 590. Springer-Verlag, Berlin, 1977.

[16] Patrick Polo. On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar. *Indag. Math. (N.S.)*, 5(4):483–493, 1994.

[17] Mark Reeder. On certain Iwahori invariants in the unramified principal series. *Pacific J. Math.*, 153(2):313–342, 1992.

[18] J. D. Rogawski. On modules over the Hecke algebra of a p-adic group. *Invent. Math.*, 79(3):443–465, 1985.

[19] W. Stein et. al. SAGE Mathematical Software, Version 4.3.2. http://www.sagemath.org, 2010.

[20] John R. Stembridge. A short derivation of the Möbius function for the Bruhat order. *J. Algebraic Combin.*, 25(2):141–148, 2007.

[21] Daya-Nand Verma. Möbius inversion for the Bruhat ordering on a Weyl group. *Ann. Sci. École Norm. Sup. (4)*, 4:393–398, 1971.