A Game Theoretic Foundation of Competitive Equilibria with Adverse Selection*

Nick Netzer
University of Zurich

Florian Scheuer
Stanford University and NBER

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Abstract

We construct a fully specified extensive form game that captures competitive markets with adverse selection. In particular, it allows firms to offer any finite set of contracts, so that cross-subsidization is not ruled out. Moreover, firms can withdraw from the market after initial contract offers have been observed. We show that a subgame perfect equilibrium always exists and that, in fact, when withdrawal is costless, the set of subgame perfect equilibrium outcomes may correspond to the entire set of feasible contracts. We then focus on robust equilibria that exist both when withdrawal costs are zero and when they are arbitrarily small but strictly positive. We show that the Miyazaki-Wilson contracts are the unique robust equilibrium outcome of our game. This outcome is always constrained efficient and involves cross-subsidization from low to high risk agents that is increasing in the share of low risks in the population under weak conditions on risk preferences.

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1 Introduction

In the wake of the recent financial crisis, considerable attention has returned to the analysis of markets with adverse selection, in particular credit and insurance markets. Much of the discussion is rooted on the seminal contribution of Rothschild and Stiglitz (1976), which has spurred a large body of work on the nature of competitive equilibria in markets with adverse selection – and unfolded an enormous influence on the development of the economics of asymmetric information in general.

However, three shortcomings of the original Rothschild-Stiglitz approach to the problem of competition under adverse selection have been pointed out: First, in their setting, an equilibrium as they define it may not exist. Second, they restrict each firm to offer a single contract only, ruling out cross-subsidization between different contracts. The resulting inefficiency results have been very influential in the normative literature on government interventions, but they are an immediate consequence of restricting the sets of contracts that firms can offer. Third, their original approach is not a (non-cooperative) game theoretic one, since they define an equilibrium directly based on properties of sets of contracts rather than as an equilibrium of a fully specified game. This raises the question to what degree the results are specific to a particular equilibrium notion rather than properties of an economic environment as captured by an extensive form game.

This paper addresses all three of these issues. In particular, we construct a fully specified dynamic game that allows firms to offer any finite number of different contracts. We show that a standard subgame-perfect equilibrium (SPE) always exists. There is in fact a multiplicity of equilibria, but by invoking an appropriate robustness criterion, we obtain as unique equilibrium outcome the so-called Miyazaki-Wilson contracts. They may involve cross-subsidization and are always constrained Pareto optimal.

Many authors have addressed various subsets of the three shortcomings raised above. For instance, Wilson (1977) and Riley (1979) have observed that the non-existence problem can be overcome by modifying the notion of equilibrium: If firms anticipate the reaction of other firms to their contract offers, in form of withdrawal (Wilson 1977) or additional offers (Riley 1979), this either sustains the Rothschild-Stiglitz equilibrium for all parameter values (Riley 1979) or gives rise to a pooling equilibrium whenever the Rothschild-Stiglitz equilibrium does not exist (Wilson 1977). Similarly, Grossman (1979) shows that the Wilson pooling equilibrium can be sustained when agents anticipate that their insurance application may be declined. However, these contributions do not include comprehensive game theoretic treatments, and they stick to the assumption that each firm can offer only a single contract.[1]

Miyazaki (1977), Engers and Fernandez (1987) and Fernandez and Rasmusen (1993) relax the latter restriction, but still define their own particular notion of equilibrium. Miyazaki (1977) considers Wilson's anticipatory equilibrium concept and allows insurers to offer menus of contracts, resulting in the so-called Miyazaki-Wilson outcome. Engers and Fernandez (1987) proceed analogously, using the reactive equilibrium concept of Riley. They also suggest a game with an infinite number of moves and argue that the Miyazaki-Wilson contracts are the outcome of a Nash equilibrium in this game, but only among many other equilibrium outcomes. Fernandez and Rasmusen (1993) also propose a framework that generates the Miyazaki-Wilson allocation, but remain within a specific contestable monopoly model and resort to a non-standard equilibrium concept.

Hellwig (1987) was the first to pursue a standard game theoretic approach to analyze (sequential) equilibria in different dynamic settings, providing a foundation for the Rothschild-Stiglitz analysis amongst others. However, he continued to restrict firms to offer no more than one contract each. Fewer papers simultaneously speak to the three shortcomings pointed out above, but they do so in frameworks that go well beyond the original market setup. Maskin and Tirole (1992) consider a very general principal-agent framework where contracts can be arbitrary mechanisms. They focus mostly on signaling settings but also provide a multiplicity result for the case of screening that is related to ours, even though less encompassing. Asheim and Nilssen (1996) allow for renegotiation between firms and customers after initial contract choice and obtain the Miyazki-Wilson contracts as equilibrium outcome. These contracts are also the outcome in Faynzilberg (2006c), in a model where firms can become insolvent, which introduces an externality between agents in a contract. In Picard (2011), a similar externality occurs because agents directly participate in the firms' profits through mutual insurance, and the Miyazaki-Wilson allocation is again obtained in equilibrium. All these settings, however, focus on additional economic mechanisms outside of the canonical framework by Rothschild and Stiglitz (1976), Wilson (1977) and Miyazaki (1977).

Instead, following the spirit of the earlier literature, we propose the following dynamic game: In a first stage, a large number of risk-neutral firms offer any finite number of contracts to a continuum of risk-averse agents with private information. In a second stage,

[1]Several other authors have provided existence results for the Rothschild-Stiglitz equilibrium through further model extensions. Dasgupta and Maskin (1986) show that equilibrium existence can be guaranteed by considering mixed-strategy equilibria. Inderst and Wambach (2001) obtain equilibrium existence in a model with capacity constraints, Guerrieri, Shimer, and Wright (2010) show that search frictions can eliminate the non-existence problem, and von Siemens and Kosfeld (2011) restore existence by introducing externalities between agents in labor markets with team production. None of these contributions obtain Miyazaki-Wilson allocations as equilibrium outcome, since cross-subsidization is ruled out.

firms observe the contracts offered in stage 1 and can decide whether to remain active or whether to withdraw from the market. Finally, agents select their preferred contracts from the set of remaining offers and the contractually specified payments are enforced. This game can be reduced to a game of complete information between firms only, as the agents' contract choices in the final stage can be incorporated into the firms' payoff functions. Solving for the set of SPE of this game, we first confirm the observation of Wilson (1977) and Miyazaki (1977) for our game theoretic setting, namely that the introduction of a withdrawal stage guarantees equilibrium existence: The above game always has an SPE. However, it turns out that the withdrawal stage in fact generates a multiplicity of SPE. In particular, non-competitive equilibria with positive profits emerge where several firms offer contracts in stage 1 only to withdraw them along the equilibrium path, but credibly threaten to remain active if they observe deviations in stage 1. In fact, we show that the *entire* set of individually rational and resource and incentive feasible contracts may be sustained as SPE outcomes if the population share of low risks is not too high, and a slightly more constrained set otherwise.

To deal with this equilibrium multiplicity and select among the set of SPE, we introduce a withdrawal cost in the second stage. The motivation is that all non-competitive equilibria described above turn out to be destroyed by arbitrarily small withdrawal costs. On the other hand, large withdrawal costs would effectively eliminate stage 2 and reintroduce equilibrium existence problems. This motivates our focus on small withdrawal costs to select *robust* equilibria, i.e. equilibria that exist both when withdrawal is costless but also when withdrawal costs are strictly positive but sufficiently small. We demonstrate that there exists only a single SPE outcome that is robust in this sense: the Miyazaki-Wilson contracts. They survive the introduction of small withdrawal costs, because they incur losses only when the low risks are attracted away by a deviation, in which case these losses are necessarily large. This renders credible the out-of-equilibrium threat of withdrawal even when there are some costs.

A number of other researchers have recently shared our interest in providing game theoretic foundations for Miyazaki-Wilson contracts, resulting in a number of contemporaneous studies that have pursued complementary approaches. The most related is the work by Mimra and Wambach (2011), which shows that results similar to ours can be obtained in a model where, instead of withdrawing from the market, firms can withdraw individual contracts and there are multiple withdrawal stages that terminate endogenously. On the other hand, Diasakos and Koufopoulos (2011) and Koufopoulos (2011) have considered a three-stage game that allows insurers to commit not to withdraw contracts, e.g. by offering pre-approved applications, or not to withdraw an individual contract unless all other contracts in the menu are also withdrawn. These papers show that the Miyazaki-Wilson allocation can result as the unique equilibrium outcome under some selection criterion: free entry during the withdrawal stages in Mimra and Wambach (2011), endogenous commitment in Diasakos and Koufopoulos (2011).

In establishing equilibrium existence and efficiency, our paper also shares a common goal with the contribution by Bisin and Gottardi (2006). However, our game theoretic approach is fundamentally different from the general equilibrium perspective that they take.[2] Assuming that agents with private information can trade consumption bundles, taking prices as given, and incorporating incentive-compatibility constraints in their definition of a competitive equilibrium, they show that an equilibrium always exists: It is given by the Rothschild-Stiglitz contracts. They then demonstrate how the potential constrained inefficiency of this equilibrium can be overcome by introducing markets for consumption rights, where agents exchange permits for trading in markets for consumption bundles before they trade these consumption bundles themselves. Under some conditions on the initial allocation of these consumption rights, it turns out that the Miyazaki-Wilson contracts result as the equilibrium outcome.

Our paper is structured as follows. Section 2 introduces the model economy and collects results about Miyazaki-Wilson contract outcomes. In Section 3, we discuss a dynamic game that has been informally proposed as giving rise to Miazaki-Wilson type contracts as SPE outcome. Section 4 contains the full equilibrium analysis of the game and demonstrates that there generally arises a multiplicity problem. It then shows that Miyazaki-Wilson contracts are the unique robust equilibrium outcome of the game under a natural robustness criterion. Section 5 concludes. Several proofs are collected in the appendix.[3]

[2]See also Prescott and Townsend (1984a,b), Gale (1992, 1996), Dubey and Geanakoplos (2002), Martin (2007) and Rusticchini and Siconolfi (2008) for general equilibrium approaches to adverse selection. Voornefeld and Weibull (2011) investigate game theoretic foundations for markets with adverse selection in a setup as proposed by Akerlof (1970).

[3]Several results in this paper have been part of our earlier discussion paper Netzer and Scheuer (2008). Netzer and Scheuer (2010), which investigates ex-ante moral hazard incentives in the spirit of Fudenberg and Tirole (1990) but with ex-post competitive markets, is also based on this discussion paper. Some of the collected results on the Miyazaki-Wilson contracts in Section 2 also appear in different form in Netzer and Scheuer (2010).

2 Miyazaki-Wilson Contracts

2.1 Setup

There is a continuum of risk-averse agents, each of whom is endowed with an amount y of a consumption good, but faces idiosyncratic risk of a damage of size $d > 0$, which reduces the endowment to $y - d$. Each agent is one of two types. Low risk agents (indicated by L) experience the damage with probability $p_L \in]0, 1[$, while high risk agents (indicated by H) experience it with larger probability $p_H \in]p_L, 1[$.[4] An agent's type is private information. The share of low risks in the population is common knowledge and denoted by $\lambda \in]0, 1[$. We assume that agents use a utility function $U : \mathbb{R} \to \mathbb{R}$ to evaluate consumption c, and they maximize expected utility. $U(c)$ is strictly increasing, strictly concave, twice continuously differentiable, and satisfies $\lim_{c \to -\infty} U(c) = -\infty$ and $\lim_{c \to \infty} U(c) = \infty$. We denote by $\Phi(U)$ the inverse of $U(c)$.

We will work in the utility space, where contracts are tuples (u_N, u_D) of consumption utilities when no damage occurs (u_N) and when the damage does occur (u_D). Let $\mathscr{C} = \{(u_N, u_D) \in \mathbb{R}^2 | u_N \geq u_D\}$ be the set of possible contracts. We denote the set of all finite, non-empty subsets of $\mathscr{C}$ by $\mathscr{O}$. Let $\mathscr{A} = \{(u_{H,N}, u_{H,D}, u_{L,N}, u_{L,D}) \in \mathbb{R}^4 | u_{H,N} \geq u_{H,D}, u_{L,N} \geq u_{L,D}\}$ be the set of quadruples representing pairs of contracts, or allocations, one for each of the two risk types.

2.2 The Miyazaki-Wilson Program

In this subsection, we describe the contractual outcomes that have been referred to as Miyazaki-Wilson contracts in the insurance literature (see Miyazaki 1977, Wilson 1977 or Spence 1978). They solve the following optimization problem, which we call problem MW (for Miyazaki-Wilson):

$$\max_{(u_{H,N}, u_{H,D}, u_{L,N}, u_{L,D}) \in \mathscr{A}} (1 - p_L)\, u_{L,N} + p_L u_{L,D} \tag{1}$$

[4]We assume that a law of large numbers applies to the continuum of random variables defined by the population facing idiosyncratic risk. That is, we assume that exactly the shares p_L and p_H of low and high risks, respectively, eventually experience a damage. While laws of large numbers for a continuum of random variables may fail due to technical complications (Judd 1985), they can be put back into force through a variety of approaches. These include the application of a weaker convergence criterion (Uhlig 1996), the redefinition of the set indexing consumers (Green 1994), or the derivation of individual risk from the desired aggregate level properties (Alós-Ferrer 2002).

subject to the constraints

$$(1-p_L)u_{L,N} + p_L u_{L,D} \geq (1-p_L)u_{H,N} + p_L u_{H,D}, \tag{2}$$

$$(1-p_H)u_{H,N} + p_H u_{H,D} \geq (1-p_H)u_{L,N} + p_H u_{L,D}, \tag{3}$$

$$\lambda\left[(1-p_L)\Phi(u_{L,N}) + p_L\Phi(u_{L,D})\right] + (1-\lambda)\left[(1-p_H)\Phi(u_{H,N}) + p_H\Phi(u_{H,D})\right] \leq R, \tag{4}$$

$$\Phi((1-p_H)u_{H,N} + p_H u_{H,D}) \geq y - p_H d, \tag{5}$$

where $R = y - [\lambda p_L + (1-\lambda)p_H]d$ are per capita resources.

Program MW prescribes maximization of the low risks' expected utility subject to three standard and one additional constraint. Constraints (2) and (3) are the usual incentive compatibility constraints, and (4) is the resource constraint. Constraint (5) additionally rules out cross-subsidization from the high to the low risks. Formally, it requires that the certainty equivalent of contract $(u_{H,N}, u_{H,D})$ for the high risks cannot be lower than their type-specific per capita (i.e. expected) resources. Hence, the low risks may cross-subsidize the high risks, but the reverse is not possible. Observe that (5) implies that $(u_{H,N}, u_{H,D})$ can earn at most zero profits when purchased only by high risks.

The following lemma summarizes well-known results about the solution to MW (see, for instance, Miyazaki 1977, Spence 1978 or Bisin and Gottardi 2006, and the discussion therein). We also include a proof, because our formulation of the cross-subsidization constraint (5) differs from earlier approaches and because we will use some intermediary results from the proof later on.[5]

Lemma 1. *MW has a unique solution* $V^{MW} = (u^{MW}_{H,N}, u^{MW}_{H,D}, u^{MW}_{L,N}, u^{MW}_{L,D})$*. It is such that* $u^{MW}_{H,N} = u^{MW}_{H,D} \equiv u^{MW}_{H}$*, the constraints (3) and (4) are binding, and (2) is slack.*

Proof. See Appendix A.1. □

In the Miyazaki-Wilson contracts, high risks obtain a full insurance contract, while the low risks are only partially insured. The high risks' incentive compatibility constraint is binding, and resources are exhausted. Moreover, the solution is unique. Constraint (5) may or may not bind in the solution.

2.3 A Characterization of Cross-Subsidization

If constraint (5) binds in the solution to MW, there is no cross-subsidization between the contracts. The Miyazaki-Wilson outcome then coincides with the Rothschild-Stiglitz out-

[5] Lemma 1 and its proof appear in different form as Lemma 2 in Netzer and Scheuer (2010).

come $V^{RS} = (u_H^{RS}, u_H^{RS}, u_{L,N}^{RS}, u_{L,D}^{RS})$, where $u_H^{RS} = U(y - p_H d)$, and $(u_{L,N}^{RS}, u_{L,D}^{RS})$ solves the high risks' binding incentive constraint

$$u_H^{RS} = p_H u_{L,D}^{RS} + (1 - p_H) u_{L,N}^{RS}$$

and the low risks' zero profit condition

$$p_L \Phi(u_{L,D}^{RS}) + (1 - p_L)\Phi(u_{L,N}^{RS}) = y - p_L d,$$

by Lemma 1. If constraint (5) does not bind in the solution to MW, cross-subsidization takes place from the low risks' partial insurance contract to the high risks' full insurance contract. Taken on its own, the high risks' contract earns losses and the low risks' contract earns strictly positive profits.

We will proceed to study comparative statics of the solution with respect to λ. We will discuss the relation to the existing literature after Lemma 2. Let us first emphasize the dependence of the program $MW(\lambda)$ and its solution $V^{MW}(\lambda)$ on the share of low risks λ. With this notation, it is useful to define the cross-subsidy for the high risks as follows. For any given $\lambda \in]0,1[$, consider the unique solution $V^{MW}(\lambda)$ of $MW(\lambda)$. The contracts given by $V^{MW}(\lambda)$ then induce the cross-subsidy

$$\chi(\lambda) \equiv \Phi\left(u_H^{MW}(\lambda)\right) - (y - p_H d)\,. \qquad (6)$$

We collect the comparative statics of this cross-subsidy in the following lemma.[6]

Lemma 2. *(i)* $\chi(\lambda)$ *is continuous in* $]0,1[$, *and it holds that* $\lim_{\lambda\to 0}\chi(\lambda) = 0$ *and* $\lim_{\lambda\to 1}\chi(\lambda) = (p_H - p_L)d > 0$.
(ii) If $\Phi'(u)$ *is convex, then there exists* $\tilde{\lambda} \in]0,1[$ *such that* $\chi(\lambda) = 0$ *for all* $\lambda \in]0,\tilde{\lambda}]$, *and* $\chi(\lambda)$ *is strictly increasing in* λ *for all* $\lambda \in [\tilde{\lambda}, 1[$.

Proof. See Appendix A.2. □

Lemma 2 reflects the fact that the efficiency of the Rothschild-Stiglitz contracts V^{RS} depends on the share of low risks λ in the population. Part (i) of the result is well-known in the insurance literature.[7] It has been observed there that, if λ is sufficiently high, an information-feasible Pareto improvement is available through partial cross-subsidization

[6] Lemma 3 in Netzer and Scheuer (2010) describes related comparative statics, focusing on the utility difference between the contracts instead of the cross-subsidy.

[7] See e.g. Wilson (1977), Crocker and Snow (1985), Eckstein, Eichenbaum, and Peled (1985) and Faynzilberg (2006a)-(2006c).

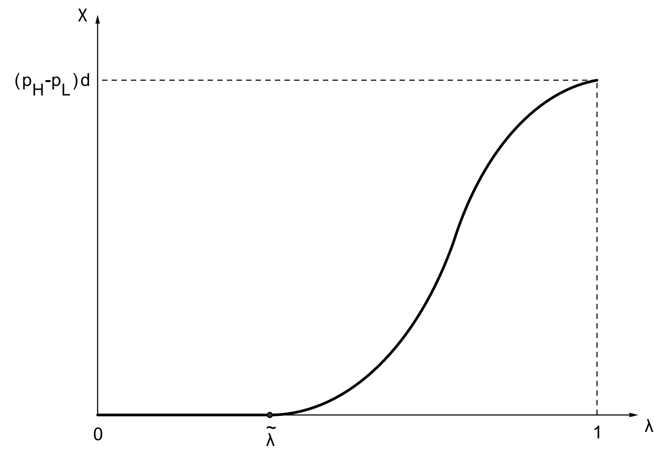

Figure 1: Optimal cross-subsidization $\chi(\lambda)$

even if the existence condition for the Rothschild-Stiglitz equilibrium is satisfied. Cross-subsidization allows to increase the low risks' insurance coverage without violating incentive compatibility. The limit as $\lambda \to 1$ corresponds to a pooling allocation in which all individuals obtain the same full insurance contract, as captured by the limiting cross-subsidization $(p_H - p_L)d$. On the other hand, as $\lambda \to 0$, cross-subsidization must vanish. Property (ii) shows that, if preferences are such that $\Phi'(u)$ is convex, the cross-subsidy is monotonically increasing in λ. This implies, in particular, the existence of a unique critical value $\tilde{\lambda}$ such that V^{MW} and V^{RS} coincide if and only if $\lambda \leq \tilde{\lambda}$. The cross-subsidy becomes positive when $\lambda > \tilde{\lambda}$, and it is strictly increasing in the population share of low risks. It is straightforward to verify that convexity of $\Phi'(u)$ is guaranteed if $U(c)$ has constant or increasing absolute risk aversion, or a constant relative risk aversion coefficient larger than one. In contrast, if risk aversion was decreasing rapidly in income, then the degree of cross-subsidization may be decreasing in λ over some range, because the low risks' willingness to subsidize the high risks in exchange for larger insurance coverage might be decreasing in λ due to income effects.[8]

The existence of a unique critical value $\tilde{\lambda}$ has been proven by Maskin and Tirole (1992) in a significantly more general principal-agent setting and under weaker conditions.[9]

[8]See Fudenberg and Tirole (1990) and Netzer and Scheuer (2010) for related arguments in a context where risk types are endogenous and thus ex-ante incentives are affected by the resulting ex-post allocation. Both contributions work with a condition that is stronger than convexity of $\Phi'(u)$, but is also implied by the sufficient conditions on risk aversion discussed above (Fudenberg and Tirole 1990, p. 1292).

[9]See Proposition 3 and its corollary in Maskin and Tirole (1992). Miyazaki (1977), Spence (1978) and Crocker and Snow (1985) also posit this result, but mostly rely on graphical arguments. Faynzilberg (2006a)-(2006c) also does not provide a formal proof for uniqueness of the threshold or monotonicity.

However, monotonicity of the cross-subsidization, while often presumed implicitly, has not been formally demonstrated to the best of our knowledge. Indeed, part (ii) of the lemma shows that a weak condition on risk preferences is required to guarantee monotonicity. Figure 1 depicts $\chi(\lambda)$ for a case where this condition is satisfied.

3 A Game Theoretic Foundation

3.1 The Dynamic Game

As discussed in the introduction, our goal is to develop a market structure that satisfies a number of requirements. First, a subgame perfect equilibrium should exist for every composition of the population $\lambda \in]0,1[$. Second, we do not want to restrict firms to offer only one contract. Third, the market structure should be captured by a simple extensive form game in line with the early ideas about markets with adverse selection. This is in contrast to most approaches in the literature on competitive insurance markets, that are either not based on a game theoretic analysis, impose the restriction of a single contract per firm, provide non-standard or only partial game theoretic arguments, or add additional aspects to the basic model.

Let $\mathscr{J} = \{0,1,2,...,m\}$ be a set of risk-neutral firms, each of which can offer up to $r \geq 2$ contracts from $\mathscr{C}$.[10] We assume that firm 0 is not a regular player of the game but always offers contract $(\bar{u}_N, \bar{u}_D)$ where $\bar{u}_N = U(y)$ and $\bar{u}_D = U(y-d)$. Choosing firm 0's contract corresponds to remaining uninsured. The extensive form we consider is the following:

Stage 1: Firms simultaneously decide on their contract offers.
Stage 2: After observing all contract offers from stage 1, firms simultaneously decide whether to remain in the market or to become inactive. Becoming inactive requires all offered contracts to be withdrawn, with a resulting payoff of $-\delta \leq 0$.
Stage 3: Agents simultaneously choose among all remaining offers.

Since we do not restrict the number of contracts that a given firm can offer in stage 1 to be one, the presence of another stage 2 in which firms can become inactive is crucial to avoid problems of equilibrium nonexistence that arise otherwise, as already observed by Miyazaki (1977).

[10] The equilibria that we construct require the existence of at least 4 active firms, so $m \geq 4$.

Note that our extensive form is different from the suggestion of insurance rejections by Grossman (1979) and also from the model by Fernandez and Rasmusen (1993), where firms can decline individual applications or withdraw individual contracts.[11] We rule out the withdrawal of individual contracts mainly because, otherwise, firms would withdraw any loss-making contract in stage 2, as pointed out by Grossman (1979), precluding any cross-subsidization in equilibrium. Other ways around this problem have been suggested in recent complementary studies, for instance by allowing for multiple withdrawal rounds with endogenous termination in Mimra and Wambach (2011), or endogenous commitment not to withdraw individual contracts from a menu in Diasakos and Koufopoulos (2011).[12] We abstract from this issue by ruling out partial withdrawal, but these papers show that analogous results can be obtained when this restriction is relaxed.

Withdrawal costs can be interpreted as a shortcut for the legal, administrative or reputational costs resulting from the fact that the firm originally offered contracts, but then foregoes the opportunity to serve them to customers on the market. Besides legal issues and claims, withdrawal may lead to adverse consequences such as consumer confusion or demand reductions in future periods or other markets in which the firm is active, effects that are not otherwise captured in our parsimonious three stage game. Alternatively, they could be motivated as a form of partial commitment not to withdraw contracts, an interpretation emphasized by Diasakos and Koufopoulos (2011) in their distinction between pre-approved versus standard applications. Methodologically, the only purpose of introducing withdrawal costs is to enable us later on to restrict attention to equilibria where firms remain active in case of indifference, so that there is no withdrawal along the equilibrium path.

We start with characterizing the agents' optimal strategies in stage 3 for any history of play up to stage 2. First, we restrict the history dependence of agents' strategies such that they are contingent only on the set of offered contracts available after stage 2, excluding the possibility that choices depend on the history of offers and withdrawals. We can then describe contract choices in stage 3 by functions $I_k : \mathscr{O} \to \mathscr{C}$, $k = L, H$, that give the

[11] As outlined above, Fernandez and Rasmusen (1993) consider a contestable monopoly model and resort to a special equilibrium concept. In their approach, exogenously given "old" contracts define a game, in which firms can decide to add "new" contracts and withdraw old but not new contracts. They are interested in subgame perfect equilibria where no new contracts are offered and no old contracts are withdrawn. Despite these differences, some of the arguments in our proofs, most notably for Proposition 2, share similarities to arguments by Fernandez and Rasmusen (1993).

[12] Note, however, that Diasakos and Koufopoulos (2011) consider a different timing where agents select a contract in stage 2 and withdrawal occurs in stage 3, so that customers who choose a contract that is withdrawn subsequently end up without insurance, as in Hellwig (1987) or the screening version of Maskin and Tirole (1992). In this case, the agents' out-of-equilibrium beliefs about whether contracts will be withdrawn become crucial, issues that are absent from our setting.

contract $I_k(Q) \in Q$ that an agent of type $k = L, H$ chooses out of any offered set Q. In particular, optimality of choice requires that for $k = L, H$ and any $Q \in \mathscr{O}$,

$$I_k^*(Q) \in \arg\max_{(u_N, u_D) \in Q} (1 - p_k) u_N + p_k u_D. \tag{7}$$

We also restrict attention to stage 3 strategies according to which the contract with smaller difference $u_N - u_D$, that is, with larger insurance coverage, is chosen in case of indifference. Moreover, we assume that, whenever the optimal contract for a type is offered by several different firms, then each firm receives the same share of these individuals.

The agents' optimal strategies can now be incorporated directly into the firms' profit functions. Stages 1 and 2 then constitute a well-defined extensive form game of complete information, denoted by Γ, in which firms are the only strategic players (we suppress the dependency of the game on the withdrawal cost parameter δ for notational convenience). Pure strategy profiles are denoted by $s = (s_0, s_1, ..., s_m) \in S$, where a firm's strategy $s_j = (s_j^1, s_j^2)$ has two components. First, s_j^1 is a set (possibly empty) of up to r contracts to be offered at stage 1. Let S_j^1 be the set of possible first period offers of firm j.[13] Then, $S^1 = \prod_{j \in \mathscr{J}} S_j^1$ is the set of possible histories to be observed at the beginning of stage 2, and we can associate a stage 2 subgame $\Gamma(\tilde{s}^1)$ to each history $\tilde{s}^1 = (\tilde{s}_0^1, ..., \tilde{s}_m^1) \in S^1$. For each subgame, s_j^2 then prescribes a withdrawal decision, i.e. $s_j^2 : S^1 \rightarrow \{NW, W\}$, where NW stands for no withdrawal and W for withdrawal. As before, denote by S_j^2 the set of firm j's possible stage 2 strategies and by $S^2 = \prod_{j \in \mathscr{J}} S_j^2$ the set of stage 2 strategy profiles.[14] Then, given a profile $s^2 \in S^2$ of functions, $s^2(\tilde{s}^1) = (s_0^2(\tilde{s}^1), ..., s_m^2(\tilde{s}^1)) \in \{NW, W\}^{m+1}$ is the vector of withdrawal decisions that s^2 prescribes after history $\tilde{s}^1$.

3.2 Payoffs

Fix a strategy profile $s = (s^1, s^2)$. We first define payoffs for each stage 2 subgame $\Gamma(\tilde{s}^1)$. Let $Q(s^2(\tilde{s}^1)|\tilde{s}^1)$ be the nonempty (due to existence of company 0), finite set of contracts that is available for choice at the end of $\Gamma(\tilde{s}^1)$ under the withdrawal decisions given by s^2. Formally,

$$Q(s^2(\tilde{s}^1)|\tilde{s}^1) = \bigcup_{\substack{j \in \mathscr{J} / \\ s_j^2(\tilde{s}^1) = NW}} \tilde{s}_j^1.$$

[13]Formally, $S_j^1 = \{Q \in \mathscr{O} | \, |Q| \leq r\} \bigcup \{\varnothing\}$ for $j = 1, ..., m$, and $S_0^1 = \{\{(\bar{u}_N, \bar{u}_D)\}\}$.

[14]Formally, $S_j^2 = \{NW, W\}^{S^1}$ for $j = 1, ..., m$, while S_0^2 is the singleton set containing only the function $s_0^2(\tilde{s}^1) = NW, \forall \tilde{s}^1 \in S^1$.

Let $\pi_k(u_N, u_D) = (1-p_k)(y - \Phi(u_N)) + p_k(y - d - \Phi(u_D))$ denote the expected profits earned with one unit of k-types in contract (u_N, u_D). Then, the payoffs of firm j in subgame $\Gamma(\tilde{s}^1)$ are given by $\Pi_j(s^2(\tilde{s}^1)|\tilde{s}^1) = -\delta$ if $s_j^2(\tilde{s}^1) = W$ and otherwise, if $s_j^2(\tilde{s}^1) = NW$, by

$$\Pi_j(s^2(\tilde{s}^1)|\tilde{s}^1) = \lambda \pi_L(I_L^*(Q)) \frac{\mathbf{1}_{\tilde{s}_j^1}(I_L^*(Q))}{\sum\limits_{\substack{i \in \mathscr{J} / \\ s_i^2(\tilde{s}^1) = NW}} \mathbf{1}_{\tilde{s}_i^1}(I_L^*(Q))} + (1-\lambda)\pi_H(I_H^*(Q)) \frac{\mathbf{1}_{\tilde{s}_j^1}(I_H^*(Q))}{\sum\limits_{\substack{i \in \mathscr{J} / \\ s_i^2(\tilde{s}^1) = NW}} \mathbf{1}_{\tilde{s}_i^1}(I_H^*(Q))},$$

where $Q = Q(s^2(\tilde{s}^1)|\tilde{s}^1)$, and $\mathbf{1}_X$ is the indicator function of set X. Given a strategy profile $s = (s^1, s^2) \in S$, the actual payoff of firm j in Γ is then $\Pi_j(s) = \Pi_j(s^2(s^1)|s^1)$. Mixed strategies and the associated payoffs can be defined analogously.

4 Equilibrium Analysis

In principle, we are interested in pure strategy subgame perfect equilibria (SPE) of Γ. However, we have to allow for randomization in some off-equilibrium path stage 2 subgames which do not have a Nash equilibrium in pure strategies.[15] We denote equilibrium candidates by σ, i.e. strategy profiles that are pure everywhere except in such off-equilibrium path subgames, and we indicate SPE of Γ by σ^*. In stage 3, agents are then faced with a nonempty set Q of available contract offers and will make their choices $I_L^*(Q)$ or $I_H^*(Q)$, respectively. Thus, any SPE σ^* is associated with an *outcome* $V^* = (u_{H,N}^*, u_{H,D}^*, u_{L,N}^*, u_{L,D}^*) \in \mathscr{A}$ that summarizes the optimal contract choices of both types in equilibrium.

4.1 Anything Goes

We start with demonstrating that, while the withdrawal phase in stage 2 guarantees equilibrium existence, it turns out that it also introduces a multiplicity of equilibria whenever withdrawal is costless ($\delta = 0$). Non-competitive equilibria emerge where firms offer competitive contracts only to withdraw them in equilibrium, but credibly threaten to remain active if they observe deviations after stage 1. The following proposition shows that, based on this idea, a very large set of contracts can in fact be sustained as SPE outcomes.

[15] In the SPE that we construct, any potentially profitable stage 1 deviation is destroyed using a pure strategy Nash equilibrium in stage 2. Thus we allow for randomization only to guarantee that there exists a Nash equilibrium in all of the (uncountably many) stage 2 subgames, including those that cannot even be reached by unilateral deviations, but we do not use randomization explicitly in our constructions.

Proposition 1. *Assume $\delta = 0$. Consider any $V^* = (u^*_{H,N}, u^*_{H,D}, u^*_{L,N}, u^*_{L,D}) \in \mathscr{A}$ that satisfies individual rationality and incentive compatibility,*

$$(u^*_{k,N}, u^*_{k,D}) = I_k\left(\{(u^*_{H,N}, u^*_{H,D}), (u^*_{L,N}, u^*_{L,D}), (\overline{u}_N, \overline{u}_D)\}\right), \;\; k \in \{H, L\}, \tag{8}$$

and the resource constraint,

$$\lambda\left[(1-p_L)\Phi(u^*_{L,N}) + p_L\Phi(u^*_{L,D})\right] + (1-\lambda)\left[(1-p_H)\Phi(u^*_{H,N}) + p_H\Phi(u^*_{H,D})\right] \leq R. \tag{9}$$

(i) Suppose λ is such that V^{RS} solves $MW(\lambda)$. Then there exists an SPE σ^* with outcome V^*.
(ii) Suppose λ is such that V^{RS} does not solve $MW(\lambda)$. Then there exists an SPE σ^* with outcome V^* if $V^* = V^{MW}$ or if V^* is Pareto dominated by V^{MW}, i.e.

$$(1-p_k)u^*_{k,N} + p_k u^*_{k,D} < (1-p_k)u^{MW}_{k,N} + p_k u^{MW}_{k,D}, \;\; k \in \{H, L\}. \tag{10}$$

Proof. See Appendix A.3. □

Proposition 1 shows that any pair of contracts on which the firms make non-negative profits together, which is incentive-compatible, and satisfies the agents' participation constraints, can be sustained as an SPE outcome of the above game, whenever λ is not too high. Note that this includes as a special case all weakly profitable pooling contracts (where $u^*_{H,N} = u^*_{L,N}$ and $u^*_{H,D} = u^*_{L,D}$) that are better than autarky for both types. If λ is so high that V^{MW} involves cross-subsidization, then the proposition delivers a slightly weaker "anything goes" result, in the sense that all contracts that satisfy the constraints mentioned above and in addition are Pareto dominated by the Miyazaki-Wilson contracts (or are identical to them) can be the outcome of an SPE. This is still a large set, including for instance all weakly profitable and individually rational pooling contracts that are worse than V^{MW} for the high risk types.

We use the following construction in order to sustain this large set of allocations as SPE outcomes. In stage 1, firm $j = 1$ offers the contracts from V^* which are to be implemented, and it remains active in stage 2. For part (i) of the result, firms $j = 2, 3$ offer the Rothschild-Stiglitz contracts in stage 1. They withdraw them in stage 2 along the equilibrium path, but credibly threaten to remain active in stage 2 whenever any deviation occurs in stage 1 (since there are two such firms, at least one remains offering the Rothschild-Stiglitz contracts after any unilateral deviation). Both withdrawal and the threat to remain active are best responses when withdrawal is costless, because the Rothschild-Stiglitz contracts earn zero profits no matter which contracts are offered in addition to them. This construction rules out any profitable deviation whenever V^{RS} is constrained efficient, because in that situation there are no contracts which can earn strictly positive profits in the presence of the Rothschild-Stiglitz contracts. A similar construction works for part (ii) of the proposition, with the main difference that $j = 2, 3$ now offer the Miyazaki-Wilson contracts and withdraw them along the equilibrium path. They threaten to remain active after any deviation, except if this would entail strict losses. Based on the arguments of Miyazaki (1977), it then again follows that deviations cannot be profitable. However, only allocations V^* that are either given by V^{MW} or make both risk types worse off than V^{MW} can be implemented in this way. If, for instance, the high risks were better off in V^* than in V^{MW}, then firms $j = 2, 3$ would not be willing to become inactive in stage 2, as they are only left with the low risks and make strictly positive profits.[16] This is the reason for the additional restriction in part (ii) of Proposition 1.

Proposition 1 demonstrates that the game Γ in itself is not able to produce Miyazaki-Wilson type contracts as the unique SPE outcome, as one may have expected. In contrast, the entire feasible set of contract outcomes, including contracts that make strictly positive profits, may arise in equilibrium. This multiplicity result is much more encompassing than related results in the literature. For instance, Hellwig (1987) argued that, if the Rothschild-Stiglitz equilibrium is inefficient, then the Rothschild-Stiglitz contracts as well as any pooling contract that Pareto dominates them can be the sequential equilibrium outcome of his three-stage game (see also Diasakos and Koufopoulos, 2011). In contrast, our multiplicity result is not confined to the case where the Rothschild-Stiglitz contracts are inefficient, and it allows for outcomes that are Pareto worse or incomparable to them. For their general screening model, Maskin and Tirole (1992) also provide a multiplicity result, but it is restricted to menus that make zero profits overall (i.e. (9) must hold with equality). Mimra and Wambach (2011) illustrate the multiplicity in their model using as an example a particular profitable full insurance contract that is only chosen by the high risks, but do not provide a further characterization.

Since the equilibria underlying the result in Proposition 1 involve firms becoming inactive along the equilibrium path, they are not robust in the sense that they are destroyed by arbitrarily small withdrawal costs. On the other hand, large withdrawal costs would effectively eliminate stage 2 and thus lead to equilibrium nonexistence. This motivates our introduction of withdrawal costs to select robust equilibria, i.e. equilibria that exist if withdrawal is costless, $\delta = 0$, but still for sufficiently small values of $\delta > 0$. It is worth emphasizing again that the purpose of considering withdrawal costs is therefore only to

[16]The same would be true if the high risks were indifferent between V^* and V^{MW} but the low risks strictly preferred V^{MW} to V^*, since then firms $j = 2, 3$ would be left with all of the low risks and part of the high risks, leading to strictly positive profits again. For this reason, (10) requires strict inequalities for both risk types whenever $V^* \neq V^{MW}$.

eliminate equilibria with withdrawal in case of indifference, and in particular along the equilibrium path. Mimra and Wambach (2011) show that an alternative selection criterion that achieves the same result in their model with multiple rounds of withdrawal and endogenous termination is to allow for entry of firms during the withdrawal stages.

4.2 Robust Equilibrium Outcomes

Let us denote the set of SPE outcomes of Γ for given cost parameter $\delta \geq 0$ by $\Omega^*(\delta) \subseteq \mathscr{A}$. That is, for every $V^* \in \Omega^*(\delta)$ there exists an SPE σ^* of the game Γ for cost parameter δ which has the outcome V^*, and conversely. From the previous section we already know that $\Omega^*(0)$ contains many elements. The main result of this subsection, which does not rest on any assumptions about λ, is the following:

Proposition 2. *(i) For any $\delta > 0$, $\Omega^*(\delta) \subseteq \{V^{MW}\} \subseteq \Omega^*(0)$.*
(ii) There exists a $\overline{\delta} > 0$ such that $V^{MW} \in \Omega^*(\delta)$ for all $0 \leq \delta < \overline{\delta}$.

Proof. See Appendix A.4. □

Part (i) of the proposition states that, whenever withdrawal costs are strictly positive, the set of SPE outcomes is either empty due to equilibrium nonexistence, which will be the case if withdrawal costs are too high, or it contains exactly the solution to MW.[17] This solution is still an equilibrium outcome if $\delta = 0$, but many additional equilibria with new outcomes emerge in this case, as illustrated by Proposition 1. All these additional outcomes are, however, not robust, i.e. they disappear for arbitrarily small withdrawal costs $\delta > 0$. On the other hand, part (ii) states that the solution to MW is actually robust, because it is an SPE outcome for sufficiently small but strictly positive values of δ. Thus, Proposition 2 shows that our market game Γ produces Miyazaki-Wilson contracts as the unique robust SPE outcome.

Why does the Miyazaki-Wilson equilibrium outcome survive the introduction of withdrawal costs? As Miyazaki (1977) has shown, there cannot be profitable deviations if the Miyazaki-Wilson contracts are withdrawn whenever a deviation has made them unprofitable. Withdrawal costs might in principle inhibit such withdrawal, even if the firms offering the Miyazaki-Wilson contracts earn losses after a deviation, which in turn could make some deviations profitable and destroy equilibria with outcome V^{MW}. However, when the Miyazaki-Wilson contracts earn losses, these losses are necessarily large, in the

[17] Observe that this is a statement about uniqueness of the equilibrium outcome, not about the equilibrium itself. There are always multiple equilibria with the same outcome, which for instance differ with respect to irrelevant contract offers that no agents chooses.

sense of being strictly bounded away from zero across all potentially profitable deviation histories. Losses can occur only when a competitor attracts away low but not high risks, in which case the profits of all firms that still offer the Miyazaki-Wilson contracts experience a discrete decline.[18] Hence withdrawal must still be optimal after any such deviation for sufficiently small but positive withdrawal costs, which is the reason for robustness of the equilibrium.

5 Conclusion

We propose a framework for competitive markets with adverse selection that satisfies three key properties: (i) firms can offer an arbitrary finite number of contracts, (ii) the analysis is based on characterizing SPE of a fully specified extensive form game, and (iii) an equilibrium always exists. In particular, it allows firms to withdraw from the market after initial contract offers have been observed. We show that, when withdrawal is costless, the set of SPE outcomes can be very large and include all contracts that are resource feasible, incentive-compatible and satisfy the agents' participation constraints (Proposition 1). We then focus on robust equilibria that exist both when withdrawal costs are zero and when they are strictly positive but small. We show that the constrained efficient Miyazaki-Wilson contracts are the unique robust equilibrium outcome of our game (Proposition 2).

Even though the extensive form game for which we derive these results is still special and may capture the strategic interaction between firms in different markets more or less well, our game theoretic approach has the advantage of shifting the focus from discussing the plausibility of equilibrium concepts to that of market structures as formalized by a fully specified game. For instance, in line with the ideas suggested by Engers and Fernandez (1987), a natural extension of our setting is to allow firms to interact for more than two periods, and offer and withdraw contracts while observing the competitors' behavior repeatedly. Pursuing this line of research, Mimra and Wambach (2011) have found that similar results to ours can be obtained in such a setting. Relatedly, Diasakos and Koufopoulos (2011) have demonstrated that distinguishing between pre-approved and standard applications for insurance and allowing firms to commit to not reject the applications for some contracts can yield similar results in a modified three stage game.

Another natural extension of our framework would be to allow for more than two risk

[18]Losses can also occur when a deviator attracts some but not all of the low risks. Since it is not possible to attract only a negligible share of the low risks, this will still have a large impact on the existing firms' profits, and the argument remains unaffected.

types. In this case, one additional complication that arises is that deviations could attract various sub-pools of the population that are profitable overall (e.g. the low and medium risks in a simple three type model). The effect of this possibility on equilibrium existence and uniqueness is not obvious. We leave these issues for future research.

References

AKERLOF, G. (1970): "The Market for Lemons: Qualitative Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84, 488–500.

ALÓS-FERRER, C. (2002): "Individual Randomness in Economic Models with a Continuum of Agents," Working paper, University of Konstanz.

ASHEIM, G., AND T. NILSSEN (1996): "Non-Discriminating Renegotiation in a Competitive Insurance Market," *European Economic Review*, 40, 1717–1736.

BISIN, A., AND P. GOTTARDI (2006): "Efficient Competitive Equilibria with Adverse Selection," *Journal of Political Economy*, 114, 485–516.

CROCKER, K., AND A. SNOW (1985): "The Efficiency of Competitive Equilibria in Insurance Markets with Asymmetric Information," *Journal of Public Economics*, 26, 207–219.

DASGUPTA, P., AND E. MASKIN (1986): "The Existence of Equilibrium in Discontinuous Games II: Applications," *Review of Economic Studies*, 53, 27–41.

DIASAKOS, T., AND K. KOUFOPOULOS (2011): "Efficient Nash Equilibrium under Adverse Selection," *Mimeo, Collegio Carlo Alberto and University of Warwick*.

DUBEY, P., AND J. GEANAKOPLOS (2002): "Competitive Pooling: Rothschild-Stiglitz Reconsidered," *Quarterly Journal of Economics*, 117, 1529–1570.

ECKSTEIN, Z., M. EICHENBAUM, AND D. PELED (1985): "Uncertain lifetimes and the welfare enhancing properties of annuity markets and social security," *Journal of Public Economics*, 26, 303–326.

ENGERS, M., AND L. FERNANDEZ (1987): "Market Equilibrium with Hidden Knowledge and Self-Selection," *Econometrica*, 55, 425–439.

FAYNZILBERG, P. (2006a): "Competitive Insurance Markets I: The Concept and Efficiency of Equilibrium," *SSRN Discussion Paper 428840*.

——— (2006b): "Competitive Insurance Markets II: The Structure of Equilibrium and Comparative Statics," *SSRN Discussion Paper 903643*.

——— (2006c): "Credible Forward Commitments and Risk-Sharing Equilibria," *SSRN Discussion Paper 903815*.

FERNANDEZ, L., AND E. RASMUSEN (1993): "Perfectly Contestable Monopoly and Adverse Selection," *Mimeo, Indiana University Bloomington*.

FUDENBERG, D., AND J. TIROLE (1990): "Moral Hazard and Renegotiation in Agency Contracts," *Econometrica*, 58, 1279–1320.

GALE, D. (1992): "A Walrasian Theory of Markets with Adverse Selection," *Review of Economic Studies*, 59, 229–255.

——— (1996): "Equilibria and Pareto Optima of Markets with Adverse Selection," *Economic Theory*, 7, 207–235.

GREEN, E. (1994): "Individual Level Randomness in a Nonatomic Population," *Mimeo, University of Minnesota*.

GROSSMAN, H. I. (1979): "Adverse Selection, Dissembling, and Competitive Equilibrium," *Bell Journal of Economics*, 10, 336–346.

GUERRIERI, V., R. SHIMER, AND R. WRIGHT (2010): "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 78, 1823–1862.

HELLWIG, M. (1987): "Some Recent Developments in the Theory of Competition in Markets with Adverse Selection," *European Economic Review*, 31, 319–325.

INDERST, R., AND A. WAMBACH (2001): "Competitive Insurance Markets Under Adverse Selection and Capacity Constraints," *European Economic Review*, 45, 1981–1992.

JUDD, K. (1985): "The Law of Large Numbers with a Continuum of iid Random Variables," *Journal of Economic Theory*, 35, 19–25.

KOUFOPOULOS, K. (2011): "Endogenous Commitment and Nash Equilibria in Competitive Markets with Adverse Selection," *Mimeo, University of Warwick*.

MARTIN, A. (2007): "On Rothschild-Stiglitz as Competitive Pooling," *Economic Theory*, 3, 371–386.

MASKIN, E., AND J. TIROLE (1992): "The Principal-Agent Relationship with an Informed Principal, II: Common Values," *Econometrica*, 60, 1–42.

MIMRA, W., AND A. WAMBACH (2011): "A Game-Theoretic Foundation for the Wilson Equilibrium in Competitive Insurance Markets with Adverse Selection," *CESifo Working Paper 3412*.

MIYAZAKI, H. (1977): "The Rat Race and Internal Labor Markets," *Bell Journal of Economics*, 8, 394–418.

NETZER, N., AND F. SCHEUER (2008): "Competitive Markets Without Commitment," SOI Discussion Paper 0814, University of Zurich.

——— (2010): "Competitive Markets Without Commitment," *Journal of Political Economy*, 118, 1079–1109.

PICARD, P. (2011): "Participating Insurance Contracts and the Rothschild-Stiglitz Equilibrium Puzzle," *Mimeo, Ecole Polytechnique*.

PRESCOTT, E., AND R. TOWNSEND (1984a): "General Competitive Equilibrium Analysis in an Economy with Private Information," *International Economic Review*, 25, 1–20.

——— (1984b): "Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard," *Econometrica*, 52, 21–46.

RILEY, J. (1979): "Informational Equilibrium," *Econometrica*, 47, 331–359.

ROTHSCHILD, M., AND J. STIGLITZ (1976): "Equilibrium in Competitive Insurance Markets: An Essay in the Economics of Incomplete Information," *Quarterly Journal of Economics*, 90, 629–649.

RUSTICCHINI, A., AND P. SICONOLFI (2008): "General Equilibrium in Economies with Adverse Selection," *Economic Theory*, 37, 1–29.

SPENCE, A. (1978): "Product Differentiation and Performance in Insurance Markets," *Journal of Public Economics*, 10, 427–447.

UHLIG, H. (1996): "A Law of Large Numbers for Large Economies," *Economic Theory*, 8, 41–50.

VON SIEMENS, F., AND M. KOSFELD (2011): "Team Production in Competitive Labor Markets with Adverse Selection," *Mimeo, University of Amsterdam*.

VOORNEFELD, M., AND J. W. WEIBULL (2011): "A Scent of Lemon – Seller Meets Buyer With a Noisy Quality Observation," *Games*, 2, 163–186.

WILSON, C. (1977): "A Model of Insurance Markets with Incomplete Information," *Journal of Economic Theory*, 12, 167–207.

A Appendix

A.1 Proof of Lemma 1

We will first show that the statement about the constraints has to be true and that the high risks' utility will not be output-dependent in any solution V^{MW} to the problem, if it exists. We then prove that the problem has a unique solution.[19]

Constraint (4). Assume that $V = (u_{H,N}, u_{H,D}, u_{L,N}, u_{L,D}) \in \mathscr{A}$ satisfies all constraints, and (4) with slack. Consider $\tilde{V} = (u_{H,N} + \epsilon_1, u_{H,D} + \epsilon_2, u_{L,N} + \epsilon_3, u_{L,D} + \epsilon_4)$ with ϵ_i, $i = 1, ..., 4$, such that

$$\frac{p_L}{1 - p_L}(\epsilon_2 - \epsilon_4) \leq \epsilon_3 - \epsilon_1 \leq \frac{p_H}{1 - p_H}(\epsilon_2 - \epsilon_4),$$

$\epsilon_1 \geq \epsilon_2 > 0$ and $\epsilon_3 \geq \epsilon_4 > 0$. By the assumptions on ϵ_i, $i = 1, ..., 4$, $\tilde{V} \in \mathscr{A}$, $\tilde{V}$ satisfies (2), (3) and (5), and $\tilde{V}$ leads to a strictly increased value of (1). To see that a set of ϵ_i, $i = 1, ..., 4$, with the required properties always exists, start by fixing any $\Delta_{24} \in \mathbb{R}^+$ and note that since $p_L < p_H$, there exists a $\Delta_{31} \in \mathbb{R}^+$ such that

$$\frac{p_L}{1 - p_L}\Delta_{24} \leq \Delta_{31} \leq \frac{p_H}{1 - p_H}\Delta_{24}.$$

Next, fix any $\epsilon_2, \epsilon_4 > 0$ such that $\epsilon_2 - \epsilon_4 = \Delta_{24}$. Clearly, it is then always possible to find $\epsilon_1, \epsilon_3 > 0$ such that $\epsilon_1 \geq \epsilon_2$, $\epsilon_3 \geq \epsilon_4$ and $\epsilon_3 - \epsilon_1 = \Delta_{31}$, which proves the claim. Finally, continuity of $\Phi(.)$ implies that (4) is still satisfied for ϵ_i sufficiently small, so that V was not a solution to MW.

Output-independent utilities for high risks. Assume that $V = (u_{H,N}, u_{H,D}, u_{L,N}, u_{L,D}) \in \mathscr{A}$ with $u_{H,N} > u_{H,D}$ satisfies all constraints, and (4) with equality. Define $\tilde{u} = (1 - p_H)u_{H,N} + p_H u_{H,D}$ and consider $\tilde{V} = (\tilde{u}, \tilde{u}, u_{L,N}, u_{L,D}) \in \mathscr{A}$. By construction, $\tilde{V}$ satisfies (3) and (5), and the value of (1) is the same under V and $\tilde{V}$. Since $p_H > p_L$ and $u_{H,N} > u_{H,D}$ (by assumption), it follows that $(1 - p_L)u_{H,N} + p_L u_{H,D} > (1 - p_H)u_{H,N} + p_H u_{H,D} = \tilde{u} = (1 - p_L)\tilde{u} + p_L\tilde{u}$, so that $\tilde{V}$ satisfies (2) as well, given that it is satisfied by V. Strict convexity of Φ implies that $(1 - p_H)\Phi(u_{H,N}) + p_H\Phi(u_{H,D}) > \Phi((1 - p_H)u_{H,N} + p_H u_{H,D}) = \Phi(\tilde{u}) = (1 - p_H)\Phi(\tilde{u}) + p_H\Phi(\tilde{u})$, so that $\tilde{V}$ satisfies (4) with slack (given that $\lambda \in]0, 1[$). From the previous argument, the value of the objective can then be increased above its value for $\tilde{V}$ and V, so that V was not a solution to MW.

Constraint (3). Let $V = (u_H, u_H, u_{L,N}, u_{L,D}) \in \mathscr{A}$ satisfy all constraints, and (4) with equality. (2) and (3) together imply $u_{L,N} \geq u_H \geq u_{L,D}$. Assume (3) is slack, which implies $u_{L,N} > u_{L,D}$.

[19]This direct proof is a combination of the proofs of Lemmas 1 and 2 in Netzer and Scheuer (2010).

Consider $\tilde{V}(\epsilon) = (u_H, u_H, u_{L,N} - \epsilon, u_{L,D} + \epsilon \frac{1-p_L}{p_L})$, $\epsilon \geq 0$, which is an element of $\mathscr{A}$ for ϵ small enough, and which satisfies $\tilde{V}(0) = V$. By construction, $\tilde{V}(\epsilon)$ satisfies (2) and (5), and the value of (1) is the same under V and $\tilde{V}(\epsilon)$, for any $\epsilon \geq 0$. (3) is also satisfied by $\tilde{V}(\epsilon)$ for ϵ small enough. Let $E_L(\epsilon) \equiv (1-p_L)\Phi(u_{L,N} - \epsilon) + p_L \Phi(u_{L,D} + \epsilon \frac{1-p_L}{p_L})$ denote the per capita expenditure for low risks in $\tilde{V}(\epsilon)$. Straightforward calculations reveal that $dE_L(\epsilon)/d\epsilon < 0$ if $0 \leq \epsilon < p_L(u_{L,N} - u_{L,D})$, so that for $\epsilon > 0$ small enough, $\tilde{V}(\epsilon)$ satisfies (4) with slack. From the above argument, V cannot be a solution to MW.

Constraint (2). Let $V = (u_H, u_H, u_{L,N}, u_{L,D}) \in \mathscr{A}$ satisfy all constraints, and (2) – (4) with equality. This implies that $u_{L,N} = u_{L,D} = u_H$, and (4) simplifies to $\Phi(u_H) = R$. Consider $\tilde{V}(\epsilon) = (u_H, u_H, u_H + \epsilon, u_H - \epsilon \frac{1-p_H}{p_H}) \in \mathscr{A}$ for $\epsilon \geq 0$, which satisfies $\tilde{V}(0) = V$. By construction and the fact that $p_L < p_H$, $\tilde{V}(\epsilon)$ satisfies (2) and (3), and the value of (1) is higher under $\tilde{V}(\epsilon)$ than under V for any $\epsilon > 0$. (5) is also satisfied by $\tilde{V}(\epsilon)$. Let $E(\epsilon) = \lambda[(1-p_L)\Phi(u_H + \epsilon) + p_L \Phi(u_H - \epsilon \frac{1-p_H}{p_H})] + (1-\lambda)\Phi(u_H)$ denote the per capita expenditure in $\tilde{V}(\epsilon)$. Straightforward calculations reveal that $dE(\epsilon)/d\epsilon < 0$ at $\epsilon = 0$, so that $\tilde{V}(\epsilon)$ satisfies (4) with slack for $\epsilon > 0$ small enough. Hence V cannot be a solution to MW.

Existence and Uniqueness. The previous results show that any solution to MW must be of the form $V = (u_H, u_H, u_{L,N}, u_{L,D})$, and that (3) becomes $u_H = (1-p_H)u_{L,N} + p_H u_{L,D}$, or equivalently $u_{L,D} = (u_H - (1-p_H)u_{L,N})/p_H$. Constraint (2) will be slack. Moreover, the condition $u_{L,N} \geq u_{L,D}$ in the definition of $\mathscr{A}$ can be reformulated as $u_{L,N} \geq u_H$, or $(u_{L,N}, u_H) \in \mathscr{C}$. We can therefore state the following modified problem MW', which has the same solution as MW:

$$\max_{(u_{L,N}, u_H) \in \mathscr{C}} \left[\left(\frac{p_L}{p_H} \right) u_H + \left(\frac{p_H - p_L}{p_H} \right) u_{L,N} \right] \tag{11}$$

subject to the binding resource constraint

$$\lambda \left[(1-p_L)\Phi(u_{L,N}) + p_L \Phi \left(\frac{u_H - (1-p_H)u_{L,N}}{p_H} \right) \right] + (1-\lambda)\Phi(u_H) = R \tag{12}$$

and the additional cross-subsidization constraint (5). Denote the LHS of (12) by $E(u_{L,N}, u_H)$. E is continuously differentiable on $\mathbb{R}^2$, and straightforward calculations reveal that it is strictly increasing in $u_{L,N}$ on $\mathscr{C}$ (including its boundary), with $\lim_{u_{L,N} \to \infty} E(u_{L,N}, u_H) = \infty$ due to convexity. E is strictly increasing in u_H globally, with $\lim_{u_H \to -\infty} = -\infty$.

The cross-subsidization constraint (5) can easily be rearranged to $u_H \geq U(y - p_H d) \equiv u^{min}$, so that it specifies the minimal choice of u_H. We next claim that $u^{max} \equiv U(R)$ (so that $u^{max} > u^{min}$) represents the largest possible choice of u_H. Consider the tuple $(u^{max}, u^{max}) \in \mathscr{C}$, which satisfies (12) by construction. Any tuple $(\tilde{u}_{L,N}, \tilde{u}_H) \in \mathscr{C}$ with $\tilde{u}_H > u^{max}$ and thus $\tilde{u}_{L,N} > u^{max}$ can be reached from (u^{max}, u^{max}) by first increasing $u_{L,N}$ from u^{max} to $\tilde{u}_{L,N}$ and then increasing u_H from u^{max} to $\tilde{u}_H$. Both moves strictly increase $E(u_{L,N}, u_H)$, so that $(\tilde{u}_{L,N}, \tilde{u}_H)$ violates (12), which proves the claim.

Now fix any $u_H \in [u^{min}, u^{max}]$. It follows that $E(u^{max}, u_H) \leq E(u^{max}, u^{max}) = R$, with strict inequality whenever $u_H < u^{max}$. Since $E(u_{L,N}, u_H)$ is strictly increasing in $u_{L,N}$ in the set $\mathscr{C}$, with $\lim_{u_{L,N}\to\infty} E(u_{L,N}, u_H) = \infty$, it follows that there exists a unique value $H(u_H)$ such that $E(H(u_H), u_H) = R$, where $H(u_H) \geq u^{max} \geq u_H$. The resulting function $H : [u^{min}, u^{max}] \to [u^{max}, \infty[$ is continuously differentiable and thus continuous by the implicit function theorem.

We can now reduce MW′ to the one-dimensional problem

$$u_H^{MW} = \arg \max_{u_H \in [u^{min}, u^{max}]} p_L u_H + (p_H - p_L) H(u_H), \tag{13}$$

for which existence of a solution follows immediately by the Weierstrass theorem. To prove uniqueness, we show strict concavity of the objective by showing that $H(u_H)$ is strictly concave. Let $(u'_{L,N}, u'_H), (u''_{L,N}, u''_H) \in \mathscr{C}$ satisfy $E(u'_{L,N}, u'_H) = E(u''_{L,N}, u''_H) = R$ and $(u'_{L,N}, u'_H) \neq (u''_{L,N}, u''_H)$. Define $u'''_{L,N} = \eta u'_{L,N} + (1-\eta) u''_{L,N}$ and $u'''_H = \eta u'_H + (1-\eta) u''_H$ for $\eta \in]0,1[$. Strict convexity of Φ then implies that $E(u'''_{L,N}, u'''_H) < R$, which in turn implies that $H(u'''_H) = H(\eta u'_H + (1-\eta) u''_H) > u'''_{L,N} = \eta u'_{L,N} + (1-\eta) u''_{L,N} = \eta H(u'_H) + (1-\eta) H(u''_H)$, which completes the proof.

A.2 Proof of Lemma 2

Property (i). We will first show that the solution $V^{MW}(\lambda)$ to MW(λ) is continuous in λ on $]0,1[$. It then follows that $\chi(\lambda)$ is continuous as well. From the proof of Lemma 1 we know that the solution to MW(λ) for $\lambda \in]0,1[$ can be found by solving the simplified problem (13):

$$u_H^{MW}(\lambda) = \arg \max_{u_H \in [u^{min}, u^{max}(\lambda)]} p_L u_H + (p_H - p_L) H(u_H, \lambda), \tag{14}$$

where $u^{max}(\lambda) = U(R(\lambda))$, and for given λ, the function H is continuously differentiable, strictly decreasing and strictly concave in u_H on $[u^{min}, u^{max}(\lambda)]$.[20] Let $\mathcal{F} = (0,1)$, $\mathcal{U} = [u^{min}, U((1-p_L)y + p_L(y-d))]$, and $\mathcal{C}(\lambda) = [u^{min}, u^{max}(\lambda)] \subset \mathcal{U}$. Clearly, the correspondence $\mathcal{C} : \mathcal{F} \rightrightarrows \mathcal{U}$ is compact-valued and continuous. Define $Z : \mathcal{U} \times \mathcal{F} \to \mathbb{R}$ by

$$Z(u_H, \lambda) = \begin{cases} H(u_H, \lambda) & \text{if} \quad u_H \leq u^{max}(\lambda), \\ u^{max}(\lambda) & \text{if} \quad u_H > u^{max}(\lambda). \end{cases} \tag{15}$$

The function Z is continuous on $\mathcal{U} \times \mathcal{F}$, because H is continuous in λ and in $u_H \in [u^{min}, u^{max}(\lambda)]$, $H(u^{max}(\lambda), \lambda) = u^{max}(\lambda)$ holds, and $u^{max}(\lambda)$ is continuous in λ. We can now rewrite the maximization problem as

$$u_H^{MW}(\lambda) = \arg \max_{u_H \in \mathcal{C}(\lambda)} p_L u_H + (p_H - p_L) Z(u_H, \lambda), \tag{16}$$

[20] The dependency of $u^{max}(\lambda)$ and $H(u_H, \lambda)$ on λ has been suppressed in earlier proofs.

and Berge's maximum principle implies that $u_H^{MW}(\lambda)$ is continuous. Then $\chi(\lambda) = \Phi(u_H^{MW}(\lambda)) - (y - p_H d)$ is continuous as well.

Now, consider first the case where $\lambda \to 0$. We will show that, as $\lambda \to 0$, constraint (5) must eventually become binding in $V^{MW}(\lambda)$, i.e. $u_H^{MW}(\lambda) = u^{min}$ for λ small enough. This implies $\chi(\lambda) = 0$. Consider the slope of the objective in (14), evaluated at $u_H = u^{min}$. Using the derivative of H with respect to u_H, obtained from implicitly differentiating (12), the condition that the objective is weakly decreasing already in $u_H = u^{min}$ (which is then the solution to (14) due to strict concavity), can be rearranged to

$$(1-\lambda)(p_H - p_L)\Phi'(u^{min}) \geq \lambda(1-p_L)p_L \left[\Phi'(H(u^{min},\lambda)) - \Phi'\left(\frac{u^{min} - (1-p_H)H(u^{min},\lambda)}{1-p_H}\right)\right]. \tag{17}$$

Fixing $u_H = u^{min}$, the budget constraint (4) can be simplified to

$$(1-p_L)\Phi(H(u^{min},\lambda)) + p_L\Phi\left(\frac{u^{min} - (1-p_H)H(u^{min},\lambda)}{p_H}\right) = (1-p_L)y + p_L(y-d),$$

which implies that $H(u^{min},\lambda)$ is independent of λ and satisfies $H(u^{min},\lambda) > u^{min}$. Hence the LHS of (17) converges to a strictly positive value as $\lambda \to 0$, while the RHS converges to zero. Hence (5) must eventually become binding.

Consider now the case where $\lambda \to 1$. From the same arguments as above it follows that (5) must become slack for sufficiently large value of λ, because (17) will eventually be violated. Observe also that $u_H = u^{max}(\lambda)$ can never be a solution to (14), for any $\lambda \in (0,1)$, as this would imply $u_{L,N} = H(u^{max}(\lambda),\lambda) = u^{max}(\lambda) = u_H$ and $u_{L,D} = u_H$, contradicting that $V^{MW}(\lambda)$ satisfies (2) with slack according to Lemma 1. Hence (14) must have an interior solution for large enough λ. Again using the derivative of H with respect to u_H from implicitly differentiating (12), the necessary and sufficient first order condition for (14) can then be rearranged to (see also Faynzilberg, 2006c, p. 25):

$$\frac{\lambda}{1-\lambda} = \frac{\Phi'(u_H^{MW}(\lambda))}{\Phi'(u_{L,N}^{MW}(\lambda)) - \Phi'(u_{L,D}^{MW}(\lambda))}\frac{p_H - p_L}{p_L(1-p_L)}. \tag{18}$$

Clearly, $u_H^{MW}(\lambda)$ is bounded below by u^{min} and above by $u^{max}(\lambda) = U(R(\lambda)$. Since $u^{max}(\lambda)$ itself is bounded above by $U((1-p_L)y + p_L(y-d))$, it must be that $u_H^{MW}(\lambda) \in [U((1-p_H)y + p_H(y-d)), U((1-p_L)y + p_L(y-d))]$ for all $\lambda \in]0,1[$. Since $\lim_{\lambda\to 1} \lambda/(1-\lambda) = \infty$ while $\Phi'(u_H^{MW}(\lambda)) \in [\Phi'(U((1-p_H)y + p_H(y-d))), \Phi'(U((1-p_L)y + p_L(y-d)))]$ for all $\lambda \in]0,1[$, we must have $\lim_{\lambda\to 1}\left(\Phi'(u_{L,N}^{MW}(\lambda)) - \Phi'(u_{L,D}^{MW}(\lambda))\right) = 0$, since otherwise the first-order condition (18) would be violated for large λ. Assume $\lim_{\lambda\to 1} u_{L,N}^{MW}(\lambda) = +\infty\,(-\infty)$. Then, incentive compatibility (3) requires $\lim_{\lambda\to 1} u_{L,D}^{MW}(\lambda) = -\infty\,(+\infty)$, and the denominator on the RHS of (18) does not go to zero. Therefore, $\lim_{\lambda\to 1}(u_{L,N}^{MW}(\lambda) - u_{L,D}^{MW}(\lambda)) = 0$ has to hold (because Φ' is strictly increasing), i.e. the low risks' contract converges towards full coverage. This implies $\lim_{\lambda\to 1} \Phi(u_{L,N}^{MW}(\lambda)) = \lim_{\lambda\to 1} \Phi(u_{L,D}^{MW}(\lambda)) = \lim_{\lambda\to 1} \Phi(u_H^{MW}(\lambda)) = \lim_{\lambda\to 1}(y - \overline{p}d)$, where $\overline{p} = \lambda p_L + (1-\lambda)p_H$ is the average damage probability. Thus $\lim_{\lambda \to 1} \chi(\lambda) = (p_H - p_L)d$.

Property (ii). Assume that $\Phi'(u)$ is convex. We will first show that both $(1 - p_L)u_{L,N}^{MW}(\lambda) + p_L u_{L,D}^{MW}(\lambda)$ and $u_H^{MW}(\lambda)$ are weakly increasing in λ, and strictly so if (5) is slack. As for $(1 - p_L)u_{L,N}^{MW}(\lambda) + p_L u_{L,D}^{MW}(\lambda)$, this holds even without convexity of Φ'. Fix a value $\lambda_0 \in (0, 1)$ and let $\lambda = \lambda_0 + \delta$ for any $\delta > 0$. In MW(λ), only the resource constraint (4) is affected. Straightforward calculations, using the fact that $V^{MW}(\lambda_0)$ satisfies (4) for λ_0 with equality, reveal that $V^{MW}(\lambda_0)$ is still feasible under λ iff

$$(p_H - p_L)d - \Big[(1 - p_L)\Phi(u_{L,N}^{MW}(\lambda_0)) + p_L\Phi(u_{L,D}^{MW}(\lambda_0)) - \Phi(u_H^{MW}(\lambda_0))\Big] \geq 0, \tag{19}$$

and satisfies the budget constraint with slack iff the inequality is strict. But the binding constraint (4) can be rearranged to

$$\lambda_0 \Big[(1 - p_L)\Phi(u_{L,N}^{MW}(\lambda_0)) + p_L\Phi(u_{L,D}^{MW}(\lambda_0)) - \Phi(u_H^{MW}(\lambda_0)) - (p_H - p_L)d\Big] + \Phi(u_H^{MW}(\lambda_0)) = y - p_H d,$$

which together with the fact that $\Phi(u_H^{MW}(\lambda_0)) \geq (1 - p_H)y + p_H(y - d)$ from (5) implies that (19) is always satisfied, and as a strict inequality whenever $\Phi(u_H^{MW}(\lambda_0)) > (1 - p_H)y + p_H(y - d)$. In this latter case, the optimal value of the objective under λ must be strictly larger than under λ_0, as argued in the proof of Lemma 1. Otherwise, given that the old contracts $V^{MW}(\lambda_0)$ are still feasible under λ, the optimal value of the objective cannot decrease.

Now consider the high risks' utility $u_H^{MW}(\lambda)$. If (5) is binding, it is given by $u_H^{MW}(\lambda) = U((1 - p_H)y + p_H(y - d))$ and is independent of λ. Assume then that (5) is slack, such that $u_H^{MW}(\lambda)$ satisfies the first-order condition (18). To arrive at a contradiction, suppose we increase λ and $u_H^{MW}(\lambda)$ decreases weakly. The binding self-selection constraint (3) can be rearranged to

$$(1 - p_L)u_{L,N}^{MW}(\lambda) + p_L u_{L,D}^{MW}(\lambda) - u_H^{MW}(\lambda) = (p_H - p_L)(u_{L,N}^{MW}(\lambda) - u_{L,D}^{MW}(\lambda)).$$

Given that $(1 - p_L)u_{L,N}^{MW}(\lambda) + p_L u_{L,D}^{MW}(\lambda)$ strictly increases in λ, as shown above, the term $u_{L,N}^{MW}(\lambda) - u_{L,D}^{MW}(\lambda)$ must also be strictly increasing. If Φ' is convex, this implies that

$$\Phi'(u_{L,N}^{MW}(\lambda)) - \Phi'(u_{L,D}^{MW}(\lambda))$$

is increasing in λ, given that $u_{L,N}^{MW}(\lambda)$ and $u_{L,D}^{MW}(\lambda)$ cannot both decrease. Collecting results, we have that, by assumption, $u_H^{MW}(\lambda)$ and hence $\Phi'(u_H^{MW}(\lambda))$ weakly decreases, while $\Phi'(u_{L,N}^{MW}(\lambda)) - \Phi'(u_{L,D}^{MW}(\lambda))$ strictly increases. But this is a contradiction to (18), as it implies that the LHS of (18) strictly increases but the RHS strictly decreases. Hence $u_H^{MW}(\lambda)$ is strictly increasing in λ whenever (5) is slack.

Finally, if (5) is slack and $u_H^{MW}(\lambda)$ is strictly increasing at some level of λ, the same clearly holds for all $\lambda' > \lambda$. Together with the previous result that (5) must be binding in $V^{MW}(\lambda)$ for sufficiently small and slack for sufficiently large values of λ, it follows that there exists a value $\tilde{\lambda} \in]0,1[$ such that for all $\lambda \leq \tilde{\lambda}$, constraint (5) will be binding in $V^{MW}(\lambda)$ and neither $V^{MW}(\lambda)$ nor $\chi(\lambda)$ change in λ, while for all $\lambda > \tilde{\lambda}$, (5) is slack and $u_H^{MW}(\lambda)$, and thus $\chi(\lambda)$, is strictly increasing in λ.

A.3 Proof of Proposition 1

The proof involves constructing an SPE σ with outcome V^*. For convenience, we omit the asterisk for equilibrium strategies, and although $\sigma = (\sigma^1, \sigma^2)$ formally is a profile of mixed strategies, we write e.g. $\sigma_j^2(s^1) = NW$ or $\sigma_j^1 = \varnothing$ to indicate a lottery placing probability 1 on a pure action.

Part (i). To prove part (i) of the proposition, we construct σ as follows. Let firm $j = 1$ offer the contracts from V^* in stage 1, where V^* satisfies the constraints (8) and (9) in the Proposition. Formally, $\sigma_1^1 = \{(u_{H,N}^*, u_{H,D}^*), (u_{L,N}^*, u_{L,D}^*)\}$. Let $\sigma_j^1 = \{(u_H^{RS}, u_H^{RS}), (u_{L,N}^{RS}, u_{L,D}^{RS})\}$ for firms $j = 2, 3$, and $\sigma_j^1 = \varnothing$ for all $j \geq 4$. Now denote the induced history by s^1. As for stage 2, first let $\sigma_1^2(s^1) = NW$ and $\sigma_j^2(s^1) = W$ for $j = 2, 3$ (for firms $j \geq 4$, the withdrawal decision after history s^1 is irrelevant). Clearly, these withdrawal decisions form a Nash equilibrium in subgame $\Gamma(s^1)$: firm 1 earns non-negative profits by the properties of V^*, and neither firm 2 nor 3 can earn profits by remaining active. Furthermore, the outcome of the candidate is the desired V^*. Any potentially profitable deviation has to take place at stage 1.

Strategies must form Nash equilibria in all off-equilibrium path subgames $\Gamma(\tilde{s}^1)$ for $\tilde{s}^1 \neq s^1$. Any such subgame is a finite normal form game, so that a Nash equilibrium exists (possibly in mixed strategies). To examine deviation incentives in stage 1, we only need to consider histories $\tilde{s}^1$ that can be reached by unilateral deviations, i.e. that differ from s^1 in exactly one coordinate i. At any such history, at least one of firms $j = 2, 3$ is a non-deviator. For a given deviation history $\tilde{s}^1$, let $J^{RS} \subseteq \{2, 3\}$ be the (nonempty) set of all firms among 2 and 3 that have not deviated from s^1. We first show there exists a Nash equilibrium in $\Gamma(\tilde{s}^1)$ where all $j \in J^{RS}$ remain active.

Lemma 3. *For any $\tilde{s}_1 \neq s_1$ that can be reached by a unilateral deviation from s_1, there exists a Nash equilibrium in $\Gamma(\tilde{s}_1)$ such that $\sigma_j^2(\tilde{s}_1) = NW$ for all $j \in J^{RS}$.*

Proof. Since V^{RS} generates zero profits no matter which types are attracted, all firms $j \in J^{RS}$ are always indifferent between withdrawing and remaining active, so that $\sigma_j^2(\tilde{s}_1) = NW$ is always a best response for them. We can then consider the reduced stage 2 subgame between firms $j \in \mathscr{J} \setminus J^{RS}$, taking as given that $\sigma_j^2(\tilde{s}_1) = NW$ for $j \in J^{RS}$. The reduced game is a finite normal form game, which has a Nash equilibrium. Together with $\sigma_j^2(\tilde{s}_1) = NW$ for $j \in J^{RS}$, it extends to a Nash equilibrium of $\Gamma(\tilde{s}^1)$. □

Hence after any unilateral deviation history, there exists a stage two Nash equilibrium where the contracts given by $V^{RS} = (u_H^{RS}, u_H^{RS}, u_{L,N}^{RS}, u_{L,D}^{RS})$ remain active. Since V^{RS} solves MW(λ) by assumption, there does not exist a set of incentive compatible contracts that makes strictly positive profits when offered in addition to only V^{RS} (see Miyazaki 1977). In particular, there does not exist a set of contracts that is profitable after all the contracts that have become unprofitable are withdrawn. However, in the considered case in which V^{RS} solves MW(λ), withdrawal does not need to be considered since V^{RS} always generates zero profits.

Now consider a deviation history $\tilde{s}^1$, where deviator i has offered $\{(\tilde{u}_{H,N}, \tilde{u}_{H,D}), (\tilde{u}_{L,N}, \tilde{u}_{L,D})\}$.[21] Consider a Nash equilibrium in $\Gamma(\tilde{s}^1)$ with the property described in Lemma 3. Profitability of the deviation would now require that $i \geq 2$, the deviator remains active, firm $j = 1$ also remains active with positive probability, and, while $\{(\tilde{u}_{H,N}, \tilde{u}_{H,D}), (\tilde{u}_{L,N}, \tilde{u}_{L,D})\}$ and $\{(u^*_{H,N}, u^*_{H,D}), (u^*_{L,N}, u^*_{L,D})\}$ jointly earn non-positive profits in the presence of the Rothschild-Stiglitz contracts, the deviator earns strictly positive (expected) profits. But this implies (expected) losses for $j = 1$, contradicting Nash equilibrium. Hence the deviation cannot be profitable.

Part (ii). To prove part (ii), we distinguish two cases. First, if $V^* = V^{MW}$, the existence of an SPE with outcome V^* for $\delta = 0$ will follow from Proposition 2. Second, for the case that V^* satisfies constraints (8) to (10), we construct the following σ. Firm $j = 1$ again offers V^*, i.e. $\sigma^1_1 = \{(u^*_{H,N}, u^*_{H,D}), (u^*_{L,N}, u^*_{L,D})\}$. For $j = 2, 3$, we let $\sigma^1_j = \{(u^{MW}_H, u^{MW}_H), (u^{MW}_{L,N}, u^{MW}_{L,D})\}$, and $\sigma^1_j = \varnothing$ for all $j \geq 4$. We again denote the induced history by s^1. As for stage 2, let $\sigma^2_1(s^1) = NW$ and $\sigma^2_j(s^1) = W$ for $j = 2, 3$ as before. Clearly, these withdrawal decisions again form a Nash equilibrium in subgame $\Gamma(s^1)$. First, firm 1 earns non-negative profits by the properties of V^*. Second, neither firm 2 nor 3 can earn profits by remaining active, because V^* becomes irrelevant whenever V^{MW} is offered due to (10), and V^{MW} involves zero profits when offered on its own. Furthermore, the outcome of the candidate is the desired V^*. Any potentially profitable deviation therefore again has to take place at stage 1.

As before, any off-equilibrium path subgame $\Gamma(\tilde{s}^1)$ for $\tilde{s}^1 \neq s^1$ is a finite normal form game so that a Nash equilibrium exists, and we again only need to consider histories $\tilde{s}^1$ that can be reached by unilateral deviations. For a given deviation history $\tilde{s}^1$, let $J^{MW} \subseteq \{2, 3\}$ be the (nonempty) set of all firms among 2 and 3 that have not deviated from s^1. Let $\{(\tilde{u}_{H,N}, \tilde{u}_{H,D}), (\tilde{u}_{L,N}, \tilde{u}_{L,D})\}$ denote the set of contracts offered by the deviator i at stage 1. Consider the set of Nash equilibria in subgame $\Gamma(\tilde{s}^1)$. Suppose first that there exists one among them in which the deviator remains active with probability less than 1. By indifference, the deviation then cannot be profitable. We therefore only need to consider further those subgames $\Gamma(\tilde{s}^1)$ in which there only exist Nash equilibria that involve the deviator remaining active with probability 1. Among those, we distinguish three cases:

Case 1. Suppose first that there exists a Nash equilibrium in which at least one $j \in J^{MW}$ remains active with probability 1. Note first that this turns 1's original offer, even if 1 is not the deviator himself and still remains active, irrelevant by (10). Hence, the deviation contracts are offered in addition to the contracts given by V^{MW}, and the latter are not earning losses. This implies that the deviation cannot be profitable, as argued earlier.

[21] This includes the possibility that the deviator i offers a single contract only, in which case $\tilde{u}_{H,N} = \tilde{u}_{L,N}$ and $\tilde{u}_{H,D} = \tilde{u}_{L,D}$, or that he offers more than two contracts, in which case $(\tilde{u}_{k,N}, \tilde{u}_{k,D})$ represents the best contract for type $k = H, L$ among those offered by i.

Case 2. If case 1 does not apply, suppose there exists a Nash equilibrium in which all $j \in J^{MW}$ strictly prefer withdrawal to remaining active, since they would make strictly negative profits if they remained active unilaterally. Again using the properties of V^{MW}, profitability of the deviation would require that $i \geq 2$, the deviator remains active, firm $j = 1$ also remains active with positive probability, and, while $\{(\tilde{u}_{H,N}, \tilde{u}_{H,D}), (\tilde{u}_{L,N}, \tilde{u}_{L,D})\}$ and $\{(u^*_{H,N}, u^*_{H,D}), (u^*_{L,N}, u^*_{L,D})\}$ jointly earn non-positive profits, the deviator earns strictly positive profits. But this implies losses for $j = 1$, contradicting Nash equilibrium. Hence the deviation cannot be profitable.

Case 3. Finally, suppose there only exist Nash equilibria in which all firms $j \in J^{MW}$ withdraw with strictly positive probability, but are indifferent between remaining active and withdrawing (if at least one of them is indifferent, then all of them are indifferent). Starting from any such Nash equilibrium, we can now construct another Nash equilibrium in which they remain active with probability 1. First, let $\sigma_j^2(\tilde{s}^1) = NW$ for all $j \in J^{MW}$, which is an alternative best response for them, given the fixed strategies of the other players. If firm 1 is not the deviator, keep $\sigma_1^2(\tilde{s}^1)$ from the original Nash equilibrium. This remains a best response, since, whenever the contracts from V^{MW} are offered, 1's offer is irrelevant by (10). As for the deviator, suppose first that a best response to the modified strategies is to remain active, as in the original Nash equilibrium. Then we have found a new Nash equilibrium which returns us to case 1 above and leads to a contradiction. Second, suppose the deviator now strictly prefers to become inactive. Then we have again found a new Nash equilibrium: If the deviator becomes inactive, the specified strategies for $j \in J^{MW}$ and $j = 1$ (if not the deviator) are best responses. However, the deviation is unprofitable in this case.

A.4 Proof of Proposition 2

We prove the proposition in two steps. First, we show that if $\delta > 0$, the outcome of any SPE of Γ must be a solution to MW. This establishes $\Omega^*(\delta) \subseteq \{V^{MW}\}$ for all $\delta > 0$, the first part of statement (i). We then show that there exists a critical value $\overline{\delta} > 0$ such that $V^{MW} \in \Omega^*(\delta)$ for all $\delta < \overline{\delta}$, including $\delta = 0$. This establishes statement (ii), and also $\{V^{MW}\} \subseteq \Omega^*(0)$, the second part of statement (i).

Throughout the proof, we adopt the following notation. First, we omit the asterisk indicating equilibrium strategies, for notational simplicity. Second, although $\sigma = (\sigma^1, \sigma^2)$ formally is a profile of mixed strategies, we write e.g. $\sigma_j^2(s^1) = NW$ or $\sigma_j^1 = \varnothing$ to indicate a lottery placing probability 1 on a pure action. Finally, we write $\sigma = (\sigma_j, \sigma_{-j})$, $\sigma^2(s^1) = (\sigma_j^2(s^1), \sigma_{-j}^2(s^1))$ and so on.

Step 1. Fix a value of withdrawal cost $\delta > 0$ and consider an SPE σ with outcome V^*. Observe first that $\sigma_j^2(s^1) = NW\ \forall j \in \mathscr{J}$, where s^1 is the history induced by σ, i.e. the profile of stage 1 offers. Otherwise, $\Pi_j(\sigma) = -\delta < 0$ for some $j \in \mathscr{J}$, and deviating to $\tilde{\sigma}_j^1 = \varnothing$ would be profitable. Observe also that $\Pi_j(\sigma) = 0$ for at least one $j \in \mathscr{J} \setminus \{0\}$. Otherwise, if $\Pi_j(\sigma) > 0$ $\forall j \in \mathscr{J} \setminus \{0\}$, any one of them, say i, could deviate to offering the contracts $(u^*_{H,N} + \epsilon, u^*_{H,D} + \epsilon)$ and $(u^*_{L,N} + \epsilon, u^*_{L,D} + \epsilon)$ in stage 1, for small $\epsilon > 0$, and remain active after the deviation. Since

$\sigma_j^2(s^1) = NW \; \forall j \in \mathscr{J}$, the contracts available in addition to $(u^*_{H,N} + \epsilon, u^*_{H,D} + \epsilon)$ and $(u^*_{L,N} + \epsilon, u^*_{L,D} + \epsilon)$, at the end of stage 2 after the deviation, are at most those available in the SPE,[22] and all agents will choose one of the deviation contracts. Also, the deviation contracts are incentive compatible, so that, for sufficiently small ϵ, the deviator could earn profits arbitrarily close to $\sum_{j \in \mathscr{J}} \Pi_j(\sigma) > \Pi_i(\sigma)$, a contradiction.

We now show that the outcome V^* must satisfy the constraints of MW, and that it must maximize the objective (1).

Constraints (2) and (3). Incentive-compatibility is satisfied by definition of V^*.

Constraint (4). Assume to the contrary that V^* violates (4). Then there must be at least one firm $j \in \mathscr{J} \setminus \{0\}$ with $\sigma_j^2(s^1) = NW$ and $\Pi_j(\sigma) < 0$.[23] $\tilde{\sigma}_j^1 = \varnothing$ would be a profitable deviation, which contradicts that V^* is an SPE outcome.

Constraint (5). Assume to the contrary that V^* violates (5), i.e. $\Phi((1-p_H)u^*_{H,N} + p_H u^*_{H,D}) < y - p_H d$. Let $\tilde{u} = (1-p_H)u^*_{H,N} + p_H u^*_{H,D} + \epsilon$, $\epsilon > 0$, with ϵ sufficiently small to guarantee $\Phi(\tilde{u}) < y - p_H d$. The contract $(\tilde{u}, \tilde{u}) \in \mathscr{C}$ then satisfies $\pi_k(\tilde{u}, \tilde{u}) = y - p_k d - \Phi(\tilde{u}) > 0$, i.e. it earns strictly positive profits if a positive mass of agents (of whatever risk) chooses it. Consider a firm $i \in \mathscr{J}$ for which $\Pi_i(\sigma) = 0$, which exists as shown above, and assume it deviates to $\tilde{\sigma}_i^1 = \{(\tilde{u}, \tilde{u})\}$ and remains active thereafter. Since $\sigma_j^2(s^1) = NW \; \forall j \in \mathscr{J}$, the contracts that are available in addition to $(\tilde{u}, \tilde{u})$ at the end of stage 2 after the deviation are at most those available in the SPE. Hence the high risks will choose $(\tilde{u}, \tilde{u})$ and make the deviation strictly profitable, which contradicts that V^* is an SPE outcome.

Maximization of (1). Assume that V^* satisfies all constraints of MW, but, to the contrary, $V^* \neq V^{MW}$. Then $(1-p_L)u^{MW}_{L,N} + p_L u^{MW}_{L,D} > (1-p_L)u^*_{L,N} + p_L u^*_{L,D}$. For $\epsilon > 0$ small enough, the contract $(u^{MW}_{L,N} - \epsilon, u^{MW}_{L,D} - \epsilon) \in \mathscr{C}$ then still satisfies $(1-p_L)(u^{MW}_{L,N} - \epsilon) + p_L(u^{MW}_{L,D} - \epsilon) > (1-p_L)u^*_{L,N} + p_L u^*_{L,D}$. Suppose a firm $i \in \mathscr{J}$ for which $\Pi_i(\sigma) = 0$ deviates to $\tilde{\sigma}_i^1 = \{(u^{MW}_{L,N} - \epsilon, u^{MW}_{L,D} - \epsilon), (u^{MW}_H - \epsilon, u^{MW}_H - \epsilon)\}$, with ϵ small enough as discussed, and remains active thereafter. The contracts that are additionally available at the end of stage 2 after the deviation are at most those available in the SPE, and thus the low risks will choose $(u^{MW}_{L,N} - \epsilon, u^{MW}_{L,D} - \epsilon)$, given that $(u^*_{L,N}, u^*_{L,D})$ was optimal before. High risks weakly prefer $(u^{MW}_H - \epsilon, u^{MW}_H - \epsilon)$ over $(u^{MW}_{L,N} - \epsilon, u^{MW}_{L,D} - \epsilon)$, since V^{MW} satisfies (3). Therefore, all high risks either choose $(u^{MW}_H - \epsilon, u^{MW}_H - \epsilon)$ or a contract offered by some other firm $j \neq i$. We claim that the deviation is strictly profitable.[24] Even if all high risks choose the contract $(u^{MW}_H - \epsilon, u^{MW}_H - \epsilon)$ in this outcome, the deviating firm i earns strictly positive profits.[25] If the high risks choose some other contract, firm i obtains only the low risks and earns strictly positive profits as well.

[22] If some non-deviating firms randomize in the stage 2 subgame reached after the deviation, this statement holds true for each possible outcome of the randomization.

[23] Clearly, $\Pi_0(\sigma) = 0$ always holds.

[24] Again, if there is randomization after the deviation, the following arguments apply to each outcome that occurs with positive probability.

[25] The contract $(u^{MW}_H - \epsilon, u^{MW}_H - \epsilon)$ might have been offered by non-deviators as well, in which case not all high risks choose firm i, but the deviation is still profitable.

Step 2. We now construct an SPE with outcome V^{MW}, which exists for sufficiently small values of δ, including $\delta = 0$.

In addition to the contracts in V^{MW}, consider the contract (u_H, u_H) that always pays the expected endowment of high risks, so that $u_H = U(y - p_H d)$. Clearly, this contract is identical to (u_H^{MW}, u_H^{MW}) if constraint (5) is binding in V^{MW}, but the latter is strictly preferred by high risks to (u_H, u_H) otherwise. We now construct an SPE σ of Γ in which $\sigma_j^1 = \{(u_H, u_H)\}$ for $j = 1, 2$, $\sigma_j^1 = \{(u_H^{MW}, u_H^{MW}), (u_{L,N}^{MW}, u_{L,D}^{MW})\}$ for $j = 3, 4$, and $\sigma_j^1 = \varnothing\ \forall j \geq 5$. Denote the induced history by s^1, and set $\sigma_j^2(s^1) = NW\ \forall j \in \mathscr{J}$.

Whenever V^{MW} satisfies (5) with slack, all agents will then spread equally among firms $j = 3, 4$, which implies $\Pi_j(\sigma^2(s^1)|s^1) = 0\ \forall j \in \mathscr{J}$. If (5) is satisfied with equality, high risks spread equally among firms $j = 1, ..., 4$, while low risks spread among firms 3 and 4 only. The fact that there is no cross-subsidization in V^{MW} again implies $\Pi_j(\sigma^2(s^1)|s^1) = 0\ \forall j \in \mathscr{J}$. Thus $\sigma_j^2(s^1) = NW$ is actually a best response for every firm in subgame $\Gamma(s^1)$, for any value of $\delta \geq 0$, and the outcome of the SPE candidate σ is V^{MW}. Any potentially profitable deviation has to take place at stage 1.

Fix a value of $\delta \geq 0$. The companies' strategies must form Nash equilibria in all off-equilibrium path subgames $\Gamma(\tilde{s}^1)$, $\tilde{s}^1 \in S^1$, $\tilde{s}^1 \neq s^1$. The fact that each subgame $\Gamma(\tilde{s}^1)$ is a finite normal form game implies that a Nash equilibrium does exist in each of them, possibly in mixed strategies. For each $\tilde{s}^1 \in S^1, \tilde{s}^1 \neq s^1$, let $\sigma^2(\tilde{s}^1)$ be such an equilibrium.[26] Now consider those stage 2 subgames $\Gamma(\tilde{s}^1)$ that can be reached after a profitable unilateral deviation, i.e. for which there exists a firm $i \in \mathscr{J}$ such that s^1 and $\tilde{s}^1$ differ in the ith coordinate only, and where $\Pi_i(\sigma^2(\tilde{s}^1)|\tilde{s}^1) > 0$. Let $\tilde{S}^1$ be the set of all histories that correspond to such subgames (suppressing the dependency on the chosen stage 2 equilibria $\sigma^2(\tilde{s}^1)$).

Lemma 4. *For each $\tilde{s}^1 \in \tilde{S}^1$, there exists a pure-strategy Nash equilibrium $\tilde{\sigma}^2(\tilde{s}^1)$ in $\Gamma(\tilde{s}^1)$.*
If $\Pi_i(\tilde{\sigma}^2(\tilde{s}^1)|\tilde{s}^1) > 0$, i.e. the deviation is still profitable under $\tilde{\sigma}^2(\tilde{s}^1)$, then $\tilde{\sigma}^2(\tilde{s}^1)$ satisfies that
(i) each non-deviator $j \neq i, j \in \{1, 2\}$ plays $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$, and
(ii) each non-deviator $j \neq i, j \in \{3, 4\}$ plays $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$ when indifferent, i.e. if $\Pi_j(NW, \tilde{\sigma}_{-j}^2(\tilde{s}^1)|\tilde{s}^1) = \Pi_j(W, \tilde{\sigma}_{-j}^2(\tilde{s}^1)|\tilde{s}^1)$.

Proof. We prove the lemma by constructing the equilibrium $\tilde{\sigma}^2(\tilde{s}^1)$ from $\sigma^2(\tilde{s}^1)$.

Consider first the case where $\delta > 0$. In the given equilibrium $\sigma^2(\tilde{s}^1)$, both the deviator i and all non-deviators $j \neq i, j \in \{1, 2\}$ remain active (with probability one). For the deviator, this is because $\Pi_i(\sigma^2(\tilde{s}^1)|\tilde{s}^1) > 0$ by assumption. Given that the contract (u_H, u_H) always earns non-negative profits, for non-deviators among $j \in \{1, 2\}$ remaining active even strictly dominates withdrawal. The same holds for firms $j \neq i, j \in \{3, 4\}$ if (5) is satisfied with equality in V^{MW}, because incentive compatibility and lack of cross-subsidization in V^{MW} then always implies zero profits when remaining active. Hence in that case $\sigma^2(\tilde{s}^1)$ is already in pure strategies,

[26] If there is more than one equilibrium in a subgame $\Gamma(\tilde{s}^1)$, let $\sigma^2(\tilde{s}^1)$ be an arbitrary one of them.

satisfies property (i), and (ii) is empty, so we have $\tilde{\sigma}^2(\tilde{s}^1) = \sigma^2(\tilde{s}^1)$. If (5) is slack in V^{MW}, but $\Pi_j(NW, \sigma^2_{-j}(\tilde{s}^1)|\tilde{s}^1) \neq \Pi_j(W, \sigma^2_{-j}(\tilde{s}^1)|\tilde{s}^1)$ for each non-deviator $j \neq i$, $j \in \{3,4\}$, property (ii) is also empty and $\sigma^2(\tilde{s}^1)$ is in pure strategies, such that we also have $\tilde{\sigma}^2(\tilde{s}^1) = \sigma^2(\tilde{s}^1)$.

Consider then the case that (5) is slack in V^{MW} and $\Pi_j(NW, \sigma^2_{-j}(\tilde{s}^1)|\tilde{s}^1) = \Pi_j(W, \sigma^2_{-j}(\tilde{s}^1)|\tilde{s}^1)$ for at least one $j \neq i$, $j \in \{3,4\}$. Assume first that $i \notin \{3,4\}$ in $\tilde{s}^1$. Let β_1 be the (non-random) payoff that one of firms $j \in \{3,4\}$ would obtain if it remained active while the other did not remain active, and all other firms' strategies were as in $\sigma^2(\tilde{s}^1)$, hence pure. Let β_2 be the analogous payoff if both $j \in \{3,4\}$ remained active, again keeping all other strategies from $\sigma^2(\tilde{s}^1)$. Indifference of (at least) one firm $j \in \{3,4\}$ in $\sigma^2(\tilde{s}^1)$ implies that $-\delta = q\beta_1 + (1-q)\beta_2$, where $q \in [0,1]$ is the probability in $\sigma^2(\tilde{s}^1)$ that the other one withdraws. It must therefore be the case that either $\beta_1 < 0$ or $\beta_2 < 0$ or both. This happens if and only if the active firm(s) among 3 and 4 obtain high risks in (u_H^{MW}, u_H^{MW}), which requires subsidization, but not enough low risks in $(u_{L,N}^{MW}, u_{L,D}^{MW})$ to break even. Also, since (u_H^{MW}, u_H^{MW}) is strictly preferred to (u_H, u_H) by high risks in the present case, firms 1 and 2 do not obtain agents whenever at least one of firms 3 and 4 is active. Hence losses for active firms $j \in \{3,4\}$ occur only if the deviator has offered a contract which is chosen by (some) low risks, in the presence of $(u_{L,N}^{MW}, u_{L,D}^{MW})$, while (u_H^{MW}, u_H^{MW}) is still the best contract for high risks.

We can now distinguish two cases: first, the deviator i's best contract for low risks in $\tilde{s}^1$ could be $(u_{L,N}^{MW}, u_{L,D}^{MW})$. In this case, the deviator did not also offer (u_H^{MW}, u_H^{MW}) in $\tilde{s}^1$, because this would imply $\Pi_i(\sigma^2(\tilde{s}^1)|\tilde{s}^1) = 0$ (irrespective of $\sigma_j^2(\tilde{s}^1)$, $j = 3,4$). Hence whenever one or both firms $j \in \{3,4\}$ are active, all high risks move only to them,[27] while all low risks spread equally between them and the deviator. The number of low risks that active firms $j \in \{3,4\}$ obtain is not large enough to break even, irrespective of whether one or both of them are active, which implies $\beta_1 < 0$ and $\beta_2 < 0$. It is also straightforward to show that $\beta_1 < \beta_2$, i.e. the individual losses are smaller if both $j = 3,4$ are active and share the losses. The second possible case is that the deviator i has offered a contract in $\tilde{s}^1$ which is strictly preferred to $(u_{L,N}^{MW}, u_{L,D}^{MW})$ by low risks.[28] The active firm(s) $j \in \{3,4\}$ then obtain only the high risks and earn strictly negative profits, irrespective of whether one or both of them are active. The losses are again smaller if they are shared, also implying $\beta_1 < \beta_2 < 0$.

With these results, we can construct $\tilde{\sigma}^2(\tilde{s}^1)$ from $\sigma^2(\tilde{s}^1)$, under the assumption that $\Pi_j(NW, \sigma^2_{-j}(\tilde{s}^1)|\, \tilde{s}^1) = \Pi_j(W, \sigma^2_{-j}(\tilde{s}^1)|\tilde{s}^1)$ for at least one $j \neq i$, $j \in \{3,4\}$. If $i \in \{3,4\}$, set $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$, and $\tilde{\sigma}_k^2(\tilde{s}^1) = \sigma_k^2(\tilde{s}^1)\ \forall k \in \mathscr{J}, k \neq j$. This simply amounts to choosing an alternative best response for the indifferent player, keeping the strategies of all others. If $i \notin \{3,4\}$, set $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$ for both $j = 3,4$, and again $\tilde{\sigma}_k^2(\tilde{s}^1) = \sigma_k^2(\tilde{s}^1)\ \forall k \in \mathscr{J}, k \notin \{3,4\}$. The fact that $\beta_1 < \beta_2 < 0$

[27] Even if the deviator has offered an output-dependent incentive contract that leaves high risks indifferent to (u_H^{MW}, u_H^{MW}), no bad type will choose it due to our tie-breaking assumptions.

[28] Any contract which leaves the low risks indifferent to $(u_{L,N}^{MW}, u_{L,D}^{MW})$ but is still chosen in the presence of $(u_{L,N}^{MW}, u_{L,D}^{MW})$, must be less high-powered and would violate incentive compatibility, given that (u_H^{MW}, u_H^{MW}) is still the best contract for high risks by assumption.

always holds, as shown above, together with $-\delta = q\beta_1 + (1-q)\beta_2$ for a given $q \in [0,1]$ implies $\beta_2 \geq -\delta$. The individual profits of firms $j = 3,4$ when jointly remaining active (β_2), still given all other players' strategies from $\sigma^2(\tilde{s}^1)$, are weakly larger than $-\delta$, making it indeed a best reply to remain active. If $\tilde{\sigma}_i^2(\tilde{s}^1) = NW$ is now still a best response for the deviator, we have arrived at the desired equilibrium, because $\tilde{\sigma}^2(\tilde{s}^1)$ is a pure strategy Nash equilibrium in which all firms $j \neq i, j \in \{1,...,4\}$ remain active. If i's unique best response is now withdrawal, set $\tilde{\sigma}_i^2(\tilde{s}^1) = W$ to arrive at the final $\tilde{\sigma}^2(\tilde{s}^1)$. It is a Nash equilibrium because $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$, as constructed above, is the unique best response for firms $j \neq i, j \in \{1,...,4\}$ if the deviator withdraws. It is in pure strategies by construction, and properties (i) and (ii) are empty due to $\Pi_i(\tilde{\sigma}^2(\tilde{s}^1)|\tilde{s}^1) = -\delta < 0$.

Assume now that $\delta = 0$. Construct $\tilde{\sigma}^2(\tilde{s}^1)$ from $\sigma^2(\tilde{s}^1)$ by first assuming that all $j \neq i, j \in \{1,2\}$ play $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$, which is always a best response for them, and initially keep all other players' strategies as in $\sigma^2(\tilde{s}^1)$. Even if this constitutes a change of strategy from $\sigma^2(\tilde{s}^1)$, the optimal behavior of non-deviators $j \neq i$, $j \in \{3,4\}$ is clearly unaffected. If $\Pi_j(NW, \sigma_{-j}^2(\tilde{s}^1)|\tilde{s}^1) \neq \Pi_j(W, \sigma_{-j}^2(\tilde{s}^1)|\tilde{s}^1)$ for all $j \neq i$, $j \in \{3,4\}$, indeed keep $\tilde{\sigma}_j^2(\tilde{s}^1) = \sigma_j^2(\tilde{s}^1)$ for them. Otherwise, if $\Pi_j(NW, \sigma_{-j}^2(\tilde{s}^1)|\tilde{s}^1) = \Pi_j(W, \sigma_{-j}^2(\tilde{s}^1)|\tilde{s}^1)$ for at least one $j \neq i$, $j \in \{3,4\}$, set $\tilde{\sigma}_j^2(\tilde{s}^1) = NW$ $\forall j \neq i$, $j \in \{3,4\}$. A similar argument as for the case $\delta > 0$ implies that they then give best responses against the profile constructed so far. If $\tilde{\sigma}_i^2(\tilde{s}^1) = NW$ is still a best response of the deviator, we have arrived at the desired equilibrium. Clearly, $\tilde{\sigma}^2(\tilde{s}^1)$ is in pure strategies, it has firms $j \neq i, j \in \{1,2\}$ remaining active, and for any firm $j \neq i, j \in \{3,4\}$ we can have $\tilde{\sigma}^2(\tilde{s}^1) = W$ only if $\Pi_j(NW, \tilde{\sigma}_{-j}^2(\tilde{s}^1)|\tilde{s}^1) < \Pi_j(W, \tilde{\sigma}_{-j}^2(\tilde{s}^1)|\tilde{s}^1)$, i.e. if there is no indifference. If, on the other hand, withdrawal is now the unique best-response of the deviator, setting $\tilde{\sigma}_i^2(\tilde{s}^1) = W$ yields the desired equilibrium, because if the deviator withdraws and $\delta = 0$, all firms $j \neq i, j \in \{1,...,4\}$ are indifferent between withdrawing and remaining active, making the above constructed pure strategies best responses. Furthermore, the fact that $\Pi_i(\tilde{\sigma}^2(\tilde{s}^1)|\tilde{s}^1) = -\delta = 0$ implies that (i) and (ii) are empty. □

For each $\tilde{s}^1 \in \tilde{S}^1$, replace the original Nash equilibrium $\sigma^2(\tilde{s}^1)$ with the pure-strategy equilibrium $\tilde{\sigma}^2(\tilde{s}^1)$.[29] In some of the corresponding subgames, using $\tilde{\sigma}^2(\tilde{s}^1)$ might already make the deviation unprofitable, i.e. $\Pi_i(\tilde{\sigma}^2(\tilde{s}^1)|\tilde{s}^1) \leq 0$. In fact, we show in the following that this is true in all $\Gamma(\tilde{s}^1)$, $\tilde{s}^1 \in \tilde{S}^1$, if δ is sufficiently small. To prove this claim, we assume to the contrary that there are still profitable deviations. The stage 2 equilibria reached after these deviations do then satisfy the properties (i) and (ii) of Lemma 4. To save on notation, relabel the newly constructed stage 2 equilibria back to $\sigma^2(\tilde{s}^1)$, for all $\tilde{s}^1 \in S^1$, and, as before, let $\tilde{S}^1$ be the set of histories that still correspond to profitable unilateral deviations from s^1 by some firm $i \in \mathscr{J}$. For each $\tilde{s}^1 \in \tilde{S}^1$, denote by $\tilde{V}(\tilde{s}^1)$ the corresponding outcome in subgame $\Gamma(\tilde{s}^1)$, i.e. the quadruple representing the two risk types' choices among the available contracts at the end of stage 2. $\tilde{V}(\tilde{s}^1)$ is well-defined because $\sigma^2(\tilde{s}^1)$ is in pure strategies.

[29]If there are several equilibria that all satisfy the properties in Lemma 4 in a subgame $\Gamma(\tilde{s}^1)$ for $\tilde{s}^1 \in \tilde{S}^1$, let $\tilde{\sigma}^2(\tilde{s}^1)$ be an arbitrary one of them.

Lemma 5. *There exists a value $\overline{\delta} > 0$ such that, if $0 \leq \delta < \overline{\delta}$, all outcomes $\tilde{V}(\tilde{s}^1), \tilde{s}^1 \in \tilde{S}^1$, satisfy the constraints of MW.*

Proof. Consider any $\tilde{s}^1 \in \tilde{S}^1$. By definition of $\tilde{V}(\tilde{s}^1)$ as being the outcome in $\Gamma(\tilde{s}^1)$ under $\sigma^2(\tilde{s}^1)$, it satisfies constraints (2) and (3). (5) must also be satisfied, because the offer (u_H, u_H) remains active by construction of $\sigma^2(\tilde{s}^1)$.

Concerning (4), assume to the contrary that for some $\tilde{s}^1 \in \tilde{S}^1$, $\tilde{V}(\tilde{s}^1)$ violates (4), and let $\widehat{S}^1 \subseteq \tilde{S}^1$ be the set of all such histories. As argued before, this implies losses for at least one active firm in $\Gamma(\tilde{s}^1)$. Then, for each $\tilde{s}^1 \in \widehat{S}^1$, let $\pi(\tilde{s}^1)$ be the (negative) profits of the active firm with the largest losses in $\Gamma(\tilde{s}^1)$. We are going to show that there exists a value $\overline{\delta} > 0$ such that $\pi(\tilde{s}^1) \leq -\overline{\delta}$ for all $\tilde{s}^1 \in \widehat{S}^1$, i.e. these losses are strictly bounded away from zero across all the histories $\tilde{s}^1 \in \widehat{S}^1$.

Consider any $\tilde{s}^1 \in \widehat{S}^1$. By assumption, $\Pi_i(\sigma^2(\tilde{s}^1)|\tilde{s}^1) > 0$, and the non-deviators $j \neq i, j \in \{1,2\}$ choose $\sigma_j^2(\tilde{s}^1) = NW$ and earn $\Pi_j(\sigma^2(\tilde{s}^1)|\tilde{s}^1) = 0$. Thus it must hold that V^{MW} satisfies (5) with slack and for at least one $j \neq i, j \in \{3,4\}$, $\sigma_j^2(\tilde{s}^1) = NW$ and $\Pi_j(\sigma^2(\tilde{s}^1)|\tilde{s}^1) < 0$ must hold. As shown in the proof of Lemma 4, there are two cases in which this can happen. First, the deviator i's best contract for low risks in $\tilde{s}^1$ could be $(u_{L,N}^{MW}, u_{L,D}^{MW})$ and he does not offer a contract that is chosen by high risks in the presence of (u_H^{MW}, u_H^{MW}). Denote by $\widehat{S}_1^1 \subset \widehat{S}^1$ the set of deviation histories with this property. Second, the deviator's best contract for low risks could be strictly preferred to $(u_{L,N}^{MW}, u_{L,D}^{MW})$ by low risks. Let $\widehat{S}_2^1 \subset \widehat{S}^1$ be the set of histories in which this is the case. Hence $\widehat{S}_1^1$ and $\widehat{S}_2^1$ form a partition of $\widehat{S}^1$.

Consider first a history $\tilde{s}^1 \in \widehat{S}_1^1$. As we have shown in the proof of Lemma 4, the profits of an active non-deviator $j \neq i$, $j \in \{3,4\}$ are then either β_1 or β_2, depending on whether one or both of them are active non-deviators, with $\beta_1 < \beta_2 < 0$. Hence we know that $\pi(\tilde{s}^1) \leq \max\{\beta_1, \beta_2\} = \beta_2 < 0$ for all $\tilde{s}^1 \in \widehat{S}_1^1$. Consider next a history $\tilde{s}^1 \in \widehat{S}_2^1$ after which active non-deviators $j \neq i$, $j \in \{3,4\}$ obtain only high risks. They earn $\pi_H(u_H^{MW}, u_H^{MW}) < 0$ with each unit of high risks agents that they obtain. Given that all high risks spread equally among at most three (and thus finitely many) firms, the losses $\pi(\tilde{s}^1)$ are strictly bounded away from zero across all $\tilde{s}^1 \in \widehat{S}_2^1$, i.e. there exists a value $\beta_3 < 0$ such that $\pi(\tilde{s}^1) \leq \beta_3$ for all $\tilde{s}^1 \in \widehat{S}_2^1$.

Putting the previous results together, we obtain that $\pi(\tilde{s}^1) \leq -\overline{\delta} := \max\{\beta_2, \beta_3\} < 0$ for all $\tilde{s}^1 \in \widehat{S}^1$, i.e. whenever the outcome after a profitable deviation violates (4), a firm earns losses larger or equal to $\overline{\delta}$ in the corresponding stage 2 Nash equilibrium. But this is a contradiction if $0 \leq \delta < \overline{\delta}$, because the firm would strictly prefer to withdraw, which implies our claim. □

Hence if withdrawal costs are sufficiently small, the outcome after any profitable deviation must satisfy the constraints of MW. We next show that the outcome cannot be a solution to MW.

Lemma 6. *If $0 \leq \delta < \overline{\delta}$, it holds that $\tilde{V}(\tilde{s}^1) \neq V^{MW}$ for all $\tilde{s}^1 \in \tilde{S}^1$.*

Proof. Assume to the contrary $\tilde{V}(\tilde{s}^1) = V^{MW}$ for some $\tilde{s}^1 \in \tilde{S}^1$. If $\Pi_i(\sigma^2(\tilde{s}^1)|\tilde{s}^1) > 0$, it must be true that V^{MW} satisfies (5) with slack, the deviator i has offered $(u_{L,N}^{MW}, u_{L,D}^{MW})$ but no contract chosen by high risks in the presence of (u_H^{MW}, u_H^{MW}), and $\sigma_j^2(\tilde{s}^1) = NW$ for at least one $j \neq i$, $j \in \{3,4\}$. But then $\Pi_j(\sigma^2(\tilde{s}^1)|\tilde{s}^1) \leq -\overline{\delta}$, as shown in the proof of Lemma 5, which cannot occur in equilibrium if $\delta < \overline{\delta}$. □

We thus know that, if $0 \leq \delta < \overline{\delta}$, after any profitable deviation history $\tilde{s}^1 \in \tilde{S}^1$ the outcome $\tilde{V}(\tilde{s}^1)$ in $\Gamma(\tilde{s}^1)$ under $\sigma^2(\tilde{s}^1)$ must satisfy the constraints of MW but is not a solution to MW. Hence low risks are strictly worse off in $\tilde{V}(\tilde{s}^1)$ than in V^{MW}, which requires that $\sigma_j^2(\tilde{s}^1) = W \; \forall j \neq i$, $j \in \{3,4\}$. But if some firm $j \neq i$, $j \in \{3,4\}$ remained active instead, it would earn non-negative profits $\Pi_j(NW, \sigma_{-j}^2(\tilde{s}^1)|\tilde{s}^1) \geq 0$. First, it would always obtain the low risks. Then, even if it obtained all high risks (in contract (u_H^{MW}, u_H^{MW})), this ensures $\Pi_j(NW, \sigma_{-j}^2(\tilde{s}^1)|\tilde{s}^1) \geq 0$. Hence remaining active is a best response (even unique if $\delta > 0$), contradicting that $\sigma_j^2(\tilde{s}^1) = W$, by construction of $\sigma^2(\tilde{s}^1)$. This final contradiction shows that there cannot be profitable deviations if $0 \leq \delta < \overline{\delta}$.