Comparing Equilibria

By Paul Milgrom and John Roberts*

We develop an ordinal approach to comparing the equilibria of economic models. Its main advantages over the traditional approach based on signing derivatives are that (i) it utilizes only a subset of the assumptions, resulting in a simpler theory that facilitates focusing attention on the economics rather than the mathematics, (ii) it applies to discrete changes, even when there are multiple equilibria and when some equilibria do not vary smoothly with the parameters, and (iii) it incorporates a formal theory of the robustness of conclusions to assumptions, which helps modelers distinguish which assumptions are "critical" to their comparative-statics conclusions. (*JEL* C60, C72)

Predictions about endogenous variables in formal economic models are frequently expressed in the form of the solution to an equation such as $f(x;t)=0$ or a fixed-point equation $x=g(x;t)$, where t represents the parameters of the model and the function f or g expresses the underlying economic relations. Analysis of the model then often consists largely of studying the solutions $x^*(t)$ to determine how they are affected by exogenous changes in the parameter t. In particular, the problem of *monotone comparative statics*—the problem of determining whether x^* is increasing in t—is a staple of economic analysis. The leading methods of comparative-statics analysis, which are based on applying the implicit-function theorem to these equations, have been a standard element of the economist's tool kit for more than half a century.

There are at least three important ways in which the traditional approach to comparative-statics analysis has proved unsatisfactory in practice. The most obvious of these is that it requires such restrictive assumptions. Models of the firm, of oligopoly, of economic growth, and of international trade are among the many that now routinely include nonconvexities as a fundamental part of the economic explanations, and these nonconvexities typically destroy the continuity and smoothness conditions that are essential for the traditional approach. Second is the problem of multiple equilibrium, which can be a source of trouble even when the equilibrium for each of the relevant parameter values is unique. Figure 1 illustrates the problem. In this figure, there is a unique solution to the equation $f(x)+t=0$ for each of the two relevant parameter values t' and t''. A local analysis of the standard type can be done at either of the values t' or t'', but to analyze the effect of a discrete change from the one to the other necessitates finding a smooth curve of solutions $x^*(t)$ connecting the two. This is impossible in the present case, and so there is no way to apply the implicit-function theorem to compare the solutions. The fact that one can *almost everywhere* analyze the effects of minute parameter changes using the implicit-function theorem in such models is of little intrinsic interest, because the theory predicts similarly minute changes in endogenous variables, assuring that the effects will be hard to detect in data and of little consequence for welfare. The ability to sign derivatives locally is valuable only when it can be done *everywhere* in

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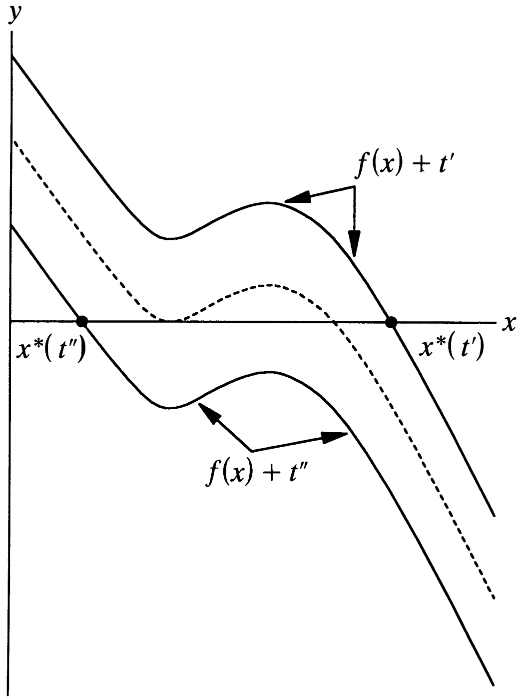

FIGURE 1. THERE IS NO CONTINUOUS PATH OF SOLUTIONS TO THE EQUATION $f(x)+t=0$ AS t VARIES FROM t'' TO t'

the relevant interval. The problems are still worse when the equations admit multiple equilibrium solutions for the relevant parameter values. Then, the local comparisons of particular equilibrium points tell little about the overall changes in the equilibrium set.

The third inadequacy of the standard method of analysis is one of omission: it offers little help in judging the relative importance of the various assumptions of a model. The role of assumptions in models is a controversial matter about which leading economists have differed widely. At one end of the spectrum, Milton Friedman (1953) dismissed the role of assumptions entirely, arguing that one should test a theory using only "the class of phenomena the hypothesis is designed to explain." He argued forcefully that "[t]ruly important and significant hypotheses will be found to have 'assumptions' that are wildly inaccurate descriptive representations of reality, and in general, the more significant the theory, the more unrealistic the assumptions" (pp. 13–14). In a sharp rebuttal, Herbert Simon (1963) advocated replacing Friedman's "principle of unreality" with a "principle of continuity of approximation" according to which "if the conditions of the real world approximate sufficiently well the assumptions of an ideal type, the derivations from these assumptions will be approximately correct.... Unreality of premises is not a virtue in scientific theory; it is a necessary evil—a concession to the finite computing capacity of the scientist that is made tolerable by the principle of continuity of approximation" (pp. 230–31).

Our approach to equilibrium comparisons rejects both views and is based instead on the premise that the complexities of the economic and social world make it infeasible to construct even approximately accurate models of any but the simplest situations. Contrary to Simon's apparent position, this does not imply that modeling is futile. The purpose of most economic modeling is to demonstrate that particular qualitative characteristics of an environment imply qualitative restrictions on the behavior of endogenous economic variables. On the other hand, in opposition to Friedman, we argue that some modeling assumptions are important. Intuitively, most models contain both *critical* assumptions and *simplifying* assumptions. We think of the latter as being made to facilitate characterizations and computations, while the former describe the essential economic mechanisms that determine the qualitative properties of the model. For a model to be useful in understanding the qualitative features of the economic world, the empirical accuracy of simplifying assumptions is unimportant. On the other hand, the critical assumptions must reflect actual features of the economic environments in which the model might be applied, or else the model fails to distinguish the environments in which the qualitative properties hold. The traditional approach provides no formal means to distinguish critical from simplifying assumptions, although economists often try to draw this distinction anyway.

The theory we present in this paper reformulates the problem of comparing equilibria to rectify these three failings. First, our analysis eschews the usual assumptions that are made to employ the implicit-function theorem and yet are both economically unwarranted in many situations and largely orthogonal to the issue of whether the solution is monotone in the parameters. Instead, we focus on the critical assumptions that actually determine the qualitative comparative-statics conclusions.

Second, our analysis focuses primarily on the global structure of the equilibrium set. As a consequence, the existence of multiple equilibria for parameter values that are intermediate between the relevant values poses no problem for our methods, although we have seen that such multiplicity precludes successful analyses based on the implicit-function theorem. When the relevant parameter values themselves admit the possibility of multiple equilibria, our analysis offers comparisons of the extreme equilibria, showing how the bounds on behavior predicted by the theory change with changing parameters. In fact, we will formulate all our results in terms of the highest and lowest equilibria. Obviously, if the equilibrium is unique then it is both the highest and the lowest equilibrium.

Third, our analysis introduces formally the idea of critical assumptions and the related ideas of robustness and context. A *context* is simply a class of models, representing what the modeler knows about the economic environment, in the same sense that an information set represents the knowledge of a player in a game. An assumption is *critical* for a particular conclusion in a particular context if its failure implies that the conclusion fails in at least one of the models in the context. In principle, this definition allows that the identification of critical assumptions could be highly sensitive to the context. The power of the methods we develop derives in part from the observation that, in fact, one can identify a priori conditions that are both critical and sufficient in a wide array of different contexts. Finally, a *robust* conclusion is one that holds in a wide context.

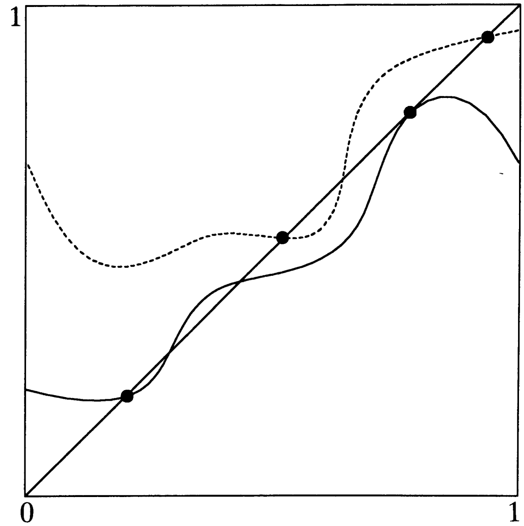

FIGURE 2. THE HIGHEST AND LOWEST FIXED POINTS OF THE HIGHER FUNCTION ARE GREATER THAN THOSE OF THE LOWER FUNCTION

To illustrate these ideas, consider the extreme solutions of the fixed-point problem illustrated in Figure 2. For concreteness, suppose the situation represented in this figure is one in which a firm chooses a level of investment in response to changing technology. Suppose, however, that the model omits some effect that is actually present, such as a special tax treatment of investments in new technologies or the limited availability of technicians skilled in the new technology or even the CEO's desire to be seen as the industry leader in adopting the most promising new technologies. Are the comparative-statics conclusions one might draw from the model robust to the inclusion of such complications? What assumptions are not merely sufficient for a comparative-statics result, but also needed if the result is to continue to be true in the broader context in which these complications might arise?

Let the base model's prediction be expressed as a fixed point of some continuous function $f(\cdot\,;t)$: $[0,1] \rightarrow [0,1]$, parameterized by t. If f is increasing in t, then its lowest and highest fixed points $x_L(t)$ and $x_H(t)$ are increasing in t as well, as Figure 2 illustrates (and as is asserted generally by Corollary 1 below). Suppose, however, that the actual reality is better represented by the two-equation model: $y = f(x,t)$ and $x = g(y,x)$, where g: $[0,1]^2 \to [0,1]$ is increasing in its first argument and continuous in both arguments. The new model includes the original determinants of the equilibrium value of x through its first equation, but there may be additional determinants as well. The fact that g is increasing in its first argument means that (substituting for y) it preserves any monotonicity with respect to t contained in the original model, although possibly in amplified or attenuated form and possibly with the addition of new effects represented by the second argument of the function g. There is no assumption that the omitted effects are small relative to the effects that are included. The fact that t enters the new model only through f means that the new model, though expanded, does not introduce any new direct effects from changes in the parameter.

Observe now that if f is increasing in t, then for any g in the class described, the lowest and highest fixed points in the more realistic model are still increasing functions of t, because the composite function $h(x,t) = g(f(x,t),x)$ is increasing in t. We conclude that the omission of these extra effects is not critical in this context and that the comparative equilibrium analysis is, to that extent, robust when f is increasing in t. Of course, our use of the g function to widen the context is merely illustrative: the main mathematical point is that many of the structural details of the specification are quite irrelevant to the qualitative comparative-statics conclusion. Provided only that f is continuous (and even that condition, as we show, can be weakened), the assumptions of the model that imply that f is increasing in t are the only ones that could be critical for the comparative equilibrium conclusion.

Is the assumption that f is increasing in t a *critical* sufficient condition? It might seem that it could not be, because there certainly do exist functions f for which the lowest and highest fixed points are increasing in the parameter t even though f is not increasing in t. Indeed, if the context for the problem consists of only one such function f, then the assumption that f is increasing in t is not critical. The new approach is most powerful in broader contexts in which one cannot simply compute the equilibrium to make comparisons, but must rely instead on qualitative reasoning.

Generally, we represent the context by a set of functions $\mathcal{F}$. We may think of $\mathcal{F}$ as representing the models that the theorist thinks plausible for the situation at hand. Actual economic models are only very rough approximations to reality, and economic modelers have only general information about the context $\mathcal{F}$, a condition that we formalize in Section I. In particular, our formalization allows that the modeler may know that f is a polynomial of degree at most n; or that it is convex or concave or S-shaped or a contraction mapping; or that it is monotone increasing or decreasing in x or t; or any of a wide array of similar conditions; or any consistent combination of such conditions. We then ask: given only such general contextual knowledge, must the modeler also know that f is increasing in t in order to conclude that the lowest fixed point $x_L(t)$ or the highest fixed point $x_H(t)$ is increasing in t? Our answer is affirmative. In any context involving only general knowledge in the sense described, the assumption that f is increasing in t is a critical sufficient condition. Without this knowledge, the modeler cannot be certain that either $x_L(t)$ or $x_H(t)$ is increasing.

A final feature of the approach we advance here is that, in comparison with the usual methods, it is remarkably simple. The mathematics are not especially sophisticated, the logic is transparent, and verifying whether the conditions of the theorems hold in applications is typically almost trivial. Our core arguments, which rely on critical assumptions only and omit inessential details, are little more than rigorous versions of the intuitive and graphical arguments on which economists have long relied.

The papers most closely related to this one are those that develop general ordinal conditions for comparing solutions in various kinds of models, because critical conditions for monotone comparative statics are always ordinal. Among the most important contributions to date are the following ones: Milgrom and Christina Shannon (1992a) and Milgrom (1993), building on earlier work by Donald Topkis (1978), identify the critical assumptions for some classes of optimization problems; Shannon (1992), extending a theorem of Milgrom and Roberts (1990b), gives ordinal sufficient conditions for comparing Nash equilibria in a class of noncooperative games; Milgrom and Shannon (1992b), building on Tatsuro Ichiishi (1990), give ordinal conditions for comparing the cores of a family of cooperative games. In addition, J. Miguel Villas-Boas (1992) has independently addressed the issue of comparing fixed points.

Four more sections of the paper follow. In Section I, we develop the general theory. After connecting the theory to the traditional differentiable approach in Section II, we apply it to the LeChatelier principle in Section III and conclude in Section IV.

I. Global Comparisons

Because qualitative comparative-statics conclusions are unaffected by monotone transformations (changes of variables that do not alter the ordering of the parameters or the endogenous variables) a natural place to begin a general theory of comparative equilibrium analysis is with conditions that are similarly invariant. The first result, Lemma 1, is of this general kind. (In reading the lemma, recall that, by definition, $\inf(\varnothing) = +\infty$ and $\sup(\varnothing) = -\infty$.)

LEMMA 1: *Let* $X \subset \mathbb{R}$ *and let* $f, g: X \to \mathbb{R}$. *Suppose that for all* $x \in X$, $f(x) \geq g(x)$. *Then* $\inf\{x \mid f(x) \leq 0\} \geq \inf\{x \mid g(x) \leq 0\}$ *and* $\sup\{x \mid f(x) \geq 0\} \geq \sup\{x \mid g(x) \geq 0\}$.

This simple result and its variations lie at the heart of all robust conclusions about comparisons of equilibria. The key to its application is to identify the conditions under which the expressions in the lemma correspond to the equilibrium of some eco-

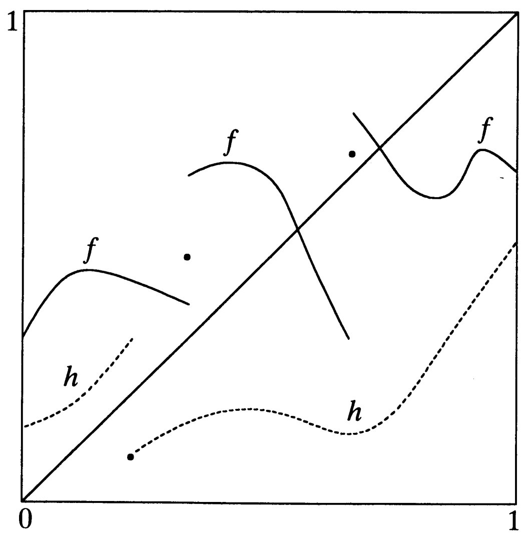

FIGURE 3. FUNCTION f, BUT NOT h, IS CONTINUOUS BUT FOR UPWARD JUMPS

nomic model. We employ a sufficient condition here which is weaker than continuity and which, unlike continuity, is preserved under order-preserving transformations of the function's domain and range.

Definition: A function $\phi: \mathbb{R} \to \mathbb{R}$ is *continuous but for upward jumps* if for all $\bar{x} \in \mathbb{R}$, $\limsup_{x \uparrow \bar{x}} \phi(x) \leq \phi(\bar{x}) \leq \liminf_{x \downarrow \bar{x}} \phi(x)$.

The function labeled f in Figure 3 is continuous but for upward jumps; that labeled h is not. It should be clear that functions with this property will continue to enjoy it after being subjected to monotone transformations of their domains and ranges, while continuous functions subjected to such transformations could lose continuity (but, of course, retain continuity but for upward jumps). This figure also suggests that functions from $[0,1]$ into itself which are continuous but for upward jumps will have fixed points. Such a function must start on or above the diagonal, and it can never jump down over the diagonal. Thus, the function starts on the diagonal [so that $f(0) = 0$, in which case 0 is a fixed point], or it stays above the diagonal always [so that $f(1) = 1$ and 1 is a fixed point], or else its graph crosses the diagonal at some point in $(0,1)$ and this is a fixed point. Similarly, a function from $[0,1]$ into $\mathbb{R}$ that is nonnegative at 0 and nonpositive at 1, if it is also continuous but for upward jumps, will have a point x satisfying $f(x)=0$. Proofs of both these results are included in Theorem 1 and Corollary 1. The first of these gives ordinal conditions that are sufficient for monotone comparative statics on the solutions of $f(x,t)=0$, where x is a real variable.

THEOREM 1: *Let $f(x,t)$: $[0,1]\times \mathbf{T}\to \mathbb{R}$, where $\mathbf{T}$ is any partially ordered set and where $f(0,t)\geq 0$ and $f(1,t)\leq 0$. Suppose that for all $t\in \mathbf{T}$, f is continuous but for upward jumps in x. Then there exists a solution for the equation $f(x,t)=0$. Moreover, $x_{\mathrm{L}}(t)\equiv \inf\{x\,|\,f(x,t)\leq 0\}$ is the lowest solution of $f(x,t)=0$ and $x_{\mathrm{H}}(t)\equiv \sup\{x\,|\,f(x,t)\geq 0\}$ is the highest solution. Suppose further that for all $x\in[0,1]$, f is monotone nondecreasing in t. Then $x_{\mathrm{L}}(t)$ and $x_{\mathrm{H}}(t)$ are monotone nondecreasing for all $t\in \mathbf{T}$. Moreover, if f is strictly increasing in t, then $x_{\mathrm{L}}(t)$ and $x_{\mathrm{H}}(t)$ are strictly increasing.*

PROOF:

By the boundary conditions of the theorem, $x_{\mathrm{L}}(t)$ and $x_{\mathrm{H}}(t)$ exist and are finite. We first need to show that they are actually solutions. We concentrate first on x_{L}. By the definition of x_{L}, the lim sup of $f(x,t)$ as $x\uparrow x_{\mathrm{L}}$ is nonnegative. Then, by continuity but for upward jumps, $f(x_{\mathrm{L}}(t),t)\geq 0$. If $x_{\mathrm{L}}(t)=1$ then, because $f(1,t)\leq 0$, we must have $f(x_{\mathrm{L}}(t),t)=0$. If $x_{\mathrm{L}}(t)<1$ and $f(x_{\mathrm{L}}(t),t)>0$, then continuity but for upward jumps implies that there is some $\varepsilon>0$ such that $f(x,t)>0$ for all $x\in[x_{\mathrm{L}}(t), x_{\mathrm{L}}(t)+\varepsilon)$, which is contrary to the definition of $x_{\mathrm{L}}(t)$. We conclude that $f(x_{\mathrm{L}}(t),t)=0$. By the condition that f is monotone in t and Lemma 1, for any $t>t'$, $x_{\mathrm{L}}(t)\geq x_{\mathrm{L}}(t')$. If f is strictly increasing in t, then there is no x such that $f(x,t)=f(x,t')=0$, so $x_{\mathrm{L}}(t)>x_{\mathrm{L}}(t')$.

The case of x_{H} is an *order dual*; that is, it is the same statement as the one for $x_{\mathrm{L}}(\cdot)$, but using the reverse orders on the domain and range of f. Order-duality arises repeatedly in this theory, and the following argument is typical. Let $g(z,t)=-f(1-z,t)$ and define $\bar{x}_{\mathrm{L}}(t)=\inf\{x\,|\,g(x,t)\leq 0\}$. Then $x_{\mathrm{H}}(t)=1-\bar{x}_{\mathrm{L}}(t)$ for all t. By the argument of the last paragraph, $g(\bar{x}_{\mathrm{L}}(t),t)=0$ and $\bar{x}_{\mathrm{L}}(\cdot)$ is monotone nonincreasing in t and strictly decreasing if f is strictly increasing in t. This establishes the claimed properties of $x_{\mathrm{H}}(\cdot)$.

The following corollary makes the corresponding statement for fixed-point models.

COROLLARY 1: *Let $g(x,t)$: $[0,1]\times\mathbf{T}\to[0,1]$, where $\mathbf{T}$ is any partially ordered set. Suppose that for all $t\in\mathbf{T}$, g is continuous but for upward jumps in x. Then $x_{\mathrm{L}}(t)=\inf\{x\,|\,g(x,t)\leq x\}$ and $x_{\mathrm{H}}(t)=\sup\{x\,|\,g(x,t)\geq x\}$ are the extreme fixed points of g, that is, the lowest and highest solutions of the equation $g(x,t)=x$. If, in addition, g is monotone nondecreasing in t for all $x\in[0,1]$, then the functions $x_{\mathrm{L}}(\cdot)$ and $x_{\mathrm{H}}(\cdot)$ are monotone nondecreasing, and if g is strictly increasing in t, then these functions are strictly increasing.*

PROOF:

Apply Theorem 1 with

$$f(x,t)=g(x,t)-x.$$

Given the possibility of using the map in the preceding proof to go back and forth between model formulations where the solution solves $f(x,t)=0$ and ones where the solution is a fixed point, we will henceforth state results only for the latter.

Note that without the condition of continuity but for upward jumps, the monotonicity results in the corollary need not hold, even when there is a unique fixed point. This is illustrated by Figure 4, where an upward shift in the function f leads to a fall in its unique fixed point.

Example 1: To illustrate Corollary 1, consider the treatment of Cournot duopoly offered by Roberts and Hugo Sonnenschein (1976). That paper emphasized that the standard assumption that the best-reply correspondences are continuous in Cournot

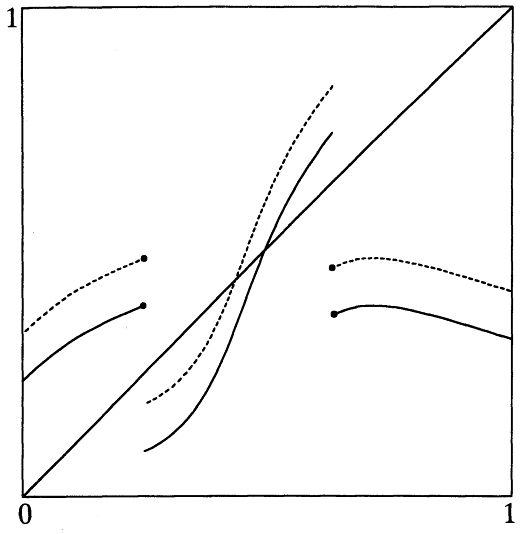

FIGURE 4. SHIFTING THE FUNCTION f UP LEADS ITS UNIQUE FIXED POINT TO SLIDE BACKWARDS DOWN THE 45° LINE

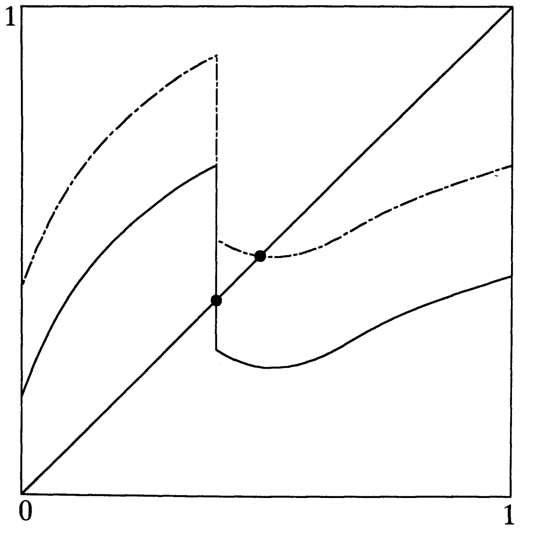

FIGURE 5. SHIFTING THIS CORRESPONDENCE UPWARD RAISES ITS UNIQUE FIXED POINT

models imposes unjustifiably strong restrictions on demand, requiring that the firm's total revenue function be strictly concave. Roberts and Sonnenschein formulated an alternative symmetric Cournot model in which the firms all have a constant marginal cost c and a finite bound b on their capacity, but in which the usual restrictions on demand are relaxed. Industry demand in that model is represented by an arbitrary upper semicontinuous function from quantities into prices. Assuming that the firm produces the largest quantity consistent with profit maximization given the competitors' choices, Roberts and Sonnenschein showed that the symmetric best-reply function, while not continuous, is nevertheless a function from $[0, b]$ into itself that is continuous but for upward jumps. The paper concluded that an equilibrium exists.[1] Corollary 1 repeats the existence conclusion and, in addition, enables one to make equilibrium comparisons. For example, a decrease in c (or an increase in $-c$) increases the extreme equilibrium quantities. One can also use the theorem to analyze the effects of changes in demand in the model.

[1]Maurice McManus (1962) had earlier arrived at a similar conclusion by somewhat less formal arguments, and very recently Nikolai Kukushkin (1992) has extended the result to situations in which all the firms have a common convex cost function and possibly differing upper bounds on their output levels.

The weak-monotonicity results of Corollary 1 can also be extended to the fixed points of certain correspondences using a weaker notion of continuity but for upward jumps. Figure 5 illustrates the kind of result that can be obtained when any vertical gaps caused by downward jumps are "filled in" to ensure that no fixed point is missed. The formal analysis begins with another definition.

Definition: Given two functions $\phi_{\rm L}, \phi_{\rm H}$: $[0,1] \to [0,1]$ with $\phi_{\rm L}(x) \le \phi_{\rm H}(x)$ for all $x \in [0,1]$, the correspondence ϕ defined by $\phi(x) = [\phi_{\rm L}(x), \phi_{\rm H}(x)]$ is *continuous but for upward jumps* if for all $\bar{x} \in [0,1]$, $\limsup_{x \uparrow \bar{x}} \phi_{\rm H}(x) \le \phi_{\rm H}(\bar{x})$ and $\phi_{\rm L}(\bar{x}) \le \liminf_{x \downarrow \bar{x}} \phi_{\rm L}(x)$.

COROLLARY 2: *Let* $\phi(x,t) = [\phi_{\rm L}(x,t), \phi_{\rm H}(x,t)]$: $[0,1] \times \mathbf{T} \to [0,1]$, *where* $\mathbf{T}$ *is any partially ordered set. Suppose that, for all* $t \in \mathbf{T}$, ϕ *is continuous but for upward jumps* *in x and that, for all $x \in [0,1]$, ϕ_L and ϕ_H are monotone nondecreasing in t. Then for all t the points $x_L(t) = \inf\{x \mid \phi_L(x,t) \leq x\}$ and $x_H(t) = \sup\{x \mid \phi_H(x,t) \geq x\}$ are the extreme fixed points of ϕ, that is, lowest and highest solutions of $x \in \phi(x,t)$. Both $x_L(\cdot)$ and $x_H(\cdot)$ are monotone nondecreasing.*

PROOF:

In view of Lemma 1, one only needs to show that x_L and x_H are fixed points, and the proof of that fact mimics the corresponding part of the proof for Theorem 1.

The preceding results entailed showing the sufficiency of the assumption that $f(x,t)$ is increasing in t provided the boundary conditions are satisfied (as in Theorem 1). Our next theorem establishes the criticality of this assumption in "general" contexts. The issue is how to formulate the notion of a general context, and we do not claim to have settled this with finality. The next theorem does give one formulation that is both tractable for analysis and useful for lending insight.

THEOREM 2: *Let $\mathcal{F}$ be a context whose elements $f(x,t)$: $[0,1] \times \mathbf{T} \to [0,1]$ are continuous in x for each t.*[2] *Suppose that for all $f \in \mathcal{F}$ and all α, β with $0 \leq \alpha < \beta \leq 1$ there exists a monotone increasing function g: $[0,1] \to [\alpha, \beta]$ such that $h \equiv g \circ f \in \mathcal{F}$. Then the lowest fixed point $x_L(t \mid f)$ [alternatively, the highest fixed point $x_H(t \mid f)$] is monotone nondecreasing in t for all $f \in \mathcal{F}$ if and only if every function in the context is nondecreasing in t. That is, a critical sufficient condition for $x_L(t \mid f)$ to be nondecreasing is that f is nondecreasing in t.*

PROOF:

The sufficiency part is a special case of Corollary 1. For necessity, suppose that there are values $t' < t''$ and $x' < x''$ such that for all $x \in [x', x'']$, $f(x,t') > f(x,t'')$. (Since the functions are continuous, if there is any x at which f decreases in t, then there will be an interval over which this holds.) Take $\alpha = x'$ and $\beta = x''$ and let h be as specified. Then the range and hence the fixed points of $h(\cdot, t)$ all lie in the interval $[x', x'']$ for both t' and t''. Regarding h as a parameterized function from $[x', x'']$ into itself and applying Lemma 1 yields the desired conclusion that $x_L(t' \mid h) > x_L(t'' \mid h)$ and $x_H(t' \mid h) > x_H(t'' \mid h)$.

The idea formalized in the theorem is that a modeler with only general knowledge of the context cannot know at which point x the minimum (or maximum) fixed point might occur. If there is any x at which $f(x,t)$ is decreasing in t, then there will exist some f in the context such that the minimum (maximum) fixed point is decreasing in t. The proof actually shows somewhat more, namely, that under these conditions there is some f such that both the maximum and minimum fixed points are decreasing in t. Note that we could have obtained a marginally weaker critical condition by allowing for a different g function for each pair of values of t and then requiring that the composition $g \circ f$ belong to the context and that its range lie in $[\alpha, \beta]$ for the corresponding values of t. The condition actually stated is obviously simpler. Note too that the theorem allows restricting g to be a linear function: $g(z) = \alpha + (\beta - \alpha)z$. In this case, the various properties mentioned in the discussion of contexts in the Introduction would all be maintained.

Example 2: Among the most elementary and best known examples of comparative statics is the fact that a competitive firm will optimally increase its output when the price offered for its product is increased. In keeping with our theme of robustness, our first question is whether an adaptively rational firm would always respond in the same way, increasing its output in response to an increase in the output price.[3] Robustness re-

[2]This theorem can be extended to accommodate functions that are continuous but for upward jumps. The critical sufficient condition is then that f be increasing in t at every "point of continuity" x of $f(x,t)$.

[3]An important precedent for this lies in the work of Richard Nelson and Sydney Winter (1982, Ch. 7), who

quires that we not be too extreme in limiting the set of adaptive rules, so we proceed by supposing that the firm's output choice at any given time depends on those things that the firm may know in the model: the current price and its past output choices.[4] These past choices may reflect either a history of actual outputs over time or a sequence of steps in a (possibly imperfect) planning process or optimization algorithm, where the starting point of the algorithm is the initial output level of the firm. Letting x_t denote the output at time t, we suppose that $x_t = g(p, x_{t-1}, \dots, x_{t-N})$. An equilibrium of the firm for price p is any output x satisfying $x = f(x,p) \equiv g(p,x,\dots,x)$. Suppose that the firm's output potential is bounded, so that the range of f is an interval $[0,\bar{x}]$, and that f is continuous but for upward jumps in x. A sufficient condition for this is that g is continuous but for upward jumps in its N quantity arguments. According to Theorem 1, if f is increasing in p, then the extreme equilibrium output levels $x_{\mathrm{L}}(p)$ and $x_{\mathrm{H}}(p)$ are increasing as well. Moreover, although the condition that f be increasing in t is not necessary for this conclusion, it is critical in the context of Theorem 2. Thus, we have a characterization of the kinds of adaptive rules that duplicate this traditional comparative static of the optimizing firm: if the adaptive dynamic process of the firm is such that its direct response to a higher price is to produce more output, which is an elementary property of even boundedly rational self-interested behavior, then the extreme long-run equilibrium output levels shift upward when prices shift upward.

The next result extends the theory by showing that the possibility of new endogenous variables leaves the robustness result for the original variable unaffected if the new variables are continuously determined by the variable on which we are focused, with no direct feedbacks other than through the determination of equilibrium.

COROLLARY 3: *Let* $\mathbf{Y}$ *be a topological space,* $\mathbf{T}$ *a partially ordered set, and* $f(x,y,t)$: $[0,1]\times\mathbf{Y}\times\mathbf{T}\to[0,1]$. *Suppose that (i) for all* $(y,t)\in\mathbf{Y}\times\mathbf{T}$, *f is continuous but for upward jumps in x, (ii) for all* $(x,t)\in\mathbf{X}\times\mathbf{T}$, *f is continuous in y, and (iii) for all* $(x,y)\in\mathbf{X}\times\mathbf{Y}$, *f is nondecreasing in t. Then for any continuous function h:* $[0,1]\to\mathbf{Y}$, *the functions* $x_{\mathrm{L}}(\cdot)$ *and* $x_{\mathrm{H}}(\cdot)$ *described by*

$$x_{\mathrm{L}}(t) \equiv \inf\{x \mid x \geq f(x,y,t) \text{ and } y = h(x)\}$$

$$x_{\mathrm{H}}(t) \equiv \sup\{x \mid x \leq f(x,y,t) \text{ and } y = h(x)\}$$

are the extreme solutions to $x = f(x,h(x),t)$. *Both* $x_{\mathrm{L}}(\cdot)$ *and* $x_{\mathrm{H}}(\cdot)$ *are monotone nondecreasing, and if f is increasing in t, then* $x_{\mathrm{L}}(\cdot)$ *and* $x_{\mathrm{H}}(\cdot)$ *are increasing.*

PROOF:

Define $\hat{f}(x,t) = f(x,h(x),t)$ and verify the conditions of Corollary 1.

There is also a version of the preceding result for correspondences.

COROLLARY 4: *Let*

$$\phi(x,y,t) = [\phi_{\mathrm{L}}(x,y,t), \phi_{\mathrm{H}}(x,y,t)]:$$

$$[0,1]\times\mathbb{R}^N\times\mathbf{T}\to[0,1]$$

where $\mathbf{T}$ *is any partially ordered set. Suppose that, for each* (y,t), ϕ *is continuous but for upward jumps in x and that, for each x,* ϕ_{L} *and* ϕ_{H} *are monotone nondecreasing in t and continuous in y. Let* $\psi(x)$: $[0,1]\to\mathbb{R}^N$ *be a correspondence with a closed graph and such that for each* $x\in[0,1]$, $\psi(x)$ *is path-connected.*[5] *Then for all t the points* $x_{\mathrm{L}}(t) = \inf\{x \mid (\exists y)\phi_{\mathrm{L}}(x,y,t) \leq x, y \in \psi(x)\}$ *and* $x_{\mathrm{H}}(t) = \sup\{x \mid (\exists y)\phi_{\mathrm{H}}(x,y,t) \geq x, y\in\psi(x)\}$ *are the lowest and the highest elements of*

explore the comparative statics of a firm under an adaptive process they call "search."

[4]The firm may also know past prices and, in general, these could have real effects on current choices even after controlling for past output choices. We rule out that dependence here in the interests of simplicity and brevity.

[5]If $N=1$, this becomes the condition that ϕ is convex-valued.

$\{x \mid (\exists y) x \in \phi(x,y,t)$ and $y \in \psi(x)\}$. Both $x_L(\cdot)$ and $x_H(\cdot)$ are monotone nondecreasing.

PROOF:

Define

$$\hat{\phi}_L(x,t) = \inf\{z \mid (\exists y) z = \phi_L(x,y,t) \text{ and } y \in \psi(x)\}$$

and

$$\hat{\phi}_H(x,t) = \sup\{z \mid (\exists y) z = \phi_H(x,y,t) \text{ and } y \in \psi(x)\}$$

and verify the conditions of Corollary 2.

Example 3: To illustrate the use of Corollaries 3 and 4, consider a monopolist faced by a competitive fringe. The market demand is $D(p,t)$, the output of the fringe is $y = h(p)$, and the cost function of the monopolist is $C(z) = F + cz$. The assumed behavior is that the monopolist sets a price p (corresponding to x in the statement of Corollary 3), the fringe places an amount y on the market that is determined by the fringe's supply function, and then the monopolist satisfies any residual demand. Assume that $D(\cdot,t)$ is continuously differentiable for each t and that increases in t raise $D(p,t)+(p-c)\partial D(p,t)/\partial p$ for $p \geq c$. Then for any continuous supply function from the fringe, the lowest and highest prices greater than c that satisfy the first-order condition for the monopolist's choice of p are increasing in t.

We would like to be able to analyze the fixed points of systems of several equations that are more general than those treated in Corollary 3 and, in particular, in which the vector of additional endogenous variables y is determined as the solution to an equation of the form $y = h(x,y)$, so that the x and y variables each influence one another. Corollary 3 covers this formulation only for the case where the solution y can be written as $y = \hat{h}(x)$ where $\hat{h}$ is continuous. At a minimum this requires that there be a unique solution y for each value of x. The difficulty with the more general case is illustrated by Figure 6, in which we have graphed $\{(x, f(x,y)+t) \mid y = h(x,y)\}$ for two different values of t. The illustrated case is one in which the equation $y = h(x,y)$ has as many as three solutions for values of x near the middle of their domain. As this figure illustrates, the unique fixed point $x^*(t)$ may be decreasing in t over some range, despite the upward shift.

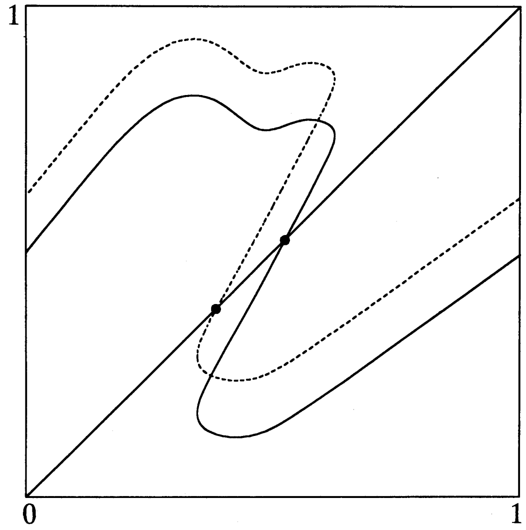

FIGURE 6. THE UNIQUE FIXED POINT OF THE HIGHER CURVE IS LOWER THAN THAT OF THE LOWER CURVE

The next result deals with a case of multivariate fixed points in which the direction of equilibrium changes is determinate even though y^* may not be a smooth function of x.

COROLLARY 5: *Let* **Y** *and* **T** *be partially ordered sets and $f(x,y,t)$: $[0,1] \times \mathbf{Y} \times \mathbf{T} \to [0,1]$. Suppose that (i) for all $(y,t) \in \mathbf{Y} \times \mathbf{T}$, f is continuous but for upward jumps in x, (ii) for all $(x,t) \in \mathrm{X} \times \mathbf{T}$, f is monotone nondecreasing in y, and (iii) for all $(x,y) \in \mathrm{X} \times \mathbf{Y}$, f is nondecreasing in t. Then for any monotone nondecreasing function h: $[0,1] \to \mathbf{Y}$, the functions $x_L(\cdot)$ and $x_H(\cdot)$ described by*

$$x_L(t,h) \equiv \min\{x \mid x = f(x,y,t) \text{ and } y = h(x)\}$$

$$x_H(t,h) \equiv \max\{x \mid x = f(x,y,t) \text{ and } y = h(x)\}$$

are well-defined and monotone nondecreasing. If f is increasing in t, then $x_L(\cdot, h)$ and $x_H(\cdot, h)$ are increasing.

PROOF:

Proceed as in the proof of Corollary 3. Define $\hat{f}(x,t) = f(x, h(x), t)$ and verify the conditions of Corollary 1.

Our results so far deal only with comparing equilibrium points involving a single real variable. The next results deal with general partially ordered sets in which the concepts of infimum and supremum are well defined. In the analysis to follow, one problem is to identify conditions under which a multivariate function has extreme fixed points similar to those of univariate functions. A second problem is to identify conditions under which the extreme fixed points vary monotonically.

Definition: A *complete lattice* $(\mathbf{X}, \geq)$ is a partially ordered set with the property that every nonempty subset $\mathbf{S}$ has a greatest lower bound $\inf(\mathbf{S}) \in \mathbf{X}$ and a least upper bound $\sup(\mathbf{S}) \in \mathbf{X}$. (For example, any interval $[a,b] \subset \mathbb{R}^N$ is a complete lattice, where (i) $1 \leq N \leq \infty$, (ii) $x \geq y$ means that $x_i \geq y_i$ for $1 \leq i \leq N$, and (iii) $[a,b] \equiv \{x \mid a \leq x \leq b\}$.)

Definition: The *highest fixed point* of a function f: $\mathbf{X} \to \mathbf{X}$ is a point x_H satisfying $f(x_H) = x_H$ and for all x such that $f(x) = x$, $x \leq x_H$, if such a point exists. The *lowest fixed point*, x_L, is defined symmetrically. Together, x_H and x_L are called the *extreme fixed points*.

THEOREM 3:[6] *Let $\mathbf{X}$ be a complete lattice, $\mathbf{T}$ a partially ordered set, and f: $\mathbf{X} \times \mathbf{T} \to \mathbf{X}$. Suppose f is monotone nondecreasing. Let $x_L(t) = \inf\{x \mid f(x,t) \leq x\}$ and $x_H(t) = \sup\{x \mid f(x,t) \geq x\}$. Then (i) $x_L(t)$ and $x_H(t)$ are extreme fixed points of $f(\cdot, t)$, (ii) $x_L(\cdot)$ and $x_H(\cdot)$ are monotone nondecreasing, and (iii) if for all $x \in \mathbf{X}$, f is increasing in t then $x_L(\cdot)$ and $x_H(\cdot)$ are increasing.*

PROOF:

Define $\mathbf{S}(t) \equiv \{x \mid f(x,t) \leq x\}$ so that $x_L(t) \equiv \inf \mathbf{S}(t)$. Since f is monotone nondecreasing, for all $x \in \mathbf{S}(t)$, $f(x_L(t),t) \leq f(x,t) \leq x$. This proves that $f(x_L(t),t)$ is a lower bound for $\mathbf{S}(t)$ and hence that (i) $f(x_L(t),t) \leq x_L(t)$, since $x_L(t)$ is the greatest lower bound of $\mathbf{S}(t)$. Applying the monotone nondecreasing function $f(\cdot, t)$ to both sides of this last inequality leads to $f(f(x_L(t),t),t) \leq f(x_L(t),t)$, which establishes that $f(x_L(t),t) \in \mathbf{S}(t)$. Since $x_L(t)$ is a lower bound of $\mathbf{S}(t)$, (ii) $x_L(t) \leq f(x_L(t),t)$. Combining (i) and (ii), $x_L(t) = f(x_L(t),t)$: $x_L(t)$ is a fixed point of $f(\cdot, t)$. By its definition, $x_L(t)$ is a lower bound for all fixed points.

Since f is monotone nondecreasing in t, the set $\mathbf{S}(t)$ becomes more exclusive as t increases. Hence, $x_L(t) = \inf \mathbf{S}(t)$ is a nondecreasing function of t. If f is increasing in t, then no x can satisfy the equation $x = f(x,t)$ for two different values of t, so $x_L(\cdot)$ must be increasing as well.

The case of $x_H(t)$ is symmetric.

Example 4: This example applies Theorem 3 to a fixed-point problem with an infinite number of variables. Consider the following variant of the arms-race game. In this two-player game, each player in each period τ decides on its current stock of armaments $x_{i\tau}$. Player 1's payoff from the game is

$$\sum_{t=1}^{\infty} \delta^\tau [M_1(x_{1\tau} - x_{2\tau}) - C_1(x_{1\tau} - \gamma x_{1,\tau-1}, t_1)]$$

and symmetrically for player 2, where the military advantage functions M_1 and M_2 are concave and where cost-of-armament functions C_1 and C_2 are convex. A strategy for a player is a sequence $\{x_{i\tau}\}$ specifying a level of armaments for each period. One can verify (see Milgrom and Roberts, 1990b) that the best-reply functions B_1 and B_2 are monotone nondecreasing in the x's and that the Nash equilibrium is unique. Consequently, Theorem 3 applies to this game. Thus, if an increase in t_1 reduces the

[6]The existence part of the theorem is a well known theorem of Alfred Tarski (1955). The monotonicity portion was reported in Milgrom and Roberts (1990b).

marginal cost of armaments, then B_1 is shifted upward by the change, and Theorem 3 implies that the equilibrium levels of armaments for both players are weakly higher in every period.

The following theorem modifies Theorem 3 in a way that is especially useful for economic applications.

THEOREM 4: *Let* $1 \le N \le \infty$. *For each* $1 \le i \le N$, *let* $f_i(x_i, \mathbf{x}_{-i}, t_i)$: $[0,1]^N \times \mathbf{T}_i \to [0,1]$ *be continuous but for upward jumps in* x_i *and nondecreasing in* $\mathbf{x}_{-i}$ *and* t_i, *where* $\mathbf{T}_i$ *is any partially ordered set. For* $1 \le i \le N$, *define the functions*

$$g_i(x,t) = \inf\{y_i | f_i(y_i, \mathbf{x}_{-i}, t) \le y_i\}$$

$$h_i(x,t) = \sup\{y_i | f_i(y_i, \mathbf{x}_{-i}, t) \ge y_i\}.$$

Then the points

$$x_{\mathrm{L}}(t) \equiv \inf\{x | g(x,t) \le x\}$$

$$x_{\mathrm{H}}(t) \equiv \sup\{x | h(x,t) \ge x\}$$

are the lowest and highest fixed points of the function $f(\cdot,t)$. *Moreover, the functions* $x_{\mathrm{L}}(\cdot)$ *and* $x_{\mathrm{H}}(\cdot)$ *are monotone nondecreasing.*

PROOF:

By Corollary 1, the function $g(\cdot,t)$ is monotone nondecreasing. Hence, by Theorem 3, $x_{\mathrm{L}}(t)$ is its lowest fixed point. For each i, by Corollary 1 and the definition of g_i, $x_{\mathrm{L}i}(t) = g_i(x,t) = f_i(x_{\mathrm{L}}, t)$, so $x_{\mathrm{L}}(t)$ is a fixed point of $f(\cdot,t)$. Let y be any other fixed point of $f(\cdot,t)$ and let $y \wedge x_{\mathrm{L}}(t)$ denote the component-wise minimum of y and $x_{\mathrm{L}}(t)$. Since g is monotone nondecreasing, $g(y \wedge x_{\mathrm{L}}(t), t) \le y$ and $g(y \wedge x_{\mathrm{L}}(t), t) \le x_{\mathrm{L}}(t)$, so $g(y \wedge x_{\mathrm{L}}(t), t) \le y \wedge x_{\mathrm{L}}(t)$. Hence, by the definition of x_{L}, $x_{\mathrm{L}}(t) \le y \wedge x_{\mathrm{L}}(t)$, which implies that $x_{\mathrm{L}}(t) \le y$, as required. The other properties of x_{L} follow from Theorem 3 and the observation that this is the lowest fixed point of the monotone nondecreasing function g. The case for $x_{\mathrm{H}}(t)$ is symmetric.

Example 5: For one illustration of Theorem 4, we will employ a standard example in general-equilibrium modeling: the case of a pure exchange economy with gross substitutes.[7] Let the goods be indexed from 0 to N ($1 \le N \le \infty$) with good 0 as numeraire.[8] Let $d_j(\mathbf{p}, t)$ denote the demand for good j when the vector of prices is $\mathbf{p}$. We specify an equilibrium in the usual way as a vector of prices such that either the excess demand for commodity j is zero or else its price is zero and excess demand is negative. Thus, an equilibrium is a fixed point of the function f defined by $f_j(\mathbf{p}, t) = \max(0, p_j + d_j(\mathbf{p}, t))$. Notice that the gross-substitutes assumption means that $f_j(p_j, \mathbf{p}_{-j}, t)$ is nondecreasing in $\mathbf{p}_{-j}$, as required by Theorem 4. Since f_j is generally decreasing in p_j, Theorem 3 does not apply. However, if preferences are strictly convex, then f_j is continuous in p_j, so Theorem 4 does apply. There are two immediate consequences.

First, equilibrium must be unique. The argument is standard. Suppose that there exist multiple equilibria. Then there is a highest equilibrium—one with the highest equilibrium prices for all goods (except, of course, the numeraire)—and a lowest equilibrium. By gross substitutes, the excess demand for the numeraire must be greater in the first equilibrium than in the second, so the market for the numeraire cannot clear in both cases. Thus, the equilibrium must be unique.

Second, any shift in t that raises demand leads to higher equilibrium prices. For example, suppose t indexes the endowment of the numeraire good for several consumers, with higher values of t corresponding to larger endowments. If the good indexed by j is a normal good, then d_j is increasing in t in that case. So, an increase in the endowment of the numeraire good leads to higher prices for all other goods, that is, a lower

[7]A beautiful formal analysis of this case using methods similar to ours is given by Michio Morishima (1964). See also Kenneth Arrow and Frank Hahn (1971).

[8]We assume that the marginal rate of substitution MRS_{0j} is never zero, so that the numeraire good's relative price is always strictly positive.

relative price of the numeraire. Of course, the same holds for each other good because the choice of numeraire is arbitrary.

The ordinal approach also makes it possible to study the dynamics of equilibrium using much more general models of price-setting than the traditional tâtonnement model, but this stability analysis takes us beyond the simple comparisons of fixed points. See Milgrom and Roberts (1991) for details.

Example 6: A second example of the use of Theorem 4 is based on a model developed by Oliver Hart and John Moore (1990) to study the optimal allocation of ownership rights in physical assets from the point of view of encouraging complementary individual investments in human capital. Our analysis combines the fixed-point techniques introduced here and the comparative-statics techniques for optimization problems of Topkis (1978).[9]

Hart and Moore (1990) consider a set $\underline{\mathbf{S}}$ of N agents, each of whom makes some investment x_i in human capital at cost $C_i(x_i)$. There is also a set of physical assets $\underline{\mathbf{A}}$ in which ownership may be assigned in various ways. An ownership assignment is indicated by a function α that associates with any possible coalition $\mathbf{S}$ of agents the set $\alpha(\mathbf{S})$ of assets that the coalition controls. It is assumed that $\mathbf{S} \subset \mathbf{S}^*$ implies $\alpha(\mathbf{S}) \subset \alpha(\mathbf{S}^*)$. Hart and Moore model the eventual bargaining among the parties as a cooperative game with transferable utility and represent its solution using the concept of the Shapley value. For this to make sense, it should be efficient for the coalition-of-the-whole to form, and we will assume this. The value of a coalition $\mathbf{S}$ controlling a set of assets $\mathbf{A}$ when investments $\mathbf{x} = (x_1, \dots, x_N)$ have been made is $v(\mathbf{S}, \mathbf{A}, \mathbf{x})$. The Shapley value for individual i is then

$$B_i(\alpha, \mathbf{x}) \equiv \sum p(\mathbf{S}) \left[v(\mathbf{S}, \alpha(\mathbf{S}), \mathbf{x}) - v(\mathbf{S} \backslash \{i\}, \alpha(\mathbf{S} \backslash \{i\}), \mathbf{x}) \right]$$

where the sum is over those $\mathbf{S}$ with $i \in \mathbf{S}$ and $p(\mathbf{S}) \equiv (\#\mathbf{S} - 1)!(N - \#\mathbf{S})!/N!$. The idea is to investigate how alternative assignments of the ownership rights affect individual investments in human capital when those investments cannot be enforced by contract. Any agreement that the agents reach concerning their investment must be self-enforcing, that is, a Nash equilibrium of the investment game.

Following Hart and Moore, we assume (i) that investments of different agents in a coalition are complementary and that an agent's investments benefit only coalitions of which he is a member. Formally, this means that $\partial^2 v / \partial x_i \partial x_j \geq 0$ for $i \neq j$ and $\partial v(\mathbf{S}, \mathbf{A}, \mathbf{x}) / \partial x_i = 0$ for $i \notin \mathbf{S}$. Assumption (i) implies that $\partial^2 B_i / \partial x_i \partial x_j \geq 0$ for $i \neq j$, which implies (by a theorem of Topkis), that i's maximum best-reply function is nondecreasing in the other agents' investment levels. Then, according to Theorem 4, there exists a largest equilibrium, that is, one with the highest level of human-capital investment by every agent.

Next, we assume in addition (ii) that the marginal products of investment $\partial v / \partial x_i$ are monotone nondecreasing functions of the inclusiveness of the coalition and its set of assets. By inspection of the formula defining $B_i(\alpha, \mathbf{x})$, this implies that each B_i is monotone nondecreasing in x_j for all $j \neq i$: there are "positive externalities." Since each agent's payoff is a nondecreasing function of the other agents' investments, each agent prefers the equilibrium with the highest investment by other agents. Hence, the maximum investment equilibrium is Pareto-preferred to all other equilibria. We assume that this Pareto-preferred equilibrium is the self-enforcing agreement that the agents would select.

According to Theorem 4,[10] any change in the ownership allocation α that increases

[9]This example also fits the framework of supermodular games. The conclusions derived here can alternatively be derived using the theorems about supermodular games of Milgrom and Roberts (1990b).

[10]This result also follows from Theorem 3.

$\partial B_i/\partial x_i$ for every agent i leads to a higher level of investment at the selected equilibrium. This is one of Hart and Moore's (1990) central conclusions, obtained here using only assumptions (i) and (ii).

A second key conclusion of their analysis is that the equilibrium levels of investment are less than the first-best levels. By the assumptions already made, $\partial v(\underline{\mathbf{S}},\underline{\mathbf{A}},\mathbf{x})/\partial x_i > \partial B_i/\partial x_i$, and the first-best investment levels are those arising at the Pareto-best Nash equilibrium of the (team) game in which each agent i chooses x_i to maximize the social objective $v(\underline{\mathbf{S}},\underline{\mathbf{A}},\mathbf{x})-C_i(x_i)$. Applying Topkis's theorem, the maximum best-reply function in this game lies above that for the actual game. Hence, by Theorem 4, the first-best investment levels exceed the investment levels of the Pareto-best equilibrium of the initial game.

The original analysis by Hart and Moore employed a large number of simplifying assumptions to reach these same conclusions. Our analysis shows that none of these additional assumptions is critical for these central conclusions in the context of their model.

We have already seen that in one-dimensional fixed-point problems, monotonicity of the fixed-point function in the parameter is a critical sufficient condition for robust comparisons. The logic of Corollaries 3 and 5 makes it clear that the same is true in multivariate problems. The next theorem indicates that, in a different context, monotone cross effects among variables are critical for multivariate equilibrium comparisons.

THEOREM 5: *Let $1 \le N \le \infty$. For each $1 \le i \le N$, let $f_i(x_i,\mathbf{x}_{-i},t_i)$: $[0,1]^N \times \mathbf{T}_i \to [0,1]$ be continuous but for upward jumps in x_i and nondecreasing in t_i, where $\mathbf{T}_i$ is any partially ordered set. Suppose there exists some i, j, $x_j' > x_j''$, $\mathbf{x}_{-ij}$, t_i, and an interval $[x_i'',x_i']$ such that for all $x_i \in [x_i'',x_i']$,*

$$f_i(x_i,x_j',\mathbf{x}_{-ij},t_i) < f_i(x_i,x_j'',\mathbf{x}_{-ij},t_i).$$

Then there exist numbers α_j and monotone nondecreasing functions $\beta_j(t_j)$ such that the extreme fixed points of the function g, where $g_j(x,t)=\alpha_j f_j(\mathbf{x},t_j)+\beta_j(t_j)$, are decreasing functions of t_j.

PROOF:

Let $\alpha_j=0$, $\beta_j(t_j')=x_j'$ and $\beta_j(t_j'')=x_j''$, $\alpha_i=x_i'-x_i''$, and $\beta_i=x_i''$, and for $k\notin\{i,j\}$, let $\alpha_k=0$ and $\beta_k=x_k'$. By Theorem 4, the extreme fixed points of this map are decreasing functions of t_j.

Because the f_i correspondences will, in many economic applications, be defined by a first-order condition, this result points to the significance of supermodularity (Topkis, 1978; Milgrom and Roberts, 1990a, b) in obtaining monotone comparative statics.

Our final theorem is a variation of LeChatelier's principle for application to fixed points. The theorem formalizes the idea that when there are more positive feedbacks at work in a monotonic equilibrium system, the system adjusts more to an exogenous change in a parameter. Here, we formulate the principle for fixed points on $[0,1]^\infty$ in the style of Theorem 4. A corresponding statement can be made in the style of Theorem 3, as well.

THEOREM 6: *Let $1 \le N \le \infty$. For each $1 \le i \le N$, let $f_i(x_i,\mathbf{x}_{-i},t)$: $[0,1]^N \times \mathbf{T} \to [0,1]$ be continuous but for upward jumps in x_i and nondecreasing in $\mathbf{x}_{-i}$ and t_i, where $\mathbf{T}$ is any partially ordered set. Suppose that $\bar{\mathbf{x}}$ satisfies $\bar{\mathbf{x}}=f(\bar{\mathbf{x}},\bar{t})$. Given a set $\mathbf{S}\subset\{1,\dots,N\}$, define $\bar{f}$: $[0,1]^N\times\mathbf{T}\to[0,1]^N$ by $\bar{f}_i(\mathbf{x},t)\equiv\bar{x}_i$ for $i\in\mathbf{S}$ and $\bar{f}_i(\mathbf{x},t)\equiv f_i(\mathbf{x},t)$ for $i\notin\mathbf{S}$. Let $\mathbf{x}_{\mathrm{L}}(t)$ and $\mathbf{x}_{\mathrm{H}}(t)$ be the extreme fixed points of f and let $\bar{\mathbf{x}}_{\mathrm{L}}(t)$ and $\bar{\mathbf{x}}_{\mathrm{H}}(t)$ be those of $\bar{f}$. (The existence of these is guaranteed by Theorem 4.) Then for $t\ge\bar{t}$, $\mathbf{x}_{\mathrm{H}}(t)\ge\bar{\mathbf{x}}_{\mathrm{H}}(t)$; and for $t\le\bar{t}$, $\mathbf{x}_{\mathrm{L}}(t)\le\bar{\mathbf{x}}_{\mathrm{L}}(t)$.*

PROOF:

As in Theorem 4, define

$$g_i(\mathbf{x},t)=\inf\{y_i \mid f_i(y_i,\mathbf{x}_{-i},t)\le y_i\}$$

and note that $x_{\mathrm{L}}(t)\equiv\inf\{\mathbf{x}\mid g(\mathbf{x},t)\le\mathbf{x}\}$. Similarly, for $i\in\mathbf{S}$ define $\bar{g}_i(\mathbf{x},t)=\bar{x}_i$ and, for

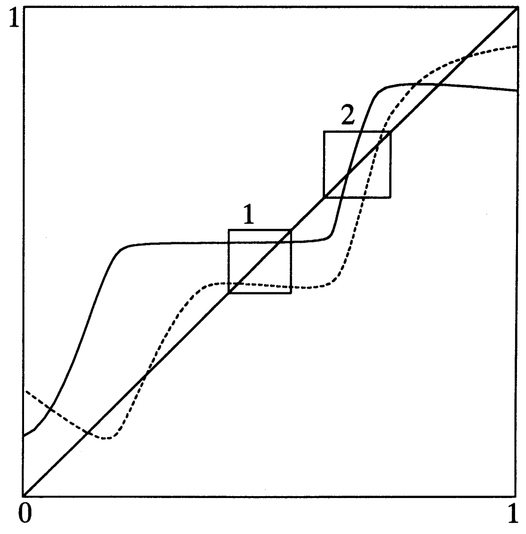

FIGURE 7. THE GLOBAL IDEAS ALSO APPLY LOCALLY USING THE FUNCTION f (BOX 1) OR f^{-1} (BOX 2)

$i \notin \mathbf{S}$, $\bar{g}_i(\mathbf{x},t) = \inf\{y_i | f_i(y_i, \mathbf{x}_{\mathrm{S}-i}, \bar{\mathbf{x}}_{\mathrm{S}}, t) \leq y_i\}$. Notice that the functions g and $\bar{g}$ are monotone nondecreasing, and hence, for $t \leq \bar{t}$, by the definition of $\bar{\mathbf{x}}$, both map the set $[\mathbf{0}, \bar{\mathbf{x}}]$ into itself. Also, for all $x \in [\mathbf{0}, \bar{\mathbf{x}}]$ and $t \leq \bar{t}$, $g(\mathbf{x},t) \leq \bar{g}(\mathbf{x},t)$, and the lowest fixed points are given by $\mathbf{x}_{\mathrm{L}}(t)$ and $\bar{\mathbf{x}}_{\mathrm{L}}(t)$. Hence, by Theorem 3, $\mathbf{x}_{\mathrm{L}}(t) \leq \bar{\mathbf{x}}_{\mathrm{L}}(t)$. The conclusion about $\mathbf{x}_{\mathrm{H}}$ is an order dual to the conclusion about $\mathbf{x}_{\mathrm{L}}$.

We shall return to the LeChatelier principle in Section III.

II. Local Comparisons

The results reported in the previous section entail global comparisons of extreme fixed points or extreme solutions to a set of equations. In this section, we show how the local results of the traditional approach based on the implicit-function theorem can be derived using the global approach. Figure 7 illustrates the main idea at a glance: the global ideas apply separately to each locale.

More precisely, suppose that the equilibrium value of the single endogenous variable in some model is characterized as any solution of an equation $x = f(x,t)$. If f is a smooth function and $(x(\bar{t}), \bar{t})$ is a regular point and a zero of $f(x,t) - x$, then by the implicit-function theorem there is a neighborhood of $\bar{t}$ on which there is a unique local solution $x(t)$ satisfying

$$x'(t) = f_t / (1 - f_x).$$

So, if $f_x < 1$, then $x(\cdot)$ is increasing in t in this neighborhood if and only if f is. Applying Corollary 1 to a neighborhood of $(\bar{t}, x(\bar{t}))$ produces the same conclusion. The conclusion for the case where $f_x > 1$ follows from (the order dual of) Theorem 1. In both cases, our results continue to apply even when f is not smooth or the equilibrium point is not a regular point, provided a unique equilibrium continues to exist in the relevant discrete neighborhood. In addition, as already noted, our results apply globally to the extreme equilibria.[11]

In multivariate equilibrium models, it is frequently possible to employ ideas from the implicit-function-theorem approach usefully to check the conditions of the ordinal approach. For example, consider a model with endogenous variables, $x \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^N$, where the solution is given by $x = f(x, \mathbf{y}, t)$ and $\mathbf{y} = g(x, \mathbf{y}, t)$, where the latter equation has a unique solution $\mathbf{y}^*(x,t)$ that is amenable to analysis by the implicit-function theorem, while the first equation may entail multiple equilibria and discontinuities. The way the extreme equilibrium values of x vary with t depends on the properties of $h(x,t) \equiv f(x, \mathbf{y}^*(x,t), t)$. In case the composite function h is smooth, Theorem 1 suggests that a sufficient condition for $x^*(t)$ to be increasing is that $h_t / (1 - h_x)$ be positive, which coincides with the Jacobian condition of the traditional analysis. This establishes that our results subsume the tra-

[11]Of course, the implicit-function-theorem approach does yield an explicit formula for the derivative $x'(t)$, and our approach does not. Note, however, that many of the conclusions that the formula might be used to derive (for example, a result that $x' > \alpha \neq 0$) are not robust: they rely on taking the specific model very seriously.

ditional ones. However, our results also cover other cases. For example, if f is monotone nondecreasing in its second argument and $\mathbf{y}^*$ is monotone nondecreasing in t, then Theorems 3 and 4 may apply. As discussed in connection with Figure 6, if f is not monotone in $\mathbf{y}$ or g is not monotone in x and if the equation $\mathbf{y} = g(x, \mathbf{y}, t)$ may have multiple solutions $\mathbf{y}$ for fixed (x, t), then the problem of comparing equilibria becomes more difficult.

III. The LeChatelier Principle

The Samuelson-LeChatelier principle represents one of the most celebrated comparative-statics conclusions in economics. In fact, Paul Samuelson proved what he considered to be two distinct versions of the principle. The more familiar is that from *The Foundations of Economic Analysis* (Samuelson, 1947), which focuses on situations modeled as optimization problems. For example, in the context of the theory of the profit-maximizing competitive firm, the usual statement of this version of the principle asserts that, starting from a point at which the firm has fully optimized relative to some specific set of limitations on its input choices, adding more such restrictions reduces the magnitude by which the firm's demand for an input falls in response to a small increase in the input's price. In particular, starting from a point on the long-run input demand curve, the demand for an input given an increase of its price is less if the usage of all inputs is free to vary (the *long run*) than if the amounts used are fixed for some goods (the *short run*), provided that the price increase is small. More formally, let good 1 be the input whose price is changing and let good 2 be fixed in the short run. Let $D(p, x_2)$ denote the demand for good 1 at price p when the quantity of good 2 is fixed at x_2 and let $x_2^*(p)$ be the long-run demand for good 2 given the varying price of good 1. Given an initial price $\bar{p}$ for good 1, the short-run demand for good 1 when the current price shifts to p is $D(p, x_2^*(\bar{p}))$. The corresponding long-run demand is $L(p) = D(p, x_2^*(p))$. Using subscripts for partial derivatives, the differential statement of LeChatelier's principle is that $L'(\bar{p}) \le D_1(\bar{p}, x_2^*(\bar{p}))$. The same statement holds when the demand relations are reinterpreted as compensated consumer demands.

Samuelson's proof of this result used the definiteness of the Hessian matrix corresponding to the second-order conditions of the optimization problem. More recently, a proof due to Eugene Silberberg (1974) that uses duality theory has become standard: see, for example, Hal Varian (1992). Both arguments are local: they compare derivatives at a point and so at most say something about the functions themselves on a neighborhood of the original price $\bar{p}$.

The formal result embodied in the Samuelson (1947) and Silberberg (1974) proofs is much weaker than what is often taught in elementary economics courses, where one describes, for example, how an increase in the price of oil leads in the long run to reduced purchases of complementary products (e.g., automobiles) and the increased purchase of substitutes (insulation, public transportation), which cause the long-run response to the price increase to exceed the short-run response.

Strikingly, the verbal argument does not seem to impose any condition that the price change be small. Moreover, it suggests that the result should be that for $p > \bar{p}$, $L(p) < D(p, x_2^*(\bar{p}))$. Analysis based on simple monotonicity arguments confirms the usual classroom arguments, given certain conditions that classroom treatments usually omit. Indeed, if good 2 is a complement for good 1, then $x_2^*(\cdot)$ is decreasing and $D(p, \cdot)$ is increasing for each p, so for $p > \bar{p}$, $L(p) = D(p, x_2^*(p)) < D(p, x_2^*(\bar{p}))$. If good 2 is a substitute for good 1, the order-dual argument applies: x_2^* is increasing and $D(p, \cdot)$ is decreasing, so again $L(p) = D(p, x_2^*(p)) < D(p, x_2^*(\bar{p}))$.[12]

Note that the monotonicity argument that long-run demand varies more than short-run demand employs an extra explicit assumption: good 2 in the analysis is either a substi-

[12] Morishima (1964) makes a related argument using the assumptions that consumers are rational (demand functions are homogeneous and symmetric) and that all goods are gross substitutes.

		Output 0	Output 1
New Capital	0	0	56 - 2p
	1	-24	32 - p

FIGURE 8. NET REVENUES WHEN THE PRICE OF OIL IS p

tute or a complement for good 1 over the entire domain in question. The conclusion of LeChatelier's principle does not apply if the two goods are substitutes over some ranges and complements over others. This is easily seen in a simple example.

Suppose a firm can produce either zero or one unit of output using oil and capital as inputs (we could allow for divisible inputs and impose the usual sort of Inada boundary conditions in this example without difficulty). Capital is fixed in the short run, but in the long run it can be increased by a unit, which costs \$24. Oil usage is variable, and zero output requires zero oil use. With the original level of capital, producing a unit of output requires two units of oil. If a unit of new capital is purchased, then producing a unit of output requires only one unit of oil. The price of the output is fixed at \$56 (see Fig. 8)

Let the price of oil initially be $p = \$20$. Then the long-run optimal plan is to produce a unit of output using only the original capital and two units of oil, earning a profit of $\$56 - 2p = \$16$. Purchasing a unit of new capital, the maximum profit would be only \$12. If the price of oil rises to \$30, the optimal short-run response is to cease production, netting zero, rather than to continue producing with the original capital and lose \$4. However, once capital is free to adjust, the optimal plan is to increase capital and use one unit of oil to produce a unit of output, netting $\$32 - p = \$2$ (see Fig. 8). Thus, in response to the oil price increase, oil consumption falls more in the short run (from two units to zero) than in the long run (where usage is one unit).

Samuelson (1960) noted that the version of LeChatelier's principle that he had initially derived from the properties of the Hessian matrices applied only locally and that an example like the one above could be constructed. In this same paper he established what he regarded as a second, distinct version of the principle for systems characterized by Minkowski matrices, including Leontief input–output systems and systems with gross substitutes. This version was, in fact, global, because these systems do not permit goods to switch between being substitutes and complements.

The critical sufficient condition in our approach—that goods are either substitutes or complements on the range where the conclusion holds—is in fact also critical for the first version of LeChatelier's principle, but it appears there in the form of the obscure and innocent-looking assumption that demand functions are differentiable.[13] To highlight this assumption, let us focus on regular points of demand, that is, prices around which demand is a single-valued, continuously differentiable function of the prices, with finite, nonzero derivatives. There is always some neighborhood around such price vectors in which the two goods in the analysis are either (compensated-demand) complements ($\partial D/\partial x > 0$ and $\partial x_2^*/\partial p < 0$) or substitutes ($\partial D/\partial x < 0$ and $\partial x_2^*/\partial p > 0$). Thus, the apparently unimportant differentiability assumptions used in proving the original version of LeChatelier's principle in fact are the key to the result, because they imply that the critical sufficient condition for the monotonicity argument holds in some neighborhood of the initial price. In contrast to Samuelson's (1947) original analysis and the one still cited in the textbooks, our analysis is not limited to small neighborhoods: the long-run response to a price increase exceeds the short-run response for price changes of any size on any domain where the two goods are

[13]The condition is obviously critical in any of the many contexts where the preceding linear example can arise or be closely approximated.

always substitutes or always complements. Meanwhile, the properties of the Hessians or the dual functions used in the standard proofs—and even maximization itself—are irrelevant for the analysis.

Once the critical monotonicity assumptions have been identified, it becomes clear that one can eschew maximization altogether and still derive a version of LeChatelier's principle in behavioral models of adaptively rational decision-making. For example, suppose a firm produces output using two inputs (capital and labor), subject to the production constraint that $x = F(k, l)$. Suppose further that the current output decision is made by a marketing manager and the current capital decision is made independently by a factory manager, with labor use varying as necessary to make the two choices consistent. Finally, suppose that the marketing manager's current output choice is a continuous-but-for-upward-jumps function of the immediate past output choice and a monotone nondecreasing function of the current price of output and the immediate past capital stock, while the factory manager's current capital choice is a continuous-but-for-upward-jumps function of the immediate past level of capital and a nondecreasing function of output ("capital is a normal input"). Suppose the price of output increases. In that case, by Theorem 4, the extreme equilibrium output and capital levels will rise. Moreover, if capital is held fixed at any equilibrium level for the old price, then the maximal equilibrium output is less than if capital is free to vary (Theorem 6).

Bounded rationality provides many modeling options, and it would be disingenuous not to point out that alternative assumptions about the organization of decision-making can yield different conclusions. One specific alternative in this example has the two managers choosing capital and labor, rather than capital and output. In that case, it is at least plausible that current input choices might be decreasing functions of the past levels of the other input (due to input substitution) and that the supply of both inputs may be increasing functions of the current price of output and a decreasing function of the price of that input. In that case, the extreme equilibrium responses of a firm to an input price change is easy to verify: the firm substitutes in favor of the input whose price has fallen. Again, LeChatelier's principle applies if one input is assumed to be fixed. However, with this organization of decisions, the response of the firm to an output price increase depends on the details of the adjustment process (Theorem 4). Thus, the organization of decision-making in an adaptively rational firm can have profound implications for the way it responds to exogenous shocks.

IV. Conclusion

The theory presented in this paper is part of an emerging new conception of comparative statics in which the robustness of results—that is, the breadth of the contexts in which they apply—is a central concern. The emerging theory emphasizes global conclusions, rather than conclusions that apply only when parameter changes are sufficiently small. It emphasizes ordinal conditions—ones that are invariant to order-preserving transformations—and suppresses conditions like convexity that are not invariant and therefore not helpful for discerning the full scope of any comparative-statics analysis.

The focus on critical assumptions has several immediate and important payoffs. First, it may often lead to unifications of diverse but related theories and to extensions of each. For example, one of our analyses unifies Samuelson's two versions of LeChatelier's principle. Another extends the Hart-Moore asset analysis by omitting most of the restrictive assumptions of the original treatment. Second, our approach can help to make intuitive analyses rigorous. For example, we justified the general intuitive principle that long-run demand is more responsive to a price change than short-run demand, while exposing the critical assumption on which the intuitive argument relies and avoiding unnecessary technicalities. Third, by establishing the full scope of a result, this focus can lead to unanticipated extensions, such as our extension of the analysis of long- and short-run demand to behavioral models of boundedly rational decision-makers. Finally, the new methods based on critical assumptions are simpler than analyses which require "simplifying assumptions," if only because fewer assumptions are required. This simplicity makes the formal logic easier to explain to nonmathematical social scientists. This last payoff might be the most important of all.

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