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A CONVERGENCE THEOREM FOR COMPETITIVE BIDDING WITH DIFFERENTIAL INFORMATION

BY PAUL R. MILGROM[1]

This paper investigates the behavior of the winning bid in a sealed bid tender auction where each bidder has private information. With an appropriate concept of value, the winning bid will converge in probability to the value of the object at auction (as the number of bidders grows large) if and only if a certain information condition is satisfied. In particular, it is not necessary for any bidder to know the value at the time the bids are submitted.

These results bear on the relationship between price and value and on the aggregation of private information by the auction mechanism.

1. INTRODUCTION

IN THIS PAPER we investigate the properties of the winning bid in a sealed bid tender auction where each player has private information. We find that it is possible for the winning bid to converge in probability to the true value of the object at auction, even though no bidder knows the true value. Necessary and sufficient conditions for this phenomenon are derived, extending and generalizing certain of Wilson's results [**3**].

We study an auction in which a seller offers to sell at the highest bid an item of unknown value V. The kth bidder receives a private signal s_k (for $k = 1, 2, \ldots$) and submits a bid without knowledge of the other signals. A finitely additive probability measure P reflects the bidders' unanimous beliefs about V and the signals. Conditional on V, the signals are independent and identically distributed. The signals take their values in some space $\mathcal{S}$.

With n bidders, a bidding strategy for k is a function $p_{nk}: \mathcal{S} \to R$. k's strategy specifies that upon receiving the signal s_k, he shall bid $p_{nk}(s_k)$.[2] Thus the winning bid is

$$(1) \qquad W_n = \max_{k \leq n} p_{nk}(s_k).$$

It is also helpful to define

$$(2) \qquad W_{ni} = \max_{\substack{k \leq n \\ k \neq i}} p_{nk}(s_k).$$

Bidder i's payoff is

$$(3) \qquad (V - p_{ni}(s_i)) I_{\{W_{ni} < p_{ni}(s_i)\}}$$

[1] The author gratefully acknowledges helpful discussions with John Christensen, David Kreps, David Siegmund, and especially, Robert Wilson. The remarks of two anonymous referees were also useful in clarifying the presentation.

[2] Mixed strategies are easily incorporated into this framework by including in the signal space a dimension unrelated to V. This dimension of each signal may be used for randomization. For a Nash equilibrium to exist, it will generally be necessary that the distribution of the signals be atomless.

where the last term is an indicator variable which equals one if i submits the highest bid and equals zero otherwise. Expression (3) implies that in the event of ties there is no winner. The introduction of a tie-breaking mechanism would complicate the notation without altering the equilibrium strategies.

To rule out the possibility of infinite expected payoffs, we impose a minimum bid of zero and assume V has finite expectation. We also assume that all bidders are risk-neutral and that V takes its values in the nowhere dense set $\mathscr{V} = \{v_1, v_2, \ldots\}$ where $0 \leq v_1 < v_2 \ldots$ and $P\{V = v_k\} > 0$ for each $v_k \in \mathscr{V}$. (Finite $\mathscr{V}$ is also acceptable.)

No measurability conditions have been imposed in this formulation. The role of measurability is discussed in the Appendix.

The above formulation defines a non-cooperative game with incomplete information. We are concerned with the behavior of W_n when the n-tuple of strategies $(p_{n1}, \ldots, p_{nn})$ forms a Nash equilibrium (if one exists). Our initial task is to determine the conditions under which W_n converges in probability to V.[3]

2. THE MOTIVATING IDEAS

Before proceeding to a rigorous treatment, let us examine the intuitive requirements for W_n to converge in probability to V.

Suppose W_n converges in probability to V. Let $\alpha = \frac{1}{2}(v_{k-1} + v_k)$. For large n, it must be much more likely that $W_n > \alpha$ when $V = v_k$ than when $V < v_k$. Thus when $V = v_k$, the winning bidder must receive a signal which leads him to bid more than α. It must be very unlikely that he would receive such a signal if V were less than v_k. A necessary condition for convergence is that there exist signals which are much more likely when $V = v_k$ than when $V < v_k$.

On the other hand, if such signals exist, then (due to independence) many bidders in a large auction will observe them. None of these bidders will (necessarily) have great faith that $V = v_k$. They may find it more likely that $V > v_k$. Nonetheless if such signals exist then whenever $V = v_k$, many bidders will be confident that $V \geq v_k$. Competition among them will force W_n up to v_k. Thus, the possibility of distinguishing the event $\{V = v_k\}$ from $\{V < v_k\}$ using signals is necessary and sufficient for $W_n \to V$ in probability.

Notice that the necessity argument makes no use of our economic assumptions. In order for the maximum bid to close to V, regardless of tastes, preferences, or even motives, the signals must meet a minimal information condition. That this quasi-measurability condition is also sufficient suggests that the auction makes excellent use of the available information, subject to the prohibition against communication.

Notice, too, that the sufficiency argument does not rely heavily on risk-neutrality or identical beliefs. In essence, we only require that the bidders' preferences lead them to compete vigorously when their information is good. This idea is developed in Sections 4 and 5.

[3] W_n converges in probability to V means $P\{|W_n - V| > \varepsilon\} \to 0$ for all positive ε. See Feller [**1**, page 253].

3. THE FORMAL DEVELOPMENT

DEFINITION: Let C and D be events and s a random variable, all in the same probability space. Then by "C can be distinguished from D using s" we mean that either (i) $P(D)=0$ or (ii) $P(C)>0$ and

$$\inf_{A} \frac{P\{s \in A|D\}}{P\{s \in A|C\}}=0. \tag{4}$$

Observe that this definition is distinctly asymmetric. It is possible that C can be distinguished from D using s but not the reverse. It is also possible that the distinction using s can be made in both directions, or in neither direction.

THEOREM 1: *$W_n \to V$ in probability if and only if* (*) *for every k the event $\{V=v_k\}$ can be distinguished from $\{V<v_k\}$ using s_1.*

Theorem 1 will be reformulated and proved under more general assumptions in Sections 4 and 5. The theorem states that certain economic consequences follow if and only if the information structure satisfies condition (*).

To improve our insight into condition (*), consider the special case where the signals are real-valued. Assume that their conditional distributions (given $V=v$) are absolutely continuous with continuous density functions $f_v(\cdot)$. (To interpret absolute continuity in the finitely additive setting, see the Appendix.)

THEOREM 2: *The following three conditions are equivalent.* (i) *$W_n \to V$ in probability.* (ii) *The sets*

$$A^n=\left\{t: \max_{i<k} f_{v_i}(t)/f_{v_k}(t)<\frac{1}{n}\right\}$$

are nonempty. (iii) *For every k there exists $\{t_m\}$ such that for every $i<k$*

$$\lim_{m\to\infty} f_{v_i}(t_m)/f_{v_k}(t_m)=0.$$

PROOF: It is straightforward that (*) is equivalent to (ii) and that (ii) is equivalent to (iii). Hence Theorem 2 follows from Theorem 1.

EXAMPLES: (i) f_v normal with mean v and variance 1. (ii) f_v normal with mean 0 and variance v^2. (iii) $f_v(t)=v\exp(-vt)$ for $t\geqslant 0$. (iv) $f_v(t)=\exp(-t/v)/v$ for $t\geqslant 0$ $(v>0)$.

For the normal densities in examples (i) and (ii), the ratios $f_w(t)/f_v(t)$ are

$$\exp[(w-v)t-\tfrac{1}{2}w^2+\tfrac{1}{2}v^2]$$

and

$$\sqrt{(v/w)}\exp[-\tfrac{1}{2}t^2(1/w^2-1/v^2)],$$

respectively. In both cases, for $w < v$, the ratio tends to 0 as $t \to \infty$. If the signals obey either of these distributions, the winning bid will converge in probability to V.

In example (iii), the ratio $f_w(t)/f_v(t)$ is bounded below by w/v if $w < v$ and by 0 if $w > v$. Convergence fails, but interestingly with this distribution for the signals $\{V = v_k\}$ can be distinguished from $\{V > v_k\}$ using s_1.

In example (iv), it is not possible to distinguish $\{V = v_k\}$ from $\{V > v_k\}$ using s_1, but it is possible to distinguish $\{V = v_k\}$ from $\{V < v_k\}$ using s_1. We check this by observing that for $w < v$, $f_w(t)/f_v(t)$ is equal to $(v/w)\exp[(1/v - 1/w)t]$ which tends to zero as $t \to \infty$. For such signals we conclude that $W_n \to V$ in probability.

Wilson [**3**] proves a theorem which establishes a sufficient condition for $W_n \to V$ almost surely.[4] His setting is somewhat different from ours, but the following corollary is similar to his result.

COLLARY: *Suppose that for each k there exists some t such that for every $i < k$*

$$f_{v_i}(t)/f_{v_k}(t) = 0.$$

Then $W_n \to V$ in probability.

PROOF: In condition (iii) of Theorem 2, let $t_m = t$ for every m.

4. A GENERALIZATION

Let $\mathscr{Z}$ be the set of possible descriptions of the object at auction. Let $\mathscr{S}$ be the set of possible signals. Assume that all bidders agree on $\mathscr{Z}$ and $\mathscr{S}$. Then the relevant "states of the world" are points in the set

$$\Omega = \mathscr{Z} \times \mathscr{S} \times \mathscr{S} \times \dots \tag{5}$$

A typical point $\omega \in \Omega$ is $(z, s_1, s_2, \dots)$ where $z \in \mathscr{Z}$ and s_i is the signal received by bidder $i (i = 1, 2, \dots)$.

Each bidder i is assumed to have a finitely additive probability measure P_i defined over all subsets of Ω.[5] The random variable Z is defined by

$$Z(z, s_1, s_2, \dots) = z. \tag{6}$$

Z represents the unknown true description of the object at auction.

A von Neumann-Morgenstern utility function $u_i(\cdot, \cdot)$ describes the preferences of bidder i. The first argument of u_i is a point $z \in \mathscr{Z}$. The second argument

[4] $W_n \to V$ almost surely means that $P\{W_n \to V\} = 1$. This is a stronger form of convergence than convergence in probability. (See Feller [**1**, page 237].)

[5] In classical probability theory, probability measures are countably additive and defined only on a σ-algebra of subsets of Ω. See Savage [**2**]. Any countably additive probability defined on a σ-algebra of subsets can be extended to a finitely additive probability defined on all subsets.

is a bid amount $b \in R_+$. We normalize the utility functions so that the utility of each losing bidder is zero.

Next we define the "value" (V_i) of Z to bidder i.

(7) $\quad V_i(z) = \sup\{b : u_i(z, b) \geqslant 0\}.$

V_i is the most i would be willing to pay if he knew Z. We regard V_i as a random variable. Finally, we define the value of the object at auction by

(8) $\quad V = \sup\{V_i : i = 1, 2, \ldots\},$

the most any informed bidder would be willing to pay.

Next we impose some substantive assumptions on $\mathscr{Z}$ and on the bidders' preferences. Let I be the set of positive integers.

DEFINITION: The family of probability measures $\{P_i, i \in I\}$ is called *consistent* if there is some finite real number b such that for every $i, j \in I$ and every subset A of Ω

(9) $\quad P_i(A) \leqslant bP_j(A).$

DEFINITION: The family of utility functions $\{u_i, i \in I\}$ is called *consistent* if there exist increasing functions $\underline{u}$ and $\bar{u}$ from R to R satisfying (a) for every positive x, $\underline{u}(x) > 0$, (b) for every negative x, $\bar{u}(x) < 0$, and (c) for every $i \in I$, $z \in \mathscr{Z}$, and $x \in R_+$,

(10) $\quad \underline{u}(V_i(z) - x) \leqslant u_i(z, x) \leqslant \bar{u}(V_i(z) - x).$

ASSUMPTION 1: The set $\mathscr{Z} = \{z_0, \ldots, z_M\}$ is finite. $P_1\{Z = z_K\} > 0$ for each $z_K \in \mathscr{Z}$ and $\Sigma_{z_K \in \mathscr{Z}} P_1\{Z = z_K\} = 1$. Let $v_K = V(z_K)$. Then $\mathscr{Z}$ is ordered so that $0 \leqslant v_0 < v_1 < \ldots < v_M$.

ASSUMPTION 2: Conditional on Z and under P_1, the signals are independent and identically distributed.

ASSUMPTION 3: The family $\{P_i, i \in I\}$ is consistent.

ASSUMPTION 4: The family $\{u_i, i \in I\}$ is consistent.

ASSUMPTION 5: The random variables V_i are bounded below by some real number $l \leqslant 0$.

ASSUMPTION 6: For every $z \in \mathscr{Z}$ and every positive δ, there are infinitely many i for which $V(z) - V_i(z) < \delta$.

ASSUMPTION 7: The minimum permissible bid is zero.

Assumption 1 serves in lieu of a regularity assumption on $P_1(s_1|Z)$. The privacy of the signals and the inability of any bidder to gather "special" information are characterized by Assumption 2.

Assumption 3 plays two major roles. First, it ensures that all bidders will agree on the veracity of statements like "W_n converges in probability to V" and "C can be distinguished from D using s." Without such agreement, Theorem 3 would be devoid of meaning. Second, the assumption guarantees that the competitors can agree on what constitutes convincing evidence.

Assumptions 4 and 5 combine to ensure that no bidder is so risk averse that he is effectively excluded from the auction. Assumption 6 promises that many bidders can make effective use of accurate information. Taken together, Assumptions 3–6 constitute an assumption of vigorous competition.

Assumption 7 guarantees that expected payoffs are finite.

THEOREM 3: *Under Assumptions* 1–7, $W_n \to V$ *in probability if and only if* (*) *for every* K *the event* $\{V = v_K\}$ *can be distinguished from* $\{V < v_K\}$ *using* s_1.

5. PROOF OF THEOREM 3

LEMMA: *Let* $\{a_m\}$ *and* $\{b_m\}$ *be sequences of nonnegative real numbers and* winning bid random variables W_n. Suppose that (*) is satisfied but W_n fails to

$$\sum_{m=1}^{n} a_m \Big/ \sum_{m=1}^{n} b_m < c.$$

Then for some $j(1 \leq j \leq n)$, $a_j/b_j < c$.

PROOF: Suppose to the contrary that for every j, either $b_j = 0$ or $a_j/b_j \geq c$. Then taking a convex combination,

$$c \leq \sum_{\substack{j=1 \\ b_j \neq 0}}^{n} (a_j/b_j) \cdot \left(b_j \Big/ \sum_{m=1}^{n} b_m\right) \leq \sum_{j=1}^{n} a_j \Big/ \sum_{m=1}^{n} b_m$$

contradicting our hypothesis. *Q.E.D.*

PROOF OF THEOREM 3: Suppose $W_n \to V$ in probability. Fix $K > 1$ and choose $\alpha = \frac{1}{2}(v_{K-1} + v_K)$. By conditional independence of the signals, for any i

$$(11) \qquad P_1\{W_n < \alpha \mid V = v_i\} = \prod_{m=1}^{n} P_1\{p_{nm}(s_m) < \alpha \mid V = v_i\}$$

$$= \prod_{m=1}^{n} (1 - P_1\{p_{nm}(s_m) \geq \alpha \mid V = v_i\})$$

$$\leq \exp\left(-\sum_{m=1}^{n} P_1\{p_{nm}(s_m) \geq \alpha \mid V = v_i\}\right).$$

Pick $i < K$ so that $v_i < \alpha$. Since $W_n \to V$ in probability, we have that $P_1\{W_n < \alpha \mid V = v_i\} \to 1$. It follows from this and (11) that for $i < K$

$$\sum_{m=1}^{n} P_1\{p_{nm}(s_m) \geq \alpha \mid V = v_i\} \to 0.$$

Hence,

$$(12)\qquad 0=\lim_{n\to\infty}\sum_{m=1}^{n}\sum_{i=1}^{K-1}P_1\{V=v_i\}P_1\{p_{nm}(s_m)\geqslant\alpha\,|\,V=v_i\}$$

$$=\lim_{n\to\infty}\sum_{m=1}^{n}P_1\{p_{nm}(s_m)\geqslant\alpha\,|\,V<v_K\}.$$

Also,

$$P_1\{W_n<\alpha\,|\,V=v_K\}\geqslant 1-\sum_{m=1}^{n}P_1\{p_{nm}(s_m)\geqslant\alpha\,|\,V=v_K\}.$$

Since the left-hand side of this inequality must tend to zero as $n\to\infty$, it follows that

$$(13)\qquad \liminf_{n\to\infty}\sum_{m=1}^{n}P_1\{p_{nm}(s_m)\geqslant\alpha\,|\,V=v_K\}\geqslant 1.$$

Combining (12) and (13) and replacing each s_m with an s_1 yields

$$(14)\qquad \lim_{n\to\infty}\frac{\sum_{m=1}^{n}P_1\{p_{nm}(s_1)\geqslant\alpha\,|\,V<v_K\}}{\sum_{m=1}^{n}P_1\{p_{nm}(s_1)\geqslant\alpha\,|\,V=v_K\}}=0.$$

Using (14) and the lemma, there must exist positive integers $j=j(n)\leqslant n$ such that

$$(15)\qquad \lim_{n\to\infty}\frac{P_1\{p_{nj}(s_1)\geqslant\alpha\,|\,V<v_K\}}{P_1\{p_{nj}(s_1)\geqslant\alpha\,|\,V=v_K\}}=0.$$

Let $A^n=p_{nj}^{-1}[\alpha,\infty)$. Then we may rewrite (15) as

$$(16)\qquad \lim_{n\to\infty}\frac{P_1\{s_1\in A^n\,|\,V<v_K\}}{P_1\{s_1\in A^n\,|\,V=v_K\}}=0.$$

This proves necessity.

For sufficiency, suppose that for each n some Nash equilibrium n-tuple $(p_{n1},\ldots,p_{nn})$ of bidding strategies is fixed. These determine corresponding winning bid random variables W_n. Suppose that (*) is satisfied but W_n fails to converge in probability to V. We must show that these assumptions lead to a contradiction.

Since W_n fails to converge in probability to V, there must exist some v_K and some positive real numbers α and $\delta<v_K/2$ such that either

$$(17a)\qquad \limsup_{n\to\infty}P_1\{W_n<v_K-2\delta\,|\,V=v_K\}>\alpha$$

or

$$(17b)\qquad \limsup_{n\to\infty}P_1\{W_n>v_K+2\delta\,|\,V=v_K\}>\alpha.$$

Suppose (17a) holds. Let v_K be the largest number in $\{v_0, \ldots, v_M\}$ such that W_n fails to converge to V under the measure $P_1\{\cdot\,|V=v_K\}$.

Since the information condition (*) is assumed satisfied, there is some subset A of $\mathcal{S}$ for which

$$(18)\qquad \frac{P_1\{s_1\in A|V<v_K\}}{P_1\{s_1\in A|V=v_k\}}\leqslant -\frac{\alpha P_1\{V=v_K\}\underline{u}(\delta)}{2b^2P_1\{V<v_K\}\underline{u}(l+\delta-v_K)}.$$

Define a bidding strategy p^* by

$$p^*(s)=\begin{cases} v_K-2\delta & \text{if } s\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Let i be any bidder for whom $V(z_K)-V_i(z_K)<\delta$. Choose $n>i$ satisfying

$$(19)\qquad P_1\{W_n<v_K-2\delta|V=v_K\}>\alpha.$$

Then i's expected winnings from p^* in the n-bidder auction is

$$(20)\qquad E_i[u_i(Z, p^*(s_i))I_{\{W_{ni}<p^*(s_i)\}}]>\alpha P_1\{V=v_K, s_1\in A\}\underline{u}(\delta)/(2b) + bP_1\{V>v_K, s_1\in A, W_{ni}<v_K-2\delta\}\underline{u}(l+\delta-v_K).$$

By our choice of v_K and the optimality of p_{ni},[6]

$$(21)\qquad \lim_{n\to\infty} P_1\{W_{ni}<v_K-2\delta|V>v_K\}=0.$$

So for large n satisfying (19), strategy p^* offers bidder i expected utility of at least $\alpha P_1\{V=v_K, s_1\in A\}\underline{u}(\delta)/(2b)$, a fixed positive real number.

In a Nash equilibrium, each bidder will have nonnegative expected utility. (Otherwise, the strategy of always bidding zero will prove superior to the strategy actually used.) Also, using Assumptions 3, 4, and 7, and defining

$$e_{ni}=E_i[u_i(Z, p_{ni}(s_i))I_{\{W_{ni}<p_{ni}(s_i)\}}],$$

we can deduce that

$$(22)\qquad \sum_{i=1}^{n} e_{ni}\leqslant bE_1[\bar{u}(V)].$$

Let $j(n)$ be the number of bidders among the first n for whom $V(z_K)-V_i(z_K)<\delta$. Then for every n, it follows from (22) that there is some bidder $i=i(n)$ among the $j(n)$ for whom

$$e_{ni}\leqslant bE_1[\bar{u}(V)]/j(n).$$

By Assumption 6, $j(n)$ tends to infinity. So $e_{ni}\to 0$. Hence for some large n satisfying (19), bidder $i(n)$ could do better with p^* than with p_{ni}. This contradicts

[6] Otherwise, the strategy given $p^*_{ni}(s)=\min(p_{ni}(s), v_k-2\delta)$ is superior to p_{ni} for some large n.

our assumptions and so proves that (17a) cannot hold in a Nash equilibrium satisfying the information condition (*).

Next suppose that for every positive δ, $P_1\{W_n < V-\delta\} \to 0$. Using this, (15b), and Assumptions 3 and 4, one can show that for some n

$$\sum_{i=1}^{n} E_i[u_i(Z, p_{ni}(s_i))I_{\{W_{ni}<p_{ni}(s_i)\}}]<0. \tag{23}$$

Hence, some bidder could improve his expected payoff by bidding zero. So contrary to our hypothesis, neither (15a) nor (15b) can hold. *Q.E.D.*

The proof of Theorem 1 is similar but does not (in the sufficiency part) require that the v_k chosen be maximal. Hence finiteness plays no role in Theorem 1.

6. PARTIAL INFORMATION

It may often happen that the possible signals bear only on some aspect X of the description Z. In such cases, by redefining the object at auction to be a lottery, Theorems 1–3 continue to be useful.

To illustrate this principle, consider again the framework of Sections 1–3. Assume that X is a sufficient statistic for the vector $(X, s_1, s_2 \ldots)$. In this finitely additive framework we take this to mean that the conditional distribution of V given X exists and equals the conditional distribution of V given $(X, s_1, s_2 \ldots)$. Assume that the support of X is $\{x_1, x_2, \ldots\}$ and define $v_j = E[V|X=x_j]$. Finally, assume that

$$v_1 < v_2 < \ldots. \tag{24}$$

THEOREM 4: *$W_n \to E[V|X]$ in probability if and only if (**) for every k the event $\{E[V|X]=v_k\}$ can be distinguished from $\{E[V|X]<v_k\}$ using s_1.*

PROOF: Let $\bar{V}=E[V|X]$. Bidder i seeks to maximize his expected payoff, which is

$$\begin{aligned} &E[(V-p_{ni}(s_i))I_{\{W_{ni}<p_{ni}(s_i)\}}] \\ &\quad = E[E[(V-p_{ni}(s_i))I_{\{W_{ni}<p_{ni}(s_i)\}}|X, s_1, \ldots, s_n]] \\ &\quad = E[(\bar{V}-p_{ni}(s_i))I_{\{W_{ni}<p_{ni}(s_i)\}}]. \end{aligned}$$

Thus each player bids as though $\bar{V}$ were the unknown value of the object at auction. Apply Theorem 1. *Q.E.D.*

7. CONCLUSIONS

As Wilson [**3**] has discussed, there are many facets to convergence results of the kind we have presented. First, we have shown the identity of price and value for large tender auctions with appropriate information structures. Second, we have shown that the auction game results in private information being aggregated in the equilibrium price. And third, this aggregation of information is reasonably efficient. The information requirements for convergence are the minimum possible requirements (as discussed in Section 2).

Stanford University

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APPENDIX

The development of Theorems 1, 3, and 4 have relied exclusively on finitely additive subjective probability measures with unrestricted domains. The theorems hold equally well when measurability conditions and measures with restricted domains are introduced.

Let $\mathcal{A}$ and $\mathcal{F}$ be σ-algebras of subsets of $\mathcal{S}$ and Ω, respectively, and let $\mathcal{B}$ be the Borel sets. Let us require that each signal (s_k, $k = 1, 2, \ldots$) be a measurable function from Ω to $\mathcal{S}$ and that the bidding strategies $p_{nk}: \mathcal{S} \to R$ be Borel measurable. These are the standard requirements in models with countably additive probability measures. Then all proofs presented in this paper remain valid with the understanding that the sets A^n are elements of $\mathcal{A}$.

Theorem 2 may be interpreted in either the classical setting described in the preceding paragraph or in a finitely additive setting. For the latter, however, one must specify a finitely additive extension of Lesbeque measure defined on all sets of real numbers. Then the requirement is that the conditional distributions of the signals have continuous density functions with respect to the specified extension, i.e.,

$$P\{s \in A \mid V = v\} = \int_A f_v(x)\mu(dx)$$

where μ is the specified extension of Lesbeque measure.

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