The Technion school lecture notes: the moving plane method, or doing PDEs in a café

Lenya Ryzhik*

September 15, 2014

1 Act I. The maximum principle enters

We will have two main characters in these notes: the maximum principle and the sliding method. The latter has a twin, the moving plane method – they are often so indistinguishable that we will count them as one character. They will be introduced separately, and then blended together to prove the symmetry properties of the solutions of elliptic equations. In this introductory section, we recall the what the maximum principle is. This material is very standard and can be found in almost any undergraduate or graduate PDE text, such as the books by Evans [7], Han and Lin [10], and Pinchover and Rubinstein [11].

We will consider equations of the form

$$\begin{aligned} &\Delta u + F(x,u) = 0 \text{ in } \Omega, \\ &u = g \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary. There are many applications where such problems appear. We will mention just two – one is in the realm of probability theory, where $u(x)$ is an equilibrium particle density for some stochastic process, and the other is in classical physics. In this short course, we will mostly appeal to the physical interpretation rather than probabilistic, because of the time restrictions. In this context, one may think of $u(x)$ as the equilibrium temperature distribution inside the domain Ω. The term $F(x,u)$ corresponds to the heat sources or sinks inside Ω, while $g(x)$ is the (prescribed) temperature on the boundary $\partial\Omega$. The maximum principle reflects a basic observations known to any child – first, if $F(x,u) = 0$ (there are neither heat sources nor sinks), or if $F(x,u) \leq 0$ (there are no heat sources but there may be heat sinks), the temeprature inside Ω may not exceed that on the boundary – without a heat source inside a room, you can not heat the interior of a room to a warmer temperature than its maximum on the boundary. Second, if one considers two prescribed boundary conditions and heat sources such that

$$g_1(x) \leq g_2(x) \text{ and } F_1(x,u) \leq F_2(x,u),$$

*Department of Mathematics, Stanford University, Stanford CA, 94305, USA; ryzhik@math.stanford.edu

then the corresponding solutions will satisfy $u_1(x) \le u_2(x)$ – stronger heating leads to warmer rooms. It is surprising how such mundane considerations may lead to rather beautiful mathematics.

The maximum principle in complex analysis

Most mathematicians are first introduced to the maximum principle in a complex analysis course. Recall that the real and imaginary parts of an analytic function $f(z)$ have the following property.

Proposition 1.1 *Let $f(z) = u(z) + iv(z)$ be an analytic function in a smooth bounded domain $\Omega \in \mathbb{C}$, continuous up to the boundary Ω. Then $u(z) = Ref(z)$, $v(z) = Imf(z)$ and $w(z) = |f(z)|$ all attain their respective maxima over Ω on its boundary. In addition, if any of these functions attains its maximum inside Ω, it has to be equal identically to a constant in Ω.*

This proposition is usually proved via the mean-value property of analytic functions (which itself is a consequence of the Cauchy integral formula): for any disk $B(z_0, r)$ contained in Ω we have

$$f(z_0) = \int_0^{2\pi} f(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \quad u(z_0) = \int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \quad v(z_0) = \int_0^{2\pi} v(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \tag{1.2}$$

and

$$w(z) \le \int_0^{2\pi} w(z_0 + re^{i\theta})\frac{d\theta}{2\pi}. \tag{1.3}$$

It is immediate to see that (1.2) and (1.3) imply that if one of the functions u, v and w attains a local maximum at a point z_0 inside Ω, it has to be equal to a constant in a disk around z_0. Thus, the set where it attains its maximum is both open and closed, hence it is all of Ω and this function equals identically to a constant.

The above argument while incredibly beautiful and simple, relies very heavily on the rigidity of analytic functions that is reflected in the mean-value principle. The same rigidity is reflected in the fact that the real and imaginary parts of an analytic function satisfy the Laplace equation

$$\Delta u = 0, \quad \Delta v = 0,$$

while $w^2 = u^2 + v^2$ is subharmonic: it satisfies

$$\Delta(w^2) \ge 0.$$

We will see next that the mean-value principle is associated to the Laplace equation and not analyticity in itself, and thus applies to harmonic (and, in a modified way, to subharmonic) functions in higher dimensions as well. This will imply the maximum principle for solutions of the Laplace equation in an arbitrary dimension. One may ask whether a version of the mean-value principle also holds for the solutions of general elliptic equations rather than just for the Laplace equation – the answer is "yes if understood properly", and the mean value property survives as the general elliptic regularity theory, an equally beautiful sister of the complex analysis which is occasionally misunderstood as "technical". We will not discuss it here.

Interlude: a probabilistic connection digression

Another good way to understand how the Laplace equation comes about, as well as many of its properties, including the maximum principle, is via its connection to the Brownian motion. It is easy to understand in terms of the discrete equations, which requires only very elementary probability theory. Consider a system of many particles on the n-dimensional integer lattice $\mathbb{Z}^n$. They all perform a symmetric random walk: at each integer time $t = k$ each particle jumps (independently from the others) from its current site $x \in \mathbb{Z}^n$ to one of its $2n$ neighbors, $x \pm e_k$ (e_k is the unit vector in the direction of the x_k-axis), with equal probability $1/(2n)$. At each step we may also insert new particles, the average number of inserted (or eliminated) particles per unit time at each site is $F(x)$. Let now $u_m(x)$ be the average number of particles at the site x at time m. The balance equation for $u_{m+1}(x)$ is

$$u_{m+1}(x) = \frac{1}{2n} \sum_{k=1}^{n} [u_n(x + e_k) + u_n(x - e_k)] + F(x).$$

If the system is in an equilibrium, so that $u_{n+1}(x) = u_n(x)$ for all x, then $u(x)$ (dropping the subscript n) satisfies the discrete equation

$$\frac{1}{2n} \sum_{k=1}^{n} [u(x + e_k) + u(x - e_k) - 2u(x)] + F(x) = 0.$$

If we now take a small mesh size h, rather than one, the above equation becomes

$$\frac{1}{2n} \sum_{k=1}^{n} [u(x + he_k) + u(x - he_k) - 2u(x)] + F(x) = 0.$$

Doing a Taylor expansion in h leads to

$$\frac{h^2}{2n} \sum_{k=1}^{n} \frac{\partial^2 u(x)}{\partial x_k^2} + F(x) = \text{lower order terms}.$$

Adjusting $F(x) \to h^2/(2n)F(x)$ – this prevents us from inserting or removing too many particles, we arrive, in the limit $h \downarrow 0$, at

$$\Delta u + F(x) = 0. \tag{1.4}$$

In this model, we interpret $u(x)$ as the local particle density, and $F(x)$ as the rate at which the particles are inserted (if $F(x) > 0$), or removed (if $F(x) < 0$). When equation (1.4) is posed in a bounded domain Ω we need to supplement it with a boundary condition

$$u(x) = g(x) \text{ on } \partial\Omega.$$

Here, it means the particle density on the boundary is prescribed – the particles are injected or removed if there "too many" or "too little" particles at the boundary, to keep $u(x)$ at the given prescribed value $g(x)$.

The mean value property for sub-harmonic and super-harmonic functions

We now return to the world of analysis. A function $u(x)$, $x \in \Omega \subset \mathbb{R}^n$ is harmonic if it satisfies the Laplace equation

$$\Delta u = 0 \text{ in } \Omega. \tag{1.5}$$

This is equation (1.1) with $F \equiv 0$, thus a harmonic function describes a heat distribution in Ω with neither heat sources nor sinks in Ω. We say that u is sub-harmonic if it satisfies

$$-\Delta u \leq 0 \text{ in } \Omega, \tag{1.6}$$

and it is super-harmonic if it satisfies

$$-\Delta u \geq 0 \text{ in } \Omega, \tag{1.7}$$

In other words, a sub-harmonic function satisfies

$$\Delta u + F(x) = 0, \text{ in } \Omega,$$

with $F(x) \leq 0$ – it describes a heat distribution in Ω with only heat sinks present, and no heat sources, while a super-harmonic function satisfies

$$\Delta u + F(x) = 0, \text{ in } \Omega,$$

with $F(x) \geq 0$ – it describes an equilibrium heat distribution in Ω with only heat sources present, and no sinks.

Exercise 1.2 *Give an interpretation of the sub-harmonic and super-harmonic functions in terms of particle probability densities.*

Note that any sub-harmonic function in one dimension is convex:

$$-u'' \leq 0,$$

and then, of course, for any $x \in \mathbb{R}$ and any $l > 0$ we have

$$u(x) \leq \frac{1}{2}\left(u(x+l) + u(x-l)\right) \leq \frac{1}{2l}\int_{x-l}^{x+l} u(y)dy.$$

The following generalization to sub-harmonic functions in higher dimensions shows that locally $u(x)$ is bounded from above by its spatial average. A super-harmonic function will be locally above its spatial average. A word on notation: for a set S we denote by $|S|$ its volume (or area), and, as before, ∂S denotes its boundary.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $B(x,r)$ be a ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ contained in Ω. Assume that the function $u(x)$ satisfies*

$$-\Delta u \leq 0, \tag{1.8}$$

for all $x \in \Omega$ and that $u \in C^2(\Omega)$. Then we have

$$u(x) \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} u dy, \quad u(x) \leq \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u dS. \tag{1.9}$$

If the function $u(x)$ is super-harmonic:

$$-\Delta u \geq 0, \tag{1.10}$$

for all $x \in \Omega$ and that $u \in C^2(\Omega)$. Then we have

$$u(x) \geq \frac{1}{|B(x,r)|}\int_{B(x,r)} u dy, \quad u(x) \geq \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u dS. \tag{1.11}$$

Moreover, if the function u is harmonic: $\Delta u = 0$, then we have equality in both inequalities in (1.9).

One reason to expect the mean-value property is from physics – if Ω is a ball with no heat sources, it is natural to expect that the temperature in the center of the ball may not exceed the average temperature over any sphere concentric with the ball. The opposite is true if there are no heat sinks (this is true for a super-harmonic function). Another can be seen from the discrete version of inequality (1.8):

$$u(x) \leq \frac{1}{2n}\sum_{j=1}^{n}(u(x+he_j)+u(x-he_j)).$$

Here, h is the mesh size, and e_j is the unit vector in the direction of the coordinate axis for x_j. This discrete equation says exactly that the value $u(x)$ is smaller than the average of the values of u at the neighbors of the point x on the lattice with mesh size h, which is similar to the statement of Theorem 1.3 (though there is no meaning to "nearest" neighbor in the continuous case).

Proof. We will only treat the case of a sub-harmonic function. Let us fix the point $x \in \Omega$ and define

$$\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(z)dS(z). \tag{1.12}$$

It is easy to see that, since $u(x)$ is continuous, we have

$$\lim_{r\downarrow 0}\phi(r) = u(x). \tag{1.13}$$

Therefore, we would be done if we knew that $\phi'(r) \geq 0$ for all $r > 0$ (and such that the ball $B(x,r)$ is contained in Ω). To this end, passing to the polar coordinates $z = x + ry$, with $y \in \partial B(0,1)$, we may rewrite (1.12) as

$$\phi(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+ry)dS(y).$$

Then, differentiating in r gives

$$\phi'(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} y\cdot\nabla u(x+ry)dS(y).$$

Going back to the z-variables gives

$$\phi'(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{1}{r}(z-x)\cdot\nabla u(z)dS(z) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial\nu}dS(z).$$

Here, we used the fact that the outward normal to $B(x,r)$ at a point $z \in \partial B(x,r)$ is

$$\nu = (z-x)/r.$$

Using Green's formula

$$\int_U \Delta g dy = \int_U \nabla \cdot (\nabla g) = \int_{\partial U} (\nu \cdot \nabla g) = \int_{\partial U} \frac{\partial g}{\partial \nu} dS,$$

gives now

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y) dy \geq 0.$$

It follows that $\phi(r)$ is a non-decreasing function of r, and then (1.13) implies that

$$u(x) \leq \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u dS, \tag{1.14}$$

which is the second identity in (1.9).

In order to prove the first equality in (1.9) we use the polar coordinates once again:

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} u dy = \frac{1}{|B(x,r)|} \int_0^r \left(\int_{\partial B(x,s)} u dS \right) ds \geq \frac{1}{|B(x,r)|} \int_0^r u(x) n\alpha(n) s^{n-1} ds$$
$$= u(x) \frac{n\alpha(n) r^n}{n\alpha(n) r^n} = u(x).$$

We used above two facts: first, the already proved identity (1.14) about averages on spherical shells, and, second, that the area of an $(n-1)$-dimensional unit sphere is $n\alpha(n)$. Now, the proof of (1.9) is complete. The proof of the mean-value property for subharmonic functions works identically. □

The weak maximum principle

The first consequence of the mean value property is the maximum principle that says that a sub-harmonic function attains its maximum over any domain on the boundary and not inside the domain[1]. Once again, in one dimension this is obvious: a smooth convex function does not have any local maxima.

Theorem 1.4 (**The weak maximum principle**) *Let $u(x)$ be a sub-harmonic function in a connected domain Ω and assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then*

$$\max_{x \in \Omega} u(x) = \max_{y \in \partial\Omega} u(y). \tag{1.15}$$

Moreover, if $u(x)$ achieves its maximum at a point x_0 in the interior of Ω, then $u(x)$ is identically equal to a constant in Ω. Similarly, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a super-harmonic function in Ω, then

$$\min_{x \in \Omega} u(x) = \min_{y \in \partial\Omega} u(y). \tag{1.16}$$

Moreover, if $u(x)$ achieves its minimum at a point x_0 in the interior of Ω, then $u(x)$ is identically equal to a constant in Ω.

[1] A sub-harmonic function is nothing but the heat distribution in a room without heat sources, hence it is very natural that it attains its maximum on the boundary (the walls of the room)

Proof. Again, we only treat the case of a sub-harmonic function. Suppose that $u(x)$ attains its maximum at an interior point $x_0 \in \Omega$, and set

$$M = u(x_0).$$

Then, for any $r > 0$ sufficiently small (so that the ball $B(x_0, r)$ is contained in Ω), we have

$$M = u(x) \leq \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u dy \leq M,$$

with the equality above holding only if $u(y) = M$ for all y in the ball $B(x_0, r)$. Therefore, the set S of points where $u(x) = M$ is open. Since $u(x)$ is continuous, this set is also closed. Since S us both open and closed in Ω, and Ω is connected, it follows that $S = \Omega$, hence $u(x) = M$ at all points $x \in \Omega$. □

We will often have to deal with slightly more general operators than the Laplacian, of the form

$$Lu = \Delta u(x) + c(x)u. \tag{1.17}$$

We may ask the same question: when is it true that the inequality

$$-\Delta u(x) - c(x)u(x) \leq 0 \text{ in } \Omega \tag{1.18}$$

guarantees that $u(x)$ attains its maximum on the boundary of Ω? It is certainly not always true that any function satisfying (1.18) attains its maximum on he boundary: consider the function $u(x) = \sin x$ on the interval $(0, \pi)$. It satisfies

$$u''(x) + u(x) = 0, \quad u(0) = u(\pi) = 0, \tag{1.19}$$

but achieves its maximum at $x = \pi/2$. In order to understand this issue a little better, consider the following exercise.

Exercise 1.5 *Consider the boundary value problem*

$$-u'' - au = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

with a given non-negative function $f(x)$, and a constant $a \geq 0$. Show that if $a < \pi^2$, then the function $u(x)$ is positive on the interval $(0, 1)$.

One possible answer to our question below (1.18) comes from our childish attempts at physics: if $u(x) \geq 0$, we may interpret $u(x)$ as a heat distribution in Ω. Then, $u(x)$ should not be able to attain its maximum inside Ω if there are no heat sources in Ω. If $u(x)$ satisfies (1.18), the only possible heat source is $c(x)u(x)$. Keeping in mind that $u(x) \geq 0$, we see that absence of heat sources is equivalent to the condition $c(x) \leq 0$ (this, in particular, rules out the counterexample (1.19)). Mathematically, this is reflected in the following.

Corollary 1.6 *Suppose that $c(x) \leq 0$ in Ω, and a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $u \geq 0$ and*

$$\Delta u(x) + c(x)u(x) \geq 0 \text{ in } \Omega.$$

Then u attains its maximum on $\partial\Omega$. Moreover, if $u(x)$ attains its maximum inside Ω then u is identically equal to a constant.

Proof. A non-negative function $u(x)$ that satisfies (1.18) is sub-harmonic, and application of Theorem 1.4 finishes the proof.

Exercise 1.7 *Give an interpretation of this result in terms of particle densities.*

2 Act II. The moving plane method

2.1 The isoperimeteric inequality and sliding

We now bring in our second set of characters, the moving plane and sliding methods. As an introduction, we show how the sliding method can work alone, without the maximum principle. Maybe the simplest situation when the sliding idea proves useful is in an elegant proof of the isoperimetric inequality. We follow here the proof given by X. Cabré in [4][2]. The isoperimetric inequality says that among all domains of a given volume the ball has the smallest perimeter.

Theorem 2.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Then,*

$$\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} \geq \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}, \tag{2.1}$$

where B_1 is the open unit ball in $\mathbb{R}^n$, $|\Omega|$ denotes the measure of Ω and $|\partial\Omega|$ is the perimeter of Ω (the $(n-1)$-dimensional measure of the boundary of Ω). In addition, equality in (2.1) holds if and only if Ω is a ball.

A technical aside: the area formula

The proof will use the area formula (see [8] for the proof), a generalization of the usual change of variables formula in the multi-variable calculus. The latter says that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth one-to-one map (a change of variables), then

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^n} g(f^{-1}(y))dy. \tag{2.2}$$

For general maps we have

Theorem 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each function $g \in L^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} g(x) \right] dy. \tag{2.3}$$

We will, in particular, need the following corollary.

Corollary 2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each measurable set $A \subset \mathbb{R}^n$ we have*

$$|f(A)| \leq \int_A Jf(x)dx. \tag{2.4}$$

[2] Readers with ordinary linguistic powers may consult [5].

Proof. For a given set S we define its characteristic functions as

$$\chi_S(x) = \begin{cases} 1, & \text{for } x \in S, \\ 0, & \text{for } x \notin S, \end{cases}$$

We use the area formula with $g(x) = \chi_A(x)$:

$$\begin{aligned} \int_A Jf(x)dx &= \int_{\mathbb{R}^n} \chi_A(x) Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} \chi_A(x) \right] dy \\ &= \int_{\mathbb{R}^n} [\#x \in A: \ f(x) = y] \, dy \geq \int_{\mathbb{R}^n} \chi_{f(A)}(y) dy = |f(A)|, \end{aligned}$$

and we are done. $\square$

A more general form of this corollary is the following.

Corollary 2.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each nonnegative function $p \in L^1(\mathbb{R}^n)$ and each measurable set A, we have*

$$\int_{f(A)} p(y)dy \leq \int_A p(f(x)) Jf(x)dx. \tag{2.5}$$

Proof. The proof is as in the previous corollary. This time, we apply the area formula to the function $g(x) = p(f(x))\chi_A(x)$:

$$\begin{aligned} &\int_A p(f(x))Jf(x)dx = \int_{\mathbb{R}^n} \chi_A(x) p(f(x)) Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} \chi_A(x) p(f(x)) \right] dy \\ &= \int_{\mathbb{R}^n} [\#x \in A: \ f(x) = y] \, p(y)dy \geq \int_{f(A)} p(y)dy, \end{aligned}$$

and we are done. $\square$

The proof of the isoperimetric inequality

We now proceed with Cabré's proof of the isoperimetric inequality in Theorem 2.1.

Step 1: sliding. Let $v(x)$ be the solution of the Neumann problem

$$\begin{aligned} &\Delta v = k, \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \end{aligned} \tag{2.6}$$

Integrating the first equation above and using the boundary condition, we obtain

$$k|\Omega| = \int_\Omega \Delta v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = |\partial\Omega|.$$

Hence, solution exists only if

$$k = \frac{|\partial\Omega|}{|\Omega|}. \tag{2.7}$$

It is a classical result that with this particular value of k there exist infinitely many solutions that differ by addition of an arbitrary constant. We let v be any of them. As Ω is a smooth domain, v is also smooth.

Let Γ_v be the lower contact set of v, that is, the set of all $x \in \Omega$ such that the tangent hyperplane to the graph of v at x lies below that graph in all of $\bar{\Omega}$. More formally, we define

$$\Gamma_v = \{x \in \Omega : \ v(y) \geq v(x) + \nabla v(x) \cdot (y - x) \text{ for all } y \in \bar{\Omega}.\} \tag{2.8}$$

The crucial observation is that

$$B_1 \subset \nabla v(\Gamma_v). \tag{2.9}$$

Here, B_1 is the open unit ball centered at the origin. The geometric reason for this is as follows: take any $p \in B_1$ and consider the graphs of the functions

$$r_c(y) = p \cdot y + c.$$

We will now slide this plane upward – we will start with a "very negative" c, and start increasing it, moving the plane up. Note that there exists $M > 0$ so that if $c < -M$, then

$$r_c(y) < v(y) - 100 \text{ for all } y \in \bar{\Omega},$$

that is, the plane is below the graph in all of Ω, and, on the other hand,

$$r_c(y) > v(y) + 100 \text{ for all } y \in \bar{\Omega},$$

in other words, the plane is above the graph in all of Ω if $c > M$. Let

$$\alpha = \sup\{c \in \mathbb{R} : \ r_c(y) < v(y) \text{ for all } y \in \bar{\Omega}\}$$

be the largest c so that the plane lies below the graph of v in all of Ω. It is easy to see that the plane $r_\alpha(y) = p \cdot y + \alpha$ has to touch the graph of v: there exists a point $y_0 \in \bar{\Omega}$ such that $r_\alpha(y_0) = v(y_0)$ and

$$r_\alpha(y) \leq v(y) \text{ for all } y \in \bar{\Omega}. \tag{2.10}$$

Furthermore, the point y_0 can not lie on the boundary $\partial\Omega$. Indeed, for all $y \in \partial\Omega$ we have

$$\left|\frac{\partial r_c}{\partial \nu}\right| = |p \cdot \nu| \leq |p| < 1 \text{ and } \frac{\partial v}{\partial \nu} = 1.$$

This means that if $r_c(y) = v(y)$ for some c, and y is on the boundary $\partial\Omega$, then there is a neighborhood $U \in \Omega$ of y such that $r_c(y) > v(y)$ for all $y \in U$. Comparing to (2.10), we see that $c \neq \alpha$, hence it is impossible that $y_0 \in \partial\Omega$. Thus, y_0 is an interior point of Ω, and, moreover, the graph of $r_\alpha(y)$ is the tangent plane to v at y_0. In particular, we have $\nabla v(y_0) = p$, and (2.10) implies that y_0 is in the contact set of v: $y_0 \in \Gamma_v$. We have now shown the inclusion (2.9): $B_1 \subset \nabla v(\Gamma_v)$. Note that the only information about the function $v(x)$ we have used so far is the Neumann boundary condition

$$\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega,$$

but not the Poisson equation for v in Ω.

Step 2: using the area formula. A trivial consequence of (2.9) is that

$$|B_1| \leq |\nabla v(\Gamma_v)|. \tag{2.11}$$

Now, we will apply Corollary 2.3 to the map $\nabla v : \ \Gamma_v \to \nabla v(\Gamma_v)$, whose Jacobian is $|\det[D^2 v]|$.

Exercise 2.5 *Show that if Γ_v is the contact set of a smooth function $v(x)$, then $\det[D^2v]$ is non-negative for $x \in \Gamma_v$, and, moreover, all eigenvalues of D^2v are nonnegative on Γ_v.*

As $\det[D^2v]$ is non-negative for $x \in \Gamma_v$, we conclude from Corollary 2.3 and (2.11) that

$$|B_1| \leq |\nabla v(\Gamma_v)| \leq \int_{\Gamma_v} \det[D^2v(x)]dx. \tag{2.12}$$

It remains to notice that by the classical arithmetic mean-geometric mean inequality applied to the (nonnegative) eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix $D^2v(x)$, $x \in \Gamma_v$ we have

$$\det[D^2v(x)] = \lambda_1\lambda_2 \dots \lambda_n \leq \left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n}\right)^n. \tag{2.13}$$

However, by a well-known formula from linear algebra,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{Tr}[D^2v],$$

and, moreover, $\mathrm{Tr}[D^2v]$ is simply the Laplacian Δv. This gives

$$\det[D^2v(x)] \leq \left(\frac{\mathrm{Tr}[D^2v]}{n}\right)^n = \left(\frac{\Delta v}{n}\right)^n \text{ for } x \in \Gamma_v. \tag{2.14}$$

However, v is the solution of (2.15):

$$\begin{aligned} &\Delta v = k, \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \end{aligned} \tag{2.15}$$

with

$$k = \frac{|\partial\Omega|}{|\Omega|}.$$

Going back to (2.12), we deduce that

$$|B_1| \leq \int_{\Gamma_v} \det[D^2v(x)]dx \leq \int_{\Gamma_v} \left(\frac{\Delta v}{n}\right)^n dx \leq \left(\frac{k}{n}\right)^n |\Gamma_v| = \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Gamma_v| \leq \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Omega|.$$

However, for the unit ball we have $|\partial B_1| = n|B_1|$, hence the above implies

$$\frac{|\partial B_1|^n}{|B_1|^{n-1}} \leq \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}, \tag{2.16}$$

which is nothing but the isoperimetric inequality (2.1).

In order to see that the inequality in (2.16) is strict unless Ω is a ball, we observe that it follows from the above argument that for the equality to hold in (2.16) we must have equality in (2.13), and, in addition, Γ_v has to coincide with Ω. This means that for each $x \in \Omega$ all eigenvalues of the matrix $D^2v(x)$ are equal to each other. That is, $D^2v(x)$ is a multiple of the identity matrix for each $x \in \Omega$.

Exercise 2.6 *Show that if $v(x)$ is a smooth function such that*

$$\frac{\partial^2 v(x)}{\partial x_i^2} = \frac{\partial^2 v(x)}{\partial x_j^2},$$

for all $1 \leq i, j \leq n$ and $x \in \Omega$, and

$$\frac{\partial^2 v(x)}{\partial x_i \partial x_j} = 0,$$

for all $i \neq j$ and $x \in \Omega$, then there exists $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$, so that

$$v(x) = b\left[(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2\right], \tag{2.17}$$

for all $x \in \Omega$.

Our function $v(x)$ satisfies the assumptions of Exercise 2.6, hence it must have the form (2.17). Finally, the boundary condition $\partial v / \partial \nu = k$ on $\partial \Omega$ implies that Ω is a ball centered at the point $a \in \mathbb{R}^n$. □

3 Act III. Their first meeting

The maximum principle returns, and we study it in a slightly greater depth. At the end of this act the maximum principle and the moving plane method are introduced to each other.

The strong maximum principle

Let us begin with the following exercises.

Exercise 3.1 *Show that if the function $u(x)$ satisfies an ODE of the form*

$$u'' + c(x)u = 0, \quad a < x < b, \tag{3.1}$$

and $u(x_0) = 0$ for some $x_0 \in (a, b)$ then u can not attain its maximum (or minimum) over the interval (a, b) at the point x_0.

This exercise is relatively easy – one has to think about the initial value problem for (3.1) with the data $u(x_0) = u'(x_0) = 0$. Now, look at the next exercise, which is slightly harder.

Exercise 3.2 *Show that, once again, in one dimension, if $u(x)$, $x \in \mathbb{R}$ satisfies an ODE of the form*

$$u'' + c(x)u \geq 0, \quad a < x < b,$$

and $u(x_0) = 0$ for some $x_0 \in (a, b)$ then u can not attain its maximum over the interval (a, b) at the point x_0.

A slightly more delicate argument leads to the strong maximum principle whose proof we omit for the sake of time (it can be found in any PDE textbook, such as [7], [10] or [11]).

Theorem 3.3 *(The Strong maximum principle) Assume that $c(x) \le 0$ in Ω, and the function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\Delta u(x) + c(x)u(x) \ge 0 \text{ in } \Omega.$$

Then, if the maximum of u over $\bar{\Omega}$ is non-negative, it may only be attained on $\partial\Omega$ unless u is a constant.

Without going into the proof of the strong maximum principle, we mention that it relies crucially on the Hopf lemma which guarantees that the point on the boundary where the maximum is attained is not a critical point of u.

Theorem 3.4 *(The Hopf Lemma) Let $B = B(y, r)$ be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$, and assume that $c(x) \le 0$ in B. Suppose that a function $u \in C^2(B) \cap C(B \cup x_0)$ satisfies*

$$\Delta u(x) + c(x)u(x) \ge 0 \text{ in } B,$$

and that $u(x) < u(x_0)$ for any $x \in B$ and $u(x_0) \ge 0$. Then, we have

$$\liminf_{t \to 0^+} \frac{u(x_0) - u(x_0 - tm)}{t} > 0$$

for each outward direction m: $m \cdot \nu(x_0) > 0$.

Remark 3.5 If the normal derivative exists at x_0 then $\dfrac{\partial u}{\partial \nu}(x_0) < 0$.

The restriction $c(x) \le 0$ may be eliminated if we know already that $u \le 0$, and only need to eliminate the possibility that $u(x) = 0$ for some x inside Ω: the next corollary applies independent of the sign of $c(x)$. Note that this statement is more delicate than our baby physics arguments – we make no assumption on whether $c(x)u(x)$ is a heat source or sink.

Corollary 3.6 *(Another version of the strong maximum principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\Delta u(x) + c(x)u(x) \ge 0 \text{ in } \Omega, \tag{3.2}$$

with $u \le 0$ in Ω, with a bounded function $c(x)$. Then either $u \equiv 0$ in Ω or $u < 0$ in Ω. Similarly, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\Delta u(x) + c(x)u(x) \le 0 \text{ in } \Omega, \tag{3.3}$$

with $u \ge 0$ in Ω, with a bounded function $c(x)$. Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

Proof. If $c(x) \le 0$, this follows directly from the strong maximum principle. As $u \le 0$ in Ω, the inequality (3.2) implies that, for any $M > 0$ we have

$$-\Delta u - c(x)u - Mu \ge -Mu \ge 0.$$

However, if $M > \|c\|_{L^\infty(\Omega)}$ then the zero order coefficient satisfies

$$c_1(x) = c(x) - M \le 0,$$

hence we may conclude, again from the strong maximum principle that either $u < 0$ in Ω or $u \equiv 0$ in Ω. The proof in the case (3.3) holds is identical. □

It is easy to understand the strong maximum principle from the point of view of (3.3) – in this case, a non-negative $u(x)$ can be interpreted as a particle density, and $c(x)u(x)$ is the rate at which the particles are inserted (where $c(x) > 0$) or eliminated (where $c(x) < 0$). The strong maximum principle says that no matter how negative $c(x)$ is, the random particles will always access any point in the domain with a positive density.

Separating sub- and super-solutions

A very common use of the strong maximum principle is to re-interpret it as the "untouchability" of a sub-solution and a super-solution of a linear or nonlinear problem – the basic principle underlying what we will see below. Assume that the functions $u(x)$ and $v(x)$ satisfy

$$\Delta u + f(x,u) \geq 0, \quad \Delta v + f(x,v) \leq 0 \text{ in } \Omega. \tag{3.4}$$

We say that $u(x)$ is a sub-solution, and $v(x)$ is a super-solution. Assume that, in addition, we know that

$$u(x) \leq v(x) \text{ for all } x \in \Omega, \tag{3.5}$$

that is, the sub-solution sits below the super-solution. In this case, we are going to rule out the possibility that they touch inside Ω (they can touch on the boundary, however): there can not be an $x_0 \in \Omega$ so that $u(x_0) = v(x_0)$. If the function $f(x,s)$ is differentiable (or Lipschitz), the quotient

$$c(x) = \frac{f(x,u(x)) - f(x,v(x))}{u(x) - v(x)}$$

is a bounded function, and the difference $w(x) = u(x) - v(x)$ satisfies

$$\Delta w + c(x)w \geq 0 \text{ in } \Omega. \tag{3.6}$$

As $w(x) \leq 0$ in all of Ω, the strong maximum principle implies that $w(x) < 0$ in Ω, that is, we have a strict inequality: $u(x) < v(x)$ for all $x \in \Omega$. In other words, a sub-solution and a super-solution can not touch at a point – this very simple principle will be extremely important in what follows.

Let us illustrate an application of the strong maximum principle, with a cameo appearance of the sliding method in a disguise as a bonus. Consider the boundary value problem

$$-u'' = e^u, \quad 0 < x < L, \tag{3.7}$$

with the boundary condition

$$u(0) = u(L) = 0. \tag{3.8}$$

If we think of $u(x)$ as a temperature distribution, then the boundary condition means that the boundary is "cold". On the other hand the positive term e^u is a "heating term", which competes with the cooling by the boundary. A nonnegative solution $u(x)$ corresponds to an equilibrium between these two effects. We would like to show that if the length of the interval L is sufficiently large, then no such equilibrium is possible – the physical reason is that the boundary is too far from the middle of the interval, so the heating term wins. This absence of an equilibrium is interpreted as an explosion, this model was introduced exactly in that context in late 30's-early 40's. It is convenient to work with the function $w = u + \varepsilon$, which satisfies

$$-w'' = e^{-\varepsilon} e^{w}, \quad 0 < x < L, \tag{3.9}$$

with the boundary condition

$$w(0) = w(L) = \varepsilon. \tag{3.10}$$

Consider a family of functions

$$v_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right), \quad \lambda \geq 0, \quad 0 < x < L.$$

These functions satisfy (for any $\lambda \geq 0$)

$$v_\lambda'' + \frac{\pi^2}{L^2} v_\lambda = 0, \quad v_\lambda(0) = v_\lambda(L) = 0. \tag{3.11}$$

Therefore, if L is so large that

$$\frac{\pi^2}{L^2} s \leq e^{-\varepsilon} e^{s}, \quad \text{for all } s \geq 0,$$

we have

$$w'' + \frac{\pi^2}{L^2} w \leq 0, \tag{3.12}$$

that is, w is a super-solution for (3.11). In addition, when $\lambda > 0$ is sufficiently small, we have

$$v_\lambda(x) \leq w(x) \text{ for all } 0 \leq x \leq L. \tag{3.13}$$

Let us now start increasing λ until the graphs of v_λ and w touch at some point:

$$\lambda_0 = \sup\{\lambda : \ v_\lambda(x) \leq w(x) \text{ for all } 0 \leq x \leq L.\} \tag{3.14}$$

The difference

$$p(x) = v_{\lambda_0}(x) - w(x)$$

satisfies

$$p'' + \frac{\pi^2}{L^2} p \geq 0,$$

and $p(x) \leq 0$ for all $0 < x < L$. In addition, there exists x_0 such that $p(x_0) = 0$, and, as $v_\lambda(0) = v_\lambda(L) = 0 < \varepsilon = w(0) = w(L)$, it is impossible that $x_0 = 0$ or $x_0 = L$. We conclude that $p(x) \equiv 0$, which is a contradiction. Hence, no solution of (3.9)-(3.10) may exist when L is sufficiently large.

In order to complete the picture, the reader may look at the following exercise.

Exercise 3.7 *Show that there exists $L_1 > 0$ so that a nonnegative solution of (3.9)-(3.10) exists for all $0 < L < L_1$, and does not exist for all $L > L_1$.*

The maximum principle for narrow domains

Before we allow the moving plane method to return, we describe the maximum principle for narrow domains, which is an indispensable tool in this method. Its proof will utilize the "ballooning method" we have seen in the analysis of the explosion problem. As we have discussed, the usual maximum principle in the form "$\Delta u + c(x)u \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ implies either $u \equiv 0$ or $u < 0$ in Ω" can be interpreted physically as follows. If u is the temperature distribution then the boundary condition $u \leq 0$ means that "the boundary is cold" while the term $c(x)u$ can be viewed as a heat source if $c(x) \geq 0$ or as a heat sink if $c(x) \leq 0$. The conditions $u \leq 0$ on $\partial\Omega$ and $c(x) \leq 0$ together mean that both the boundary is cold and there are no heat sources – therefore, the temperature is cold everywhere, and we get $u \leq 0$. On the other hand, if the domain is such that each point inside Ω is "close to the boundary" then the effect of the cold boundary can dominate over a heat source, and then, even if $c(x) \geq 0$ at some (or all) points $x \in \Omega$, the maximum principle still holds.

Mathematically, the first step in that direction is the maximum principle for narrow domains. We use the notation $c^+(x) = \max[0, c(x)]$.

Theorem 3.8 *(The maximum principle for narrow domains) Let e be a unit vector. There exists $d_0 > 0$ that depends on the L^∞-norm $\|c^+\|_\infty$ so that if $|(y-x)\cdot e| < d_0$ for all $(x, y) \in \Omega$ then the maximum principle holds for the operator $\Delta + c(x)$. That is, if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\Delta u(x) + c(x)u(x) \geq 0 \text{ in } \Omega,$$

and $u \leq 0$ on $\partial\Omega$ then either $u \equiv 0$ or $u < 0$ in Ω.

The main observation here is that in a narrow domain we need not assume $c \leq 0$ – but "the largest possible narrowness", depends, of course, on the size of the positive part $c^+(x)$ that competes against it.

Proof. Note that, according to the strong maximum principle in the form of Corollary 3.6 (which has no assumptions on the sign of $c(x)$), it suffices to show that $u(x) \leq 0$ in Ω. For the sake of contradiction, suppose that

$$\sup_{x\in\Omega} u(x) > 0. \tag{3.15}$$

Without loss of generality we may assume that e is the unit vector in the direction x_1 so that

$$\bar{\Omega} \subset \{0 < x_1 < d\}.$$

Suppose that d is so small that

$$c(x) \leq \pi^2/d^2, \quad \text{for all } x \in \Omega, \tag{3.16}$$

and consider the function

$$w(x) = \sin\big(\frac{\pi x_1}{d}\big).$$

It satisfies

$$\Delta w + \frac{\pi^2}{d^2} w = 0, \tag{3.17}$$

and $w(x) > 0$ in $\bar{\Omega}$, in particular

$$\inf_{\bar{\Omega}} w(x) > 0. \tag{3.18}$$

A consequence of the above is

$$\Delta w + c(x)w \leq 0, \tag{3.19}$$

Given $\lambda \geq 0$, let us set $w_\lambda(x) = \lambda w(x)$. As a consequence of (3.18), there exists $\Lambda > 0$ so large that $\Lambda w(x) > u(x)$ for all $x \in \Omega$. Now we are going to push w_λ down until it touches $u(x)$: set

$$\lambda_0 = \inf\{\lambda : \ w_\lambda(x) > u(x) \text{ for all } x \in \Omega.\}$$

Note, that, because of (3.15), we know that $\lambda_0 > 0$. The difference

$$v(x) = u(x) - w_{\lambda_0}(x)$$

satisfies

$$\Delta v + c(x)v \geq 0.$$

The difference between $u(x)$, which satisfies the same inequality, and $v(x)$ is that we know already that $v(x) \leq 0$ – hence, we may conclude from the strong maximum principle (Corollary 3.6 again) that $v(x) < 0$ in Ω. As $v(x) < 0$ also on the boundary $\partial\Omega$, there exists $\varepsilon_0 > 0$ so that $v(x) < -\varepsilon_0$ for all $x \in \bar{\Omega}$, that is,

$$u(x) + \varepsilon_0 < w_{\lambda_0}(x) \text{ for all } x \in \bar{\Omega}.$$

But then we may choose $\lambda' < \lambda_0$ so that we still have

$$w_{\lambda'}(x) > u(x) \text{ for all } x \in \Omega.$$

This contradicts the minimality of λ_0. Thus, it is impossible that $u(x) > 0$ for some $x \in \Omega$, and we are done. □

The maximum principle for small domains

The maximum principle for narrow domains can be extended, dropping the requirement that the domain is narrow and replacing it by the condition that the domain has a small volume. We begin with the following lemma, which measures how far from the maximum principle a force can push you.

Lemma 3.9 *(The baby ABP Maximum Principle) Assume that $c(x) \leq 0$ for all $x \in \Omega$, and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\Delta u + c(x)u \geq f \text{ in } \Omega, \tag{3.20}$$

and $u \leq 0$ on $\partial\Omega$. Then

$$\sup_{\Omega} u \leq C\text{diam}(\Omega) \left\|f^-\right\|_{L^n(\Omega)}, \tag{3.21}$$

with the constant C that depends only on the dimension n (but not on the function $c(x) \leq 0$).

Proof. The idea is very similar to what we did in the proof of the isoperimetric inequality. If $M := \sup_\Omega u \le 0$, then there is nothing to prove, hence we assume that $M > 0$. The maximum is achieved at an interior point $x_0 \in \Omega$, $M = u(x_0)$, as $u(x) \le 0$ on $\partial\Omega$. Consider the function $v = -u^+$, then $v \le 0$ in Ω, $v \equiv 0$ on $\partial\Omega$ and

$$-M = \inf_\Omega v = v(x_0).$$

We proceed as in the proof of the isoperimetric inequality. Let Γ be the lower contact set of the function v. As $v \le 0$ in Ω, we have $v < 0$ on Γ, hence v is smooth on Γ, and

$$\Delta v = -\Delta u \le -f(x) + c(x)u \le -f(x), \text{ for } x \in \Gamma, \tag{3.22}$$

as $c(x) \le 0$ and $u(x) \ge 0$ on Γ. The analog of the inclusion (2.9) that we will now prove is

$$B(0; M/d) \subset \nabla v(\Gamma), \tag{3.23}$$

with $d = \text{diam}(\Omega)$ and $B(0, M/d)$ the open ball centered at the origin of radius M/d. One way to see that is by sliding: let $p \in B(0; M/d)$ and consider the hyperplane that is the graph of

$$z_k(x) = p \cdot x - k.$$

Clearly, $z_k(x) < v(x)$ for k sufficiently large. As we decrease k, sliding the plane up, let $\bar{k}$ be the first value when the graphs of $v(x)$ and $z_{\bar{k}}(x)$ touch at a point x_1. Then we have $v(x) \ge z_{\bar{k}}(x)$ for all $x \in \Omega$. If x_1 is on the boundary $\partial\Omega$ then $v(x_1) = z_{\bar{k}}(x_1) = 0$, and we have

$$p \cdot (x_0 - x_1) = z_k(x_0) - z_k(x_1) \le v(x_0) - 0 = -M,$$

whence $|p| \ge M/d$, which is a contradiction. Therefore, x_1 is an interior point, which means that $x_1 \in \Gamma$ (by the definition of the lower contact set), and $p = \nabla v(x_1)$. This proves the inclusion (3.23).

Mimicking the proof of the isoperimetric inequality we use the area formula (c_n is the volume of the unit bal in $\mathbb{R}^n$):

$$c_n \left(\frac{M}{d}\right)^n = |B(0; M/d)| \le |\nabla v(\Gamma)| \le \int_\Gamma |\det(D^2 v(x))| dx. \tag{3.24}$$

Now, as in the aforementioned proof, for every point x in the contact set Γ, the matrix $D^2 v(x)$ is non-negative definite, hence (note that (3.22) implies that $f(x) \le 0$ on Γ)

$$|\det[D^2 v(x)]| \le \left(\frac{\Delta v}{n}\right)^n \le \frac{(-f(x))^n}{n^n}. \tag{3.25}$$

Integrating (3.25) and using (3.24), we get

$$M^n \le \frac{(\text{diam}(\Omega))^n}{c_n n^n} \int_\Gamma |f^-(x)|^n dx, \tag{3.26}$$

which is (3.21). □

An important consequence of Lemma 3.9 is a maximum principle for a domain with a small volume [1]. Despite a simple proof and beautiful applications it has been observed only fairly recently, at least in the West where it was discovered in the 1990's by Varadhan[3].

[3] It was first noted by Bakelman in USSR.

Theorem 3.10 *(The maximum principle for domains of a small volume) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\Delta u(x) + c(x)u(x) \geq 0 \text{ in } \Omega,$$

and assume that $u \leq 0$ on $\partial\Omega$. Then there exists a positive constant δ which depends on the spatial dimension n, the diameter of Ω, and $\|c^+\|_{L^\infty}$, so that if $|\Omega| \leq \delta$ then $u \leq 0$ in Ω.

Proof. If $c \leq 0$ then $u \leq 0$ by the standard maximum principle. In general, assume that $u^+ \not\equiv 0$, and write $c = c^+ - c^-$. We have

$$\Delta u - c^- u \geq -c^+ u.$$

Lemma 3.9 implies that (with a constant C that depends only on the dimension n)

$$\sup_\Omega u \leq C \text{diam}(\Omega) \|c^+ u^+\|_{L^n(\Omega)} \leq C \text{diam}(\Omega) \|c^+\|_\infty |\Omega|^{1/n} \sup_\Omega u \leq \frac{1}{2} \sup_\Omega u,$$

when the volume of Ω is sufficiently small:

$$|\Omega| \leq \frac{1}{(2C \text{diam}(\Omega) \|c^+\|_\infty)^n}. \tag{3.27}$$

We deduce that $\sup_\Omega u \leq 0$ contradicting the assumption $u^+ \not\equiv 0$, Hence, we have $u \leq 0$ in Ω under the condition (3.27). □

4 Act IV. Dancing together

We will now use a combination of the maximum principle (mostly for small domains) and the moving plane method to prove some results on the symmetry of the solutions to elliptic problems. We show just the tip of the iceberg – a curious reader will find many other results in the literature, the most famous being, perhaps, the De Giorgi conjecture, a beautiful connection between geometry and applied mathematics.

4.1 The Gidas-Ni-Nirenberg theorem

The following result on the radial symmetry of non-negative solutions is due to Gidas, Ni and Nirenberg. It is a basic example of a general phenomenon that positive solutions of elliptic equations tend to be monotonic in one form or other. We present the proof of the Gidas-Ni-Nirenberg theorem from [3]. The proof uses the moving plane method combined with the maximum principles for narrow domains, and domains of small volume.

Theorem 4.1 *Let $B_1 \in \mathbb{R}^n$ be the unit ball, and $u \in C(\bar{B}_1) \cap C^2(B_1)$ be a positive solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } B_1 \\ &u = 0 \text{ on } \partial B_1 \end{aligned} \tag{4.1}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then, the function u is radially symmetric in B_1 and

$$\frac{\partial u}{\partial r}(x) < 0 \text{ for } x \neq 0.$$

Exercise 4.2 *Show that the conclusion that u is radially symmetric is false without the assumption that the function u is positive.*

The proof is based on the following lemma, which applies to general domains with a planar symmetry, not just balls.

Lemma 4.3 *Let Ω be a bounded domain that is convex in the x_1-direction and symmetric with respect to the plane $\{x_1 = 0\}$. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ be a positive solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } \Omega \\ &u = 0 \ \text{on } \partial\Omega \end{aligned} \tag{4.2}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then, the function u is symmetric with respect to x_1 and

$$\frac{\partial u}{\partial x_1}(x) < 0 \text{ for any } x \in \Omega \text{ with } x_1 > 0.$$

Proof of Theorem 4.1. Theorem 4.1 follows immediately from Lemma 4.3. Indeed, Lemma 4.3 implies that $u(x)$ is decreasing in any given radial direction, since the unit ball is symmetric with respect to any plane passing through the origin. It also follows from the same lemma that $u(x)$ is invariant under a reflection with respect to any hyperplane passing through the origin – this trivially implies that u is radially symmetric. □

Proof of Lemma 4.3

We use the coordinate system $x = (x_1, y) \in \Omega$ with $y \in \mathbb{R}^{n-1}$. We will prove that

$$u(x_1, y) < u(x_1^*, y) \text{ for all } x_1 > 0 \text{ and } -x_1 < x_1^* < x_1. \tag{4.3}$$

This, obviously, implies monotonicity in x_1 for $x_1 > 0$. Next, letting $x_1^* \to -x_1$, we get the inequality

$$u(x_1, y) \leq u(-x_1, y) \text{ for any } x_1 > 0.$$

Changing the direction, we get the reflection symmetry: $u(x_1, y) = u(-x_1, y)$.

We now prove (4.3). Given any $\lambda \in (0, a)$, with $a = \sup_\Omega x_1$, we take the "moving plane"

$$T_\lambda = \{x_1 = \lambda\},$$

and consider the part of Ω that is "to the right" of T_λ:

$$\Sigma_\lambda = \{x \in \Omega : \ x_1 > \lambda\}.$$

Finally, given a point x, we let x_λ be the reflection of $x = (x_1, x_2, \ldots, x_n)$ with respect to T_λ:

$$x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n).$$

Consider the difference

$$w_\lambda(x) = u(x) - u(x_\lambda) \text{ for } x \in \Sigma_\lambda.$$

The mean value theorem implies that w_λ satisfies

$$\Delta w_\lambda = f(u(x_\lambda)) - f(u(x)) = \frac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} w_\lambda = -c(x,\lambda) w_\lambda$$

in Σ_λ. This is a recurring trick: the difference of two solutions of a semi-linear equation satisfies a "linear" equation with an unknown function c. However, we know a priori that the function c is bounded:

$$|c(x)| \le \mathrm{Lip}(f), \text{ for all } x \in \Omega. \tag{4.4}$$

The boundary $\partial\Sigma_\lambda$ consists of a piece of $\partial\Omega$, where $w_\lambda = -u(x_\lambda) < 0$ and of a part of T_λ, where $x = x_\lambda$, thus $w_\lambda = 0$. Summarizing, we have

$$\begin{aligned} &\Delta w_\lambda + c(x,\lambda) w_\lambda = 0 \text{ in } \Sigma_\lambda \\ &w_\lambda \le 0 \text{ and } w_\lambda \not\equiv 0 \text{ on } \partial\Sigma_\lambda, \end{aligned} \tag{4.5}$$

with a bounded function $c(x,\lambda)$. We will show that

$$w_\lambda < 0 \text{ inside } \Sigma_\lambda \text{ for all } \lambda \in (0,a). \tag{4.6}$$

This implies in particular that w_λ assumes its maximum (equal to zero) over $\bar{\Sigma}_\lambda$ along T_λ. The Hopf lemma implies then

$$\left.\frac{\partial w_\lambda}{\partial x_1}\right|_{x_1=\lambda} = 2 \left.\frac{\partial u}{\partial x_1}\right|_{x_1=\lambda} < 0.$$

Given that λ is arbitrary, we conclude that

$$\frac{\partial u}{\partial x_1} < 0, \text{ for any } x \in \Omega \text{ such that } x_1 > 0.$$

Therefore, it remains only to show that $w_\lambda < 0$ inside Σ_λ to establish monotonicity of u in x_1 for $x_1 > 0$. Another consequence of (4.6) is that

$$u(x_1, x') < u(2\lambda - x_1, x') \text{ for all } \lambda \text{ such that } x \in \Sigma_\lambda,$$

that is, for all $\lambda \in (0, x_1)$, which is the same as (4.3).

In order to show that $w_\lambda < 0$ one would like to apply the maximum principle to the boundary value problem (4.5). However, a priori the function $c(x,\lambda)$ does not have a sign, so the usual maximum principle may not be used. On the other hand, there exists δ_c such that the maximum principle for narrow domains holds for the operator

$$Lu = \Delta u + c(x)u,$$

and domains of the width δ_c in the x_1-direction. Note that δ_c depends only on $\|c\|_{L^\infty}$ that is controlled in our case by (4.4). Moreover, when λ is sufficiently close to a:

$$a - \delta_c < \lambda < a,$$

the domain Σ_λ does have the width in the x_1-direction which is smaller than δ_0. Thus, for such λ the maximum principle for narrow domains implies that $w_\lambda < 0$ inside Σ_λ. This is because $w_\lambda \le 0$ on $\partial\Sigma_\lambda$, and $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$.

Let us now decrease λ (move the plane T_λ to the left, hence the name "the moving plane" method), and let (λ_0, a) be the largest interval of values so that $w_\lambda < 0$ inside Σ_λ for all $\lambda \in (\lambda_0, a)$. If $\lambda_0 = 0$ then we are done – (4.6) follows. Next, assume, for the sake of a contradiction, that $\lambda_0 > 0$. Then, by continuity, we still know that

$$w_{\lambda_0} \le 0 \text{ in } \Sigma_{\lambda_0}.$$

Moreover, w_{λ_0} is not identically equal to zero on $\partial\Sigma_{\lambda_0}$. The strong maximum principle implies that

$$w_{\lambda_0} < 0 \text{ in } \Sigma_{\lambda_0}. \tag{4.7}$$

We will show that then

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \tag{4.8}$$

for sufficiently small $\varepsilon < \varepsilon_0$. This will contradict our choice of λ_0 (unless $\lambda_0 = 0$).

Here is the key idea and the reason why the maximum principle for domains of small volume is useful: choose a simply connected closed set K in Σ_{λ_0}, with a smooth boundary, which is "nearly all" of Σ_{λ_0}, in the sense that

$$|\Sigma_{\lambda_0} \backslash K| < \delta/2$$

with $\delta > 0$ to be determined. Inequality (4.7) implies that there exists $\eta > 0$ so that

$$w_{\lambda_0} \le -\eta < 0 \text{ for any } x \in K.$$

By continuity, we have

$$w_{\lambda_0-\varepsilon} < -\frac{\eta}{2} < 0 \text{ for any } x \in K. \tag{4.9}$$

Let us now see what happens in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. As far as the boundary is concerned, we have

$$w_{\lambda_0-\varepsilon} \le 0$$

on $\partial\Sigma_{\lambda_0-\varepsilon}$ – this is true for $\partial\Sigma_\lambda$ for all $\lambda \in (0, a)$, and, in addition,

$$w_{\lambda_0-\varepsilon} < 0 \text{ on } \partial K,$$

because of (4.9) We conclude that

$$w_{\lambda_0-\varepsilon} < 0 \text{ on } \partial(\Sigma_{\lambda_0-\varepsilon} \backslash K).$$

However, when ε is sufficiently small we have $|\Sigma_{\lambda_0-\varepsilon} \backslash K| < \delta$. Choose δ (once again, solely determined by $\|c\|_{L^\infty(\Omega)}$), so small that we may apply the maximum principle for domains of small volume to the function $w_{\lambda_0-\varepsilon}$ in the domain $\Sigma_{\lambda_0-\varepsilon} \backslash K$. Then, we obtain

$$w_{\lambda_0-\varepsilon} \le 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \backslash K.$$

The strong maximum principle implies that

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \backslash K.$$

Putting two and two together we see that (4.8) holds. This, however, contradicts the choice of λ_0. The proof of the Gidas-Ni-Nirenberg theorem is complete. □

4.2 The sliding method

The sliding method differs from the moving plane method in that one compares translations of a function rather than its reflections with respect to a plane. We will illustrate it on an example taken from [3], which is maybe the simplest application of the method.

Theorem 4.4 *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$ which is convex in the x_1-direction. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \text{ in } \Omega \\ &u = \eta(x) \text{ on } \partial\Omega \end{aligned} \tag{4.10}$$

with $f \in C^1$. Assume that for any three points $x' = (x_1', y)$, $x = (x_1, y)$ and $x'' = (x_1'', y)$ lying on a segment parallel to the x_1-axis, $x_1' < x_1 < x_1''$ with $x', x'' \in \partial\Omega$, the following hold:

$$\eta(x') < u(x) < \eta(x'') \text{ if } x \in \Omega \tag{4.11}$$

and

$$\eta(x') \leq \eta(x) \leq \eta(x'') \text{ if } x \in \partial\Omega. \tag{4.12}$$

Then u is monotone in x_1 in Ω:

$$u(x_1 + \tau) > u(x_1, y) \text{ for } (x_1, y), (x_1 + \tau, y) \in \Omega \text{ and } \tau > 0.$$

Finally, u is the unique solution of (4.10) in $C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (4.11).

Assumption (4.11) is usually checked in applications from the maximum principle and is not as unverifiable and restrictive in practice as it might seem at a first glance. For instance, consider (4.10) in a rectangle $D = [-a, a]_x \times [0, 1]_y$ with the Dirichlet data

$$\eta(-a, y) = 0, \quad \eta(a, y) = 1,$$

prescribed at the vertical boundaries, while the data prescribed along the horizontal lines $y = 0$ and $y = 1$: $\eta_0(x) = u(x, 0)$ and $\eta_1(x) = u(x, 1)$ are monotonic in x. The function f is assumed to vanish at $u = 0$ and $u = 1$:

$$f(0) = f(1) = 0, \; f(s) \leq 0 \text{ for } u \notin [0, 1].$$

The maximum principle implies that then $0 \leq u \leq 1$ so that both (4.11) and (4.12) hold. Then Theorem 4.4 implies that the solution $u(x, y)$ is monotonic in x.

Proof. The philosophy of the proof is very similar to what we did in the proof of the Gidas-Ni-Nirenberg theorem. For $\tau \geq 0$, we let $u^\tau(x_1, y) = u(x_1 + \tau, y)$ be a shift of u to the left. The function u^τ is defined on the set $\Omega^\tau = \Omega - \tau \mathbf{e}_1$ obtained from Ω by sliding it to the left a distance τ parallel to the x_1-axis. The monotonicity of u may be restated as

$$u^\tau > u \text{ in } D^\tau = \Omega^\tau \cap \Omega \text{ for any } \tau > 0, \tag{4.13}$$

and this is what we will prove. As before, we first establish (4.13) for τ close to the largest value τ_0 – that is, those that have been slid almost all the way to the left, and the domain D^τ

is both narrow and small. This will be done using the maximum principle for domains of a small volume. Then we will start decreasing τ, sliding the domain Ω^τ to the right, and will show that you may go all the way to $\tau = 0$ keeping (4.13) enforced.

Consider the function

$$w^\tau(x) = u^\tau(x) - u(x) = u(x_1 + \tau, y) - u(x_1, y),$$

defined in D^τ. Since u^τ satisfies the same equation as u, we have from the mean value theorem

$$\begin{aligned} &\Delta w^\tau + c^\tau(x) w^\tau = 0 \text{ in } D^\tau \\ &w^\tau \geq 0 \text{ on } \partial D^\tau \end{aligned} \tag{4.14}$$

where

$$c^\tau(x) = \frac{f(u^\tau(x)) - f(u(x))}{u^\tau(x) - u(x)}$$

is a uniformly bounded function:

$$|c^\tau(x)| \leq \mathrm{Lip}(f). \tag{4.15}$$

The inequality on the boundary ∂D^τ in (4.14) follows from assumptions (4.11) and (4.12). Let

$$\tau_0 = \sup\{\tau > 0 : \ D^\tau \neq \emptyset\}$$

be the largest shift of Ω to the left that we can make so that Ω and Ω^τ still have a non-zero intersection. The volume $|D^\tau|$ is small when τ is close to τ_0. As in the moving plane method, since the function $c^\tau(x)$ is uniformly bounded by (4.15), we may apply the maximum principle for small domains to w^τ in D^τ for τ close to τ_0, and conclude that $w^\tau > 0$ for such τ.

Then we start sliding Ω^τ back to the right, that is, we decrease τ from τ_0 to a critical position τ_1: let (τ_1, τ_0) be a maximal interval with $\tau_1 \geq 0$ so that

$$w^\tau \geq 0 \text{ in } D^\tau \text{ for all } \tau \in (\tau_1, \tau_0].$$

We want to show that $\tau_1 = 0$ and argue by contradiction assuming that $\tau_1 > 0$.

Continuity implies that $w^{\tau_1} \geq 0$ in D^{τ_1}. Furthermore, (4.11) implies that

$$w^{\tau_1}(x) > 0 \text{ for all } x \in \Omega \cap \partial D^{\tau_1}.$$

The strong maximum principle then implies that $w^{\tau_1} > 0$ in D^{τ_1}.

Now we use the same idea as in the proof of Lemma 4.3: choose $\delta > 0$ so that the maximum principle holds for any solution of (4.14) in a domain of volume less than δ. Carve out of D^{τ_1} a closed set $K \subset D^{\tau_1}$ so that

$$|D^{\tau_1} \backslash K| < \delta/2.$$

We know that $w^{\tau_1} > 0$ on K, hence for ε small $w^{\tau_1 - \varepsilon}$ is also positive on K. Moreover, for $\varepsilon > 0$ small, we have

$$|D^{\tau_1 - \varepsilon} \backslash K| < \delta.$$

Furthermore, since

$$\partial(D^{\tau_1 - \varepsilon} \backslash K) \subset \partial D^{\tau_1 - \varepsilon} \cup K,$$

we see that

$$w^{\tau_1-\varepsilon} \geq 0 \text{ on } \partial(D^{\tau_1-\varepsilon}\backslash K).$$

Thus, $w^{\tau_1-\varepsilon}$ satisfies

$$\begin{aligned}&\Delta w^{\tau_1-\varepsilon} + c^{\tau_1-\varepsilon}(x) w^{\tau_1-\varepsilon} = 0 \text{ in } D^{\tau_1-\varepsilon}\backslash K \\ &w^{\tau_1-\varepsilon} \geq 0 \text{ on } \partial(D^{\tau_1-\varepsilon}\backslash K).\end{aligned} \tag{4.16}$$

The maximum principle for domains of small volume implies that

$$w^{\tau_1-\varepsilon} \geq 0 \text{ on } D^{\tau_1-\varepsilon}\backslash K.$$

Hence, we have

$$w^{\tau_1-\varepsilon} \geq 0 \text{ in all of } D^{\tau_1-\varepsilon},$$

and, as

$$w^{\tau_1-\varepsilon} \not\equiv 0 \text{ on } \partial D^{\tau_1-\varepsilon},$$

it is positive in $D^{\tau_1-\varepsilon}$. However, this contradicts the choice of τ_1. Therefore, $\tau_1 = 0$ and the function u is monotone in the x_1-variable.

Finally, to show that such solution u is unique, we suppose that v is another solution. We argue exactly as before but with $w^\tau = u^\tau - v$. The same proof shows that $u^\tau \geq v$ for all $\tau \geq 0$. In particular, $u \geq v$. Interchanging the role of u and v we conclude that $u = v$. □

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