On the effective stress in unsaturated porous continua with double porosity

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A B S T R A C T

Using mixture theory we formulate the balance laws for unsaturated porous media composed of a double-porosity solid matrix infiltrated by liquid and gas. In this context, the term 'double porosity' pertains to the microstructural characteristic that allows the pore spaces in a continuum to be classified into two pore subspaces. We use the first law of thermodynamics to identify energy-conjugate variables and derive an expression for the 'effective', or constitutive, stress that is energy-conjugate to the rate of deformation of the solid matrix. The effective stress has the form $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + B\overline{p}\mathbf{1}$, where $\boldsymbol{\sigma}$ is the total Cauchy stress tensor, B is the Biot coefficient, and $\overline{p}$ is the mean fluid pressure weighted according to the local degrees of saturation and pore fractions. We identify other emerging energy-conjugate pairs relevant for constitutive modeling of double-porosity unsaturated continua, including the local suction versus degree of saturation pair and the pore volume fraction versus weighted pore pressure difference pair. Finally, we use the second law of thermodynamics to determine conditions for maximum plastic dissipation in the regime of inelastic deformation for the unsaturated two-porosity mixture.

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1. Introduction

The mechanical constitutive study of multi-phase porous media requires, as a first step, the determination of an 'effective stress' in the solid skeleton. While much debate has transpired over the real physical meaning of the term 'effective stress' in a mixture composed of two or more types of fluid, we will use such term, even without the single quotes, in parity with the complete constitutive stress tensor that is energy-conjugate to the rate of deformation of the solid. Being a combination of the external stresses and the internal fluid pressures, the effective stress may be used for constitutive description of the solid matrix in particular and for material modeling of the total solid–fluid mixture in general (Borja, 2004, 2006; Houlsby, 1979, 1997). Over the years, the concept of effective stress has been widely evaluated for porous media with a single dominant porosity including one saturating fluid (Terzaghi, 1936; Biot, 1941; Carroll and Katsube, 1983; Skempton, 1961; Nur and Byerlee, 1971), or two immiscible fluids (Borja, 2004, 2006; Houlsby, 1979, 1997; Bishop, 1959; Hutter et al., 1999; Loret and Khalili, 2000; Khalili et al., 2004; Nuth and Laloui, 2007; Oka, 1996). However, many natural geomaterials, such as aggregated soils or fissured rocks, exhibit a pore size distribution with at least two dominant values of porosity (Al-Mukhtar, 1995; Delage et al., 1996; Didwania, 2002; Koliji et al., 2006). The two scales of porosity correspond to the micropores (intra-aggregate or matrix pores) and macropores (inter-aggregate pores or fissures) which can each be

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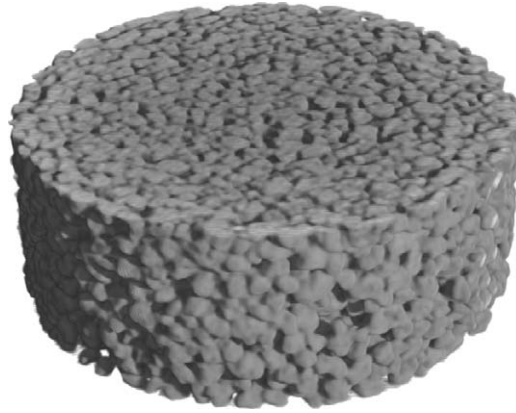   *E-mail address:* borja@stanford.edu (R.I. Borja).

Fig. 1. Reconstructed computed tomography volume of an aggregated silty clay, diameter $= 80$ mm, height $= 35$ mm. The aggregates are composed of much smaller silty clay solid particles and intra-aggregate pores. Visible spaces between aggregates are the inter-aggregate pores. After Koliji (2008).

filled up with more than one type of fluid. Fig. 1 shows a computed tomography (CT) image of an aggregated silty clay in which the porous aggregates and the large inter-aggregate pores can be clearly distinguished.

An important issue in the investigation of the behavior of materials exhibiting two scales of porosity, or double porosity, is the evolution of the internal structure and the proportional changes of micro- and macroporosity during the course of loading. Using a combination of tomography techniques and image analysis, Koliji et al. (2009) showed that the volume and structure of the macropores are strongly affected by the mechanical loading of an aggregated soil while the corresponding changes in the micropores are almost insignificant. They showed that significant change in macroporosity, and hence, a major structural evolution, occurs only with plastic straining of the soil. On the basis of the experimental results, they then proposed an evolution law linking the macroporosity alteration to plastic strain (Koliji, 2008; Koliji et al., 2008).

Because of the particular structure of materials exhibiting double porosity, their behavior should be addressed with proper account of the two scales of porosity (Barrenblatt et al., 1960; Barrenblatt, 1963). In addition, an overall measure of the effective stress in such media should be determined considering the different sources of fluid pressures. Based on phenomenological approaches, Khalili and Valliappan (1996) and Callari and Federico (2000) evaluated the effective stress parameters for double porous media with one saturating fluid. More recently, Pao and Lewis (2002) and Khalili et al. (2005) extended their analysis to the case of double porous media with two immiscible fluids. In the latter work, Khalili et al. (2005) expressed the effective stress parameters in terms of air entry suction value of the pores and measurable compressibility values of the solid, of the porous matrix, and of the whole porous medium. Despite the above major previous contributions, a definition of the effective stress in multi-phase double porous media is still an unresolved issue, and mathematical developments still suffer from the lack of a thermodynamic basis like those presented by Hutter et al. (1999), Hassanizadeh and Gray (1990), Gray and Schrefler (2001, 2007), Borja (2004, 2006), and Houlsby (1979, 1997) for porous media with one dominant porosity.

The present paper uses the continuum principles of thermodynamics to derive an expression for the effective stress tensor in multi-phase double porous media including a solid and one or more types of fluid. The effective stress is derived from the first law of thermodynamics by identifying an energy-conjugate stress variable to the rate of deformation of the solid. The first law also identifies a suction stress versus degree of saturation pair for each porosity scale, as well as a micropore volume fraction versus mean pore pressure difference pair emerging from the double-porosity formulation. The second law of thermodynamics leads to the principle of maximum plastic dissipation in the regime of inelastic deformation. In the context of classical theory of plasticity, we conclude that the yield function should depend not only on the effective stresses and the stress-like plastic internal variables, but also on the local suction stresses in the two scales of porosity and the mean pore pressure difference. Developments presented in this paper could provide important insight into the constitutive framework for materials with two porosity scales.

2. Double-porosity model and conservation laws

We consider a mixture of solid, liquid and gas. We assume that the solid forms a matrix able to resist load, with continuous pores filled with either liquid or gas. 'Double porosity' pertains to two porosity scales present in the mixture. For soils, double porosity pertains to the intra-aggregate and inter-aggregate pores shown in Fig. 1; for rocks, it may be used to characterize the porosity of the intact rock mass as well as that of the fissures. The pore space in each porosity scale may be filled with liquid and/or gas. Common terminology in soil mechanics characterizes a mixture as 'saturated' if the liquid fills the entire pore space, or 'unsaturated' (or 'partially saturated') if the pore space is partly filled by liquid and partly by gas. However, in earth sciences and multi-phase flow literature, a mixture is called 'saturated' only if there is no empty

pore. According to this latter definition, referred to as physical saturation, a 'physically unsaturated' mixture possesses a fraction of volume whose space is massless (Ehlers, 1993; Wilmanski, 1995; De Boer, 1996).

To clarify the existing terminology as well as underscore the need for a double-porosity characterization, we consider the case of aggregated soils or fissured rocks exhibiting two porosity scales. The pores and the contained fluids are divided into two parts corresponding to the macropores (inter-aggregate pores) and micropores (intra-aggregate pores). Fluids in the micropores are enclosed and separated from the free fluids in the macropores by the aggregate boundaries. Despite possible differences in the physical properties, the chemical composition of fluids remains more or less the same at the two porosity scales. Hence, they cannot be considered as different fluid phases, and so a model of single multi-phase mixture is inadequate to describe such a material.

As an alternative approach, and to elucidate the concept of double porosity, Fig. 2 shows a ternary mixture of solid, liquid and gas divided into two distinct but overlapping submixtures of the same volume with interactions between them. Submixture 1 (Fig. 2b) represents the porous aggregates consisting of solid and fluids within the micropores, while submixture 2 (Fig. 2c) represents the macropores consisting of only fluid constituents in the macropores. The total physically saturated mixture with two porosity scales is the superimposed continuum of two physically unsaturated submixtures. Fluid phases are assumed to be continuous and interact between the submixtures through different supply terms of thermodynamic variables.

Consider a solid material point X_s whose reference placement in the solid configuration is given by the position vector $\boldsymbol{X}_s$. Because of the special role played by the solid, we shall drop the subscript 's' and take X to mean X_s, $\boldsymbol{X}$ to mean $\boldsymbol{X}_s$, etc. Let $\boldsymbol{x}$ denote the current position of X. We assume that there is a mapping between $\boldsymbol{x}$ and the reference solid position vector $\boldsymbol{X}$ of the form

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t), \tag{1}$$

where t denotes time. The solid velocity and acceleration are given by

$$\boldsymbol{v}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}, t)}{\partial t}, \quad \boldsymbol{a}(\boldsymbol{X}, t) = \frac{\partial^2 \boldsymbol{\varphi}(\boldsymbol{X}, t)}{\partial t^2}. \tag{2}$$

Fig. 2 suggests that fluid α in submixture m may also occupy the same position $\boldsymbol{x}$ through a mapping of the form

$$\boldsymbol{x} = \boldsymbol{\varphi}_{\alpha m}(\boldsymbol{X}_{\alpha m}, t), \tag{3}$$

	a		b		c
Elementary volume representation		=		+	
Aggregate (Solid, liquid , gas)					
Macropore (Liquid, gas)					

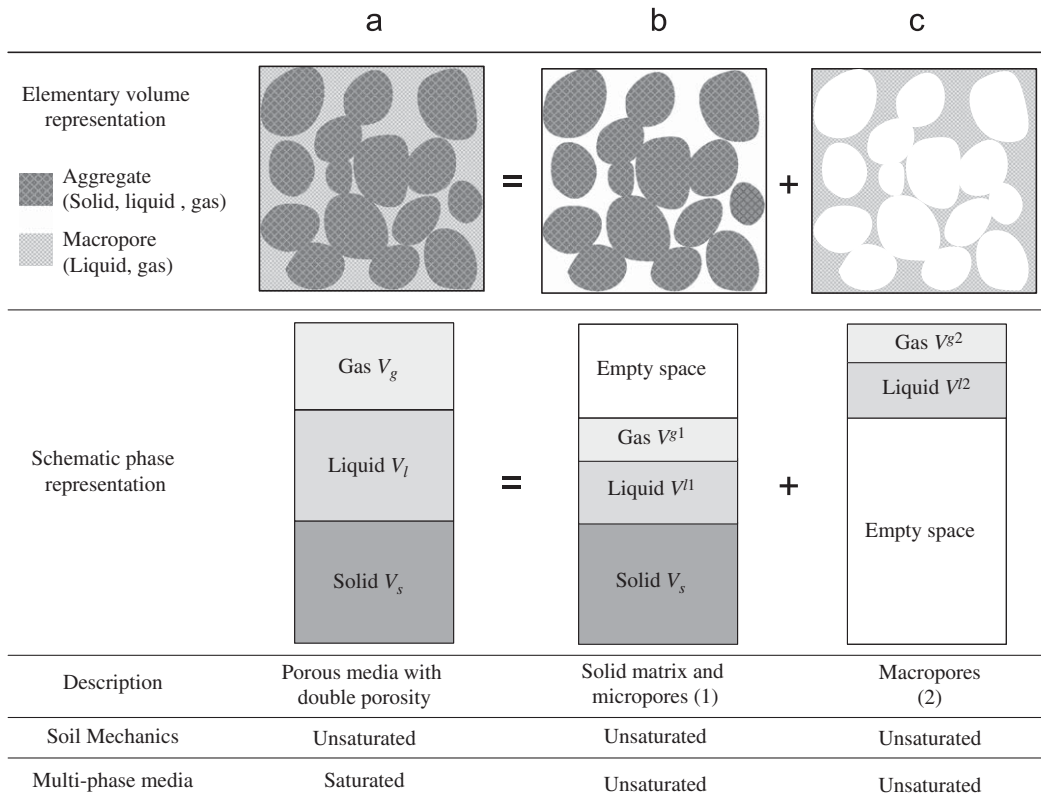|					
Schematic phase representation	Gas V_g / Liquid V_l / Solid V_s	=	Empty space / Gas V^{g1} / Liquid V^{l1} / Solid V_s	+	Gas V^{g2} / Liquid V^{l2} / Empty space
Description	Porous media with double porosity		Solid matrix and micropores (1)		Macropores (2)
Soil Mechanics	Unsaturated		Unsaturated		Unsaturated
Multi-phase media	Saturated		Unsaturated		Unsaturated

Fig. 2. Schematic representation of a mixture with double porosity: (a) total physically saturated mixture of solid, liquid, and gas; (b) physically unsaturated mixture of solid, liquid, and gas; and (c) physically unsaturated mixture of liquid and gas.

where $\boldsymbol{X}_{\alpha m}$ is the reference placement of this fluid and $\boldsymbol{\varphi}_{\alpha m}$ defines its motion. The fluid velocity and acceleration are then given by

$$\boldsymbol{v}_{\alpha m}(\boldsymbol{X}_{\alpha m}, t) = \frac{\partial \boldsymbol{\varphi}_{\alpha m}(\boldsymbol{X}_{\alpha m}, t)}{\partial t}, \quad \boldsymbol{a}_{\alpha m}(\boldsymbol{X}_{\alpha m}, t) = \frac{\partial^2 \boldsymbol{\varphi}_{\alpha m}(\boldsymbol{X}_{\alpha m}, t)}{\partial t^2}. \tag{4}$$

Let dV denote the elementary volume of the whole mixture. The two submixtures have the same elementary volume, and so the volume fractions of the constituents are defined as

$$\phi^s = dV_s/dV, \quad \phi^{\alpha m} = dV_{\alpha m}/dV, \quad \alpha = l, g; \; m = 1, 2. \tag{5}$$

The volume fractions are subject to the constraint

$$\phi^s + \sum_{m=1}^{2} \sum_{\alpha = l, g} \phi^{\alpha m} = 1. \tag{6}$$

This means that the volume of empty pores in submixture 1 is equal to the sum of the volume of constituents in submixture 2.

We denote the material time derivative following the solid motion by $d(\cdot)/dt$, and the material time derivative following the motion of fluid α in submixture m by $d^{\alpha m}(\cdot)/dt$. The balance of mass for the solid and all fluids may be written as

$$\frac{d\rho^s}{dt} + \rho^s \text{div}(\boldsymbol{v}) = 0, \tag{7}$$

$$\frac{d^{\alpha m} \rho^{\alpha m}}{dt} + \rho^{\alpha m} \text{div}(\boldsymbol{v}_{\alpha m}) = c^{\alpha m}, \quad \alpha = l, g, \; m = 1, 2. \tag{8}$$

In the above equations, ρ^s and $\rho^{\alpha m}$ are the partial mass densities of the solid and fluid, respectively, and are given by

$$\rho^s = \phi^s \rho_s, \quad \rho^{\alpha m} = \phi^{\alpha m} \rho_{\alpha m}, \tag{9}$$

where ρ_s and $\rho_{\alpha m}$ are the respective intrinsic mass densities of solid and fluid, $\boldsymbol{v}$ and $\boldsymbol{v}_{\alpha m}$ are the solid and fluid velocities, and $c^{\alpha m}$ is the rate of mass density exchange for fluid α in submixture m. Thus, mass exchange is limited to the fluids, and the solid does not exchange mass with them. The mass exchange terms satisfy the closure condition

$$\sum_{m=1}^{2} \sum_{\alpha} c^{\alpha m} = 0. \tag{10}$$

In some cases, however, one can write the stronger condition

$$\sum_{m=1}^{2} c^{\alpha m} = 0, \quad \alpha = l, g. \tag{11}$$

The above equation implies that mass transfer between the submixtures is limited to fluids of the same type, see Passman et al. (1984).

Balance of linear momentum for the unsaturated continuum with two porosity scales takes the form

$$\text{div}(\boldsymbol{\sigma}^s) + \rho^s \boldsymbol{g} + \boldsymbol{h}^s = \rho^s \boldsymbol{a}, \tag{12}$$

$$\text{div}(\boldsymbol{\sigma}^{\alpha m}) + \rho^{\alpha m} \boldsymbol{g} + \boldsymbol{h}^{\alpha m} = c^{\alpha m} \boldsymbol{v}_{\alpha m} + \rho^{\alpha m} \boldsymbol{a}_{\alpha m}, \tag{13}$$

where $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^{\alpha m}$ are the solid and fluid partial Cauchy stress tensors, respectively, $\boldsymbol{a}$ and $\boldsymbol{a}_{\alpha m}$ are the solid and fluid accelerations, and $\boldsymbol{h}^s$ and $\boldsymbol{h}^{\alpha m}$ are momentum supplies representing the drag on the relevant constituent by the surrounding constituents. The latter body force vectors satisfy the closure condition

$$\boldsymbol{h}^s + \sum_{m=1}^{2} \sum_{\alpha = l, g} \boldsymbol{h}^{\alpha m} = \boldsymbol{0}. \tag{14}$$

The balance of linear momentum for the total mixture is obtained by summing (12) for the solid and (13) for all fluids, yielding

$$\text{div}(\boldsymbol{\sigma}) + \rho \boldsymbol{g} = \rho \boldsymbol{a} + \sum_{m=1}^{2} \sum_{\alpha = l, g} \rho^{\alpha m} (\widetilde{\boldsymbol{a}}_{\alpha m} + c^{\alpha m} \boldsymbol{v}_{\alpha m}), \tag{15}$$

where

$$\widetilde{\boldsymbol{a}}_{\alpha m} = \boldsymbol{a}_{\alpha m} - \boldsymbol{a} \tag{16}$$

is the relative acceleration of fluid α in submixture m with respect to the solid. The partial Cauchy stress tensors and partial mass densities combine additively to form the total Cauchy stress tensor $\boldsymbol{\sigma}$ and total mass density ρ, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^s + \sum_{m=1}^{2}\sum_{\alpha=l,g}\boldsymbol{\sigma}^{\alpha m}, \quad \rho = \rho^s + \sum_{m=1}^{2}\sum_{\alpha=l,g}\rho^{\alpha m}. \tag{17}$$

Clearly, the above theory is sufficiently general to accommodate any similar mixture with multiple porosity scales and fluid types simply by changing the indices of summation m and α.

3. Balance of energy for compressible flows

Let K denote the kinetic energy, I the internal energy, and P the total power in a solid–fluid mixture exhibiting double porosity. Balance of energy for the mixture may be written as

$$\dot{K} + \dot{I} = P, \tag{18}$$

where the superimposed dot denotes a rate of change. The solid–fluid mixture is an open system in that fluid is allowed to flow in and out of the system, and so K and I are herein treated as extensive variables (Borja, 2008; Coussy, 1995). This means that any rate of change of these variables must account for the motion of each constituent in the whole system.

The total kinetic energy in the mixture is written as

$$K = \int_V \frac{1}{2}\rho^s \boldsymbol{v} \cdot \boldsymbol{v}\,\mathrm{d}V + \sum_{m=1}^{2}\sum_{\alpha=l,g}\int_V \frac{1}{2}\rho^{\alpha m}\boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m}\,\mathrm{d}V. \tag{19}$$

The rate of change of kinetic energy is then given by

$$\begin{aligned}\dot{K} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\rho^s \boldsymbol{v} \cdot \boldsymbol{v}\,\mathrm{d}V + \sum_{m=1}^{2}\sum_{\alpha=l,g}\frac{\mathrm{d}^{\alpha m}}{\mathrm{d}t}\int_V \frac{1}{2}\rho^{\alpha m}\boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m}\,\mathrm{d}V \\ &= \int_V \rho^s \boldsymbol{a} \cdot \boldsymbol{v}\,\mathrm{d}V + \int_V \sum_{m=1}^{2}\sum_{\alpha=l,g}\left(\frac{1}{2}c^{\alpha m}\boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m} + \rho^{\alpha m}\boldsymbol{a}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m}\right)\mathrm{d}V. \end{aligned} \tag{20}$$

Furthermore, we denote the rate of change of internal energy by the integral

$$\dot{I} = \int_V \rho \dot{e}\,\mathrm{d}V, \tag{21}$$

where $\dot{e}$ is the rate of change of internal energy per unit total mass of the mixture.

The total power P in the mixture is the sum of the mechanical power P^m and non-mechanical power P^n,

$$P = P^m + P^n. \tag{22}$$

The mechanical power is the sum of the powers of the surface tractions and body forces,

$$\begin{aligned}P^m &= \int_A \boldsymbol{\sigma}^s : \boldsymbol{n} \otimes \boldsymbol{v}\,\mathrm{d}A + \int_V (\boldsymbol{h}^s \cdot \boldsymbol{v} + \rho^s \boldsymbol{g} \cdot \boldsymbol{v})\,\mathrm{d}V + \int_A \sum_{m=1}^{2}\sum_{\alpha=l,g}\boldsymbol{\sigma}^{\alpha m} : \boldsymbol{n} \otimes \boldsymbol{v}_{\alpha m}\,\mathrm{d}A \\ &\quad + \int_V \sum_{m=1}^{2}\sum_{\alpha=l,g}(\boldsymbol{h}^{\alpha m} \cdot \boldsymbol{v}_{\alpha m} + \rho^{\alpha m}\boldsymbol{g} \cdot \boldsymbol{v}_{\alpha m})\,\mathrm{d}V, \end{aligned} \tag{23}$$

where A is the surface area of the volume V, and $\boldsymbol{n}$ is the outward unit normal vector to $\mathrm{d}A$. The area integral can be converted into a volume integral via Gauss theorem, yielding the following result:

$$\begin{aligned}P^m &= \int_V \left(\boldsymbol{\sigma}^s : \boldsymbol{d} + \sum_{m=1}^{2}\sum_{\alpha=l,g}\boldsymbol{\sigma}^{\alpha m} : \boldsymbol{d}_{\alpha m}\right)\mathrm{d}V + \int_V [\mathrm{div}(\boldsymbol{\sigma}^s) \cdot \boldsymbol{v} + \boldsymbol{h}^s \cdot \boldsymbol{v} + \rho^s \boldsymbol{g} \cdot \boldsymbol{v}]\,\mathrm{d}V \\ &\quad + \int_V \sum_{m=1}^{2}\sum_{\alpha=l,g}[\mathrm{div}(\boldsymbol{\sigma}^{\alpha m}) \cdot \boldsymbol{v}_{\alpha m} + \boldsymbol{h}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m} + \rho^{\alpha m}\boldsymbol{g} \cdot \boldsymbol{v}_{\alpha m}]\,\mathrm{d}V, \end{aligned} \tag{24}$$

where $\boldsymbol{d}$ and $\boldsymbol{d}_{\alpha m}$ are the rate of deformation tensors for the solid and fluid, respectively.

The non-mechanical power is the sum of heat supply into the volume V and the heat flux crossing the surface area A. Because of the open structure of a porous medium, it is often more convenient to specify the heat transport into and out from the solid–fluid mixture as a whole, rather than segregate them according to the constituent phases. Thus, letting the heat supply per unit volume of the mixture be denoted by $r(\boldsymbol{x}, t)$ and the heat flux across a surface with unit normal $\boldsymbol{n}$ by $h(\boldsymbol{x}, t, \boldsymbol{n})$, the non-mechanical power for a solid–fluid mixture is given by

$$P^n = \int_V r\,\mathrm{d}V + \int_A h\,\mathrm{d}A. \tag{25}$$

Alternatively, we can write $h = -\boldsymbol{q} \cdot \boldsymbol{n}$, where $\boldsymbol{q}$ is the heat flux vector. Using the Gauss theorem then gives

$$P^{\mathrm{n}} = \int_V r \, \mathrm{d}V - \int_A \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_V [r - \mathrm{div}(\boldsymbol{q})] \, \mathrm{d}V. \tag{26}$$

By reverting back to balance of energy (18), imposing the balance of linear momentum, and noting that V is arbitrary, we obtain the following localized expression for the rate of change of internal energy:

$$\rho \dot{e} = \boldsymbol{\sigma}^{\mathrm{s}} : \boldsymbol{d} + \sum_{m=1}^{2} \sum_{\alpha=\mathrm{l,g}} \left(\boldsymbol{\sigma}^{\alpha m} : \boldsymbol{d}_{\alpha m} + \frac{1}{2} c^{\alpha m} \boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m} \right) + r - \mathrm{div}(\boldsymbol{q}). \tag{27}$$

If $c^{\alpha m} = 0$, then the mechanical component of the rate of internal energy density is simply equal to the sum of the mechanical powers produced by the partial stress tensors.

Transitioning to the development of a mathematical expression for the effective stress, we introduce intrinsic constitutive laws governing the bulk stiffnesses of the solid and fluid. To this end, we consider the case of barotropic (isothermal) flows for the solid and fluid, for which the constitutive properties are given by the functional relations (Malvern, 1969)

$$f_{\mathrm{s}} = f_{\mathrm{s}}(p_{\mathrm{s}}, \rho_{\mathrm{s}}) = 0, \quad f_{\alpha m} = f_{\alpha m}(p_{\alpha m}, \rho_{\alpha m}) = 0, \tag{28}$$

where p_{s} and ρ_{s} are, respectively, the intrinsic pressure and mass density of the solid, and $p_{\alpha m}$ and $\rho_{\alpha m}$ are, respectively, the intrinsic pressure and mass density of phase α in submixture m. The functional forms presented above are a particular case of an equation of state independent of temperature and indicate a one-to-one relation between the intrinsic pressure and mass density of the solid and fluid. The material time derivative following the motion of the relevant constituent is readily evaluated as

$$\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} = p_{\mathrm{s}}'(\rho_{\mathrm{s}}) \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}t}, \quad \frac{\mathrm{d}^{\alpha m} p_{\alpha m}}{\mathrm{d}t} = p_{\alpha m}'(\rho_{\alpha m}) \frac{\mathrm{d}^{\alpha m} \rho_{\alpha m}}{\mathrm{d}t}, \tag{29}$$

where the primes denote an ordinary differentiation. From the above functional relations we recover the intrinsic bulk relations (Malvern, 1969)

$$\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} = K_{\mathrm{s}} \left(\frac{1}{\rho_{\mathrm{s}}} \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}t} \right), \quad \frac{\mathrm{d}^{\alpha m} p_{\alpha m}}{\mathrm{d}t} = K_{\alpha m} \left(\frac{1}{\rho_{\alpha m}} \frac{\mathrm{d}^{\alpha m} \rho_{\alpha m}}{\mathrm{d}t} \right), \tag{30}$$

where

$$K_{\mathrm{s}} = \rho_{\mathrm{s}} p_{\mathrm{s}}'(\rho_{\mathrm{s}}), \quad K_{\alpha m} = \rho_{\alpha m} p_{\alpha m}'(\rho_{\alpha m}) \tag{31}$$

are the intrinsic bulk moduli of the solid and fluid constituents, respectively. We recognize the expression $(\mathrm{d}^{\alpha m} \rho^{\alpha m}/\mathrm{d}t)/\rho^{\alpha m}$ as the intrinsic volumetric strain rate of the relevant constituent, i.e.,

$$\frac{1}{\rho_{\alpha m}} \frac{\mathrm{d}^{\alpha m} \rho_{\alpha m}}{\mathrm{d}t} = \frac{1}{\rho_{\alpha m}} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\frac{M_{\alpha m}}{V_{\alpha m} - \Delta V_{\alpha m}} - \frac{M_{\alpha m}}{V_{\alpha m}} \right) = \frac{1}{V_{\alpha m}} \frac{\mathrm{d}^{\alpha m} V_{\alpha m}}{\mathrm{d}t}. \tag{32}$$

Note in the formulation above that $M_{\alpha m}$ is conserved by the motion of the relevant constituent, thus justifying the name intrinsic bulk moduli.

Since $\rho^{\mathrm{s}} = \phi^{\mathrm{s}} \rho_{\mathrm{s}}$ and $\rho^{\alpha m} = \phi^{\alpha m} \rho_{\alpha m}$, we have

$$\frac{\mathrm{d}\rho^{\mathrm{s}}}{\mathrm{d}t} = \frac{\phi^{\mathrm{s}}}{p_{\mathrm{s}}'(\rho_{\mathrm{s}})} \frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \rho_{\mathrm{s}} \frac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}t}, \tag{33}$$

$$\frac{\mathrm{d}^{\alpha m} \rho^{\alpha m}}{\mathrm{d}t} = \frac{\phi^{\alpha m}}{p_{\alpha}'(\rho_{\alpha})} \frac{\mathrm{d}^{\alpha m} p_{\alpha m}}{\mathrm{d}t} + \rho_{\alpha m} \frac{\mathrm{d}^{\alpha m} \phi^{\alpha m}}{\mathrm{d}t}. \tag{34}$$

Introducing the intrinsic bulk moduli into the balance of mass yields

$$\frac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}t} + \frac{\phi^{\mathrm{s}}}{K_{\mathrm{s}}} \frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \phi^{\mathrm{s}} \mathrm{div}(\boldsymbol{v}) = 0, \tag{35}$$

$$\frac{\mathrm{d}^{\alpha m} \phi^{\alpha m}}{\mathrm{d}t} + \frac{\phi^{\alpha m}}{K_{\alpha m}} \frac{\mathrm{d}^{\alpha m} p_{\alpha m}}{\mathrm{d}t} + \phi^{\alpha m} \mathrm{div}(\boldsymbol{v}_{\alpha m}) = c^{\alpha m}. \tag{36}$$

In addition to the intrinsic bulk moduli introduced earlier, we also consider the elastic bulk modulus of the solid matrix. Following Borja (2006), for an elastic solid matrix we consider a functional relation of the form

$$F_{\mathrm{s}}(p^{\mathrm{s}}, \rho^{\mathrm{s}}) = F_{\mathrm{s}}(\phi^{\mathrm{s}} p_{\mathrm{s}}, \rho^{\mathrm{s}}) = 0, \tag{37}$$

which is equivalent to the constitutive relation

$$p_{\mathrm{s}} = \widetilde{p}_{\mathrm{s}}(\rho^{\mathrm{s}}, \phi^{\mathrm{s}}). \tag{38}$$

Taking the material time derivative in the direction of the solid motion gives

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = \frac{\partial \widetilde{p}_s}{\partial \rho^s}\frac{\mathrm{d}\rho^s}{\mathrm{d}t} + \frac{\partial \widetilde{p}_s}{\partial \phi^s}\frac{\mathrm{d}\phi^s}{\mathrm{d}t}. \tag{39}$$

It is easy to verify that

$$\frac{1}{\rho^s}\frac{\mathrm{d}\rho^s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left(\frac{\rho^s}{\rho_0^s}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}t}(\ln J^{-1}) = -\mathrm{div}(\boldsymbol{v}), \tag{40}$$

where $\rho_0^s = J\rho^s$ is the pull-back solid partial mass density. Furthermore, Eq. (35) gives

$$\frac{\mathrm{d}\phi^s}{\mathrm{d}t} = -\frac{\phi^s}{K_s}\frac{\mathrm{d}p_s}{\mathrm{d}t} - \phi^s\,\mathrm{div}(\boldsymbol{v}). \tag{41}$$

Substituting these last two equations into (39) gives

$$\left(1 + \frac{\partial \widetilde{p}_s}{\partial \phi^s}\frac{\phi^s}{K_s}\right)\frac{\mathrm{d}p_s}{\mathrm{d}t} = -\left(\rho^s\frac{\partial \widetilde{p}_s}{\partial \rho^s} + \rho^s\frac{\partial \widetilde{p}_s}{\partial \phi^s}\right)\mathrm{div}(\boldsymbol{v}) \tag{42}$$

or

$$\phi^s\frac{\mathrm{d}p_s}{\mathrm{d}t} = -\overline{K}\,\mathrm{div}(\boldsymbol{v}), \tag{43}$$

where

$$\overline{K} = \phi^s\left(\rho^s\frac{\partial \widetilde{p}_s}{\partial \rho^s} + \rho^s\frac{\partial \widetilde{p}_s}{\partial \phi^s}\right)\bigg/\left(1 + \frac{\partial \widetilde{p}_s}{\partial \phi^s}\frac{\phi^s}{K_s}\right) \tag{44}$$

is the elastic bulk modulus of the solid matrix (Borja, 2006).

4. Energy-conjugate pairs and the effective stress

To identify energy-conjugate pairs we rewrite the expression for the rate of change of internal energy density in a form that does not involve the partial stress tensors. To do this, we first substitute (43) into (35) and write

$$\frac{\mathrm{d}\phi^s}{\mathrm{d}t} + (\phi^s - b)\mathrm{div}(\boldsymbol{v}) = 0, \quad b = \overline{K}/K_s. \tag{45}$$

The balance of mass for fluid α contained in submixture m may be written in terms of the material time derivative with respect to the solid motion,

$$\frac{\mathrm{d}\phi^{\alpha m}}{\mathrm{d}t} + \frac{\phi^{\alpha m}}{K_{\alpha m}}\frac{\mathrm{d}p_{\alpha m}}{\mathrm{d}t} + \phi^{\alpha m}\mathrm{div}(\boldsymbol{v}) = c^{\alpha m} - \phi^{\alpha m}\mathrm{div}(\widetilde{\boldsymbol{v}}_{\alpha m}) - \widetilde{\boldsymbol{v}}_{\alpha m}\cdot\mathrm{grad}(\phi^{\alpha m}) - \frac{\phi^{\alpha m}}{K_{\alpha m}}\widetilde{\boldsymbol{v}}_{\alpha m}\cdot\mathrm{grad}(p_{\alpha m}), \tag{46}$$

where

$$\widetilde{\boldsymbol{v}}_{\alpha m} = \boldsymbol{v}_{\alpha m} - \boldsymbol{v} \tag{47}$$

is the relative velocity of fluid phase α in submixture m relative to the solid motion.

We now introduce the pore fraction ψ^m for submixture m, defined as the ratio between the pore volume occupied by submixture m in relation to the total volume of the pores in the entire mixture, i.e.,

$$\psi^m = \frac{\mathrm{d}V_m}{\sum_{m=1}^{2}\mathrm{d}V_m} = \frac{\mathrm{d}V_m}{\mathrm{d}V - \mathrm{d}V_s} = \frac{\phi^m}{1 - \phi^s}, \tag{48}$$

where ϕ^m and ϕ^s are, respectively, the volume fraction for the pores of submixture m and the volume fraction of the solid. It follows from the definition above that

$$\sum_{m=1}^{2}\psi^m = 1. \tag{49}$$

Next we define the local saturation $S^{\alpha m}$, defined as the ratio between the volume occupied by fluid phase α in submixture m in relation to the volume occupied by all fluids in the same submixture, i.e.,

$$S^{\alpha m} = \frac{\mathrm{d}V_{\alpha m}}{\mathrm{d}V_m} = \frac{\phi^{\alpha m}}{\phi^m} = \frac{\phi^{\alpha m}}{\psi^m(1 - \phi^s)}, \tag{50}$$

where $\phi^{\alpha m} = \mathrm{d}V_{\alpha m}/\mathrm{d}V$. The local saturation satisfies the condition

$$\sum_{\alpha=\mathrm{l,g}} S^{\alpha m} = 1 \tag{51}$$

for $m = 1, 2$. It follows that

$$\phi^{\alpha m} = S^{\alpha m}\psi^m(1 - \phi^s). \tag{52}$$

For completeness we also take the time derivative of the above equation in the direction of the solid motion,

$$\frac{d\phi^{\alpha m}}{dt} = \frac{dS^{\alpha m}}{dt}\psi^m(1 - \phi^s) + S^{\alpha m}\frac{d\psi^m}{dt}(1 - \phi^s) - S^{\alpha m}\psi^m\frac{d\phi^s}{dt}. \tag{53}$$

Combining (45) and (53) gives

$$\frac{d\phi^{\alpha m}}{dt} = \frac{dS^{\alpha m}\psi^m}{dt}(1 - \phi^s) + S^{\alpha m}\frac{d\psi^m}{dt}(1 - \phi^s) + S^{\alpha m}\psi^m(\phi^s - b)\mathrm{div}(\boldsymbol{v}). \tag{54}$$

Eliminating $d\phi^{\alpha m}/dt$ from (46) and (54), and solving, yields

$$\phi^{\alpha m}\mathrm{div}(\widetilde{\boldsymbol{v}}_{\alpha m}) = c^{\alpha m} - [S^{\alpha m}\psi^m(\phi^s - b) + \phi^{\alpha m}]\mathrm{div}(\boldsymbol{v}) - \frac{dS^{\alpha m}}{dt}\psi^m(1 - \phi^s) - S^{\alpha m}\frac{d\psi^m}{dt}(1 - \phi^s) - \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(\phi^{\alpha m})$$
$$- \frac{\phi^{\alpha m}}{K_{\alpha m}}\left[\frac{dp_{\alpha m}}{dt} + \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(p_{\alpha m})\right]. \tag{55}$$

The term on the left-hand side of the above equation will now be used to obtain an alternative expression for the energy balance equation.

For a mixture with two porosity scales and infiltrated by liquid and gas the rate of internal energy density is (cf. (27))

$$\rho\dot{e} = \boldsymbol{\sigma}^s : \boldsymbol{d} + \sum_{m=1}^{2}\sum_{\alpha=l,g}\boldsymbol{\sigma}^{\alpha m} : \boldsymbol{d}_{\alpha m} + \frac{1}{2}\sum_{m=1}^{2}\sum_{\alpha=l,g}c^{\alpha m}\boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m} + r - \mathrm{div}(\boldsymbol{q}). \tag{56}$$

Assuming an isotropic fluid, the fluid partial Cauchy stress tensors may be written in the form

$$\boldsymbol{\sigma}^{\alpha m} = -\phi^{\alpha m}p_{\alpha m}\mathbf{1}. \tag{57}$$

Substituting the above expression into (56) gives

$$\rho\dot{e} = \boldsymbol{\sigma} : \boldsymbol{d} - \sum_{m=1}^{2}\sum_{\alpha=l,g}\phi^{\alpha m}\mathrm{div}(\widetilde{\boldsymbol{v}}_{\alpha m})p_{\alpha m} + \frac{1}{2}\sum_{m=1}^{2}\sum_{\alpha l,g}c^{\alpha m}\boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m} + r - \mathrm{div}(\boldsymbol{q}). \tag{58}$$

Next we insert (55) into the first double summations in (58). In doing so, we observe from (52) that

$$S^{\alpha m}\psi^m(\phi^s - b) + \phi^{\alpha m} = S^{\alpha m}\psi^m(\phi^s - b) + S^{\alpha m}\psi^m(1 - \phi^s) \equiv S^{\alpha m}\psi^m B, \tag{59}$$

where

$$B = 1 - b = 1 - \frac{\overline{K}}{K_s} \tag{60}$$

is the familiar Biot coefficient. Therefore, the rate of change of internal energy density becomes

$$\rho\dot{e} = \overline{\boldsymbol{\sigma}} : \boldsymbol{d} + \sum_{m=1}^{2}\sum_{\alpha=l,g}G_{\alpha m}p_{\alpha m} + \frac{1}{2}\sum_{m=1}^{2}\sum_{\alpha=l,g}c^{\alpha m}\boldsymbol{v}_{\alpha m} \cdot \boldsymbol{v}_{\alpha m} + r - \mathrm{div}(\boldsymbol{q}). \tag{61}$$

In the above energy equation we identify the energy-conjugate pair $\langle\overline{\boldsymbol{\sigma}}, \boldsymbol{d}\rangle$, in which $\overline{\boldsymbol{\sigma}}$ is the 'effective', or constitutive, stress, a stress tensor that is energy-conjugate to the rate of deformation tensor $\boldsymbol{d}$ of the solid. The 'effective' stress has the form

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + B\overline{p}\mathbf{1}, \tag{62}$$

where

$$\overline{p} = \sum_{m=1}^{2}\psi^m \sum_{\alpha=l,g}S^{\alpha m}p_{\alpha m} \equiv \sum_{m=1}^{2}\psi^m\overline{p}_m \tag{63}$$

is the mean pore fluid pressure for the entire mixture, and

$$\overline{p}_m = \sum_{\alpha=l,g}S^{\alpha m}p_{\alpha m} \tag{64}$$

is the mean pore pressure in submixture m obtained from the intrinsic liquid and gas pressures weighted according to the local degree of saturation in that submixture. The effective stress defined in (62) has a form remarkably similar to the one proposed by Khalili et al. (2005) except that the latter authors employed compressibility ratios and not volume/pore fractions to obtain the weighted mean pore pressures. This suggests a possible correlation between the compressibility of the double porous medium with the actual porosity of the material and the degree of liquid saturation at both porosity scales.

The effective stress described in (62) reduces to more familiar forms under various simplifying assumptions. For unsaturated porous continua with one dominant porosity we can set $m = 1$ and $\psi^m = 1$ to obtain

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + B \sum_{\alpha = l, g} S^\alpha p_\alpha \mathbf{1}, \tag{65}$$

where S^α and p_α are the liquid/gas saturation and fluid pressure, respectively. This equation was derived earlier by Borja (2006) for an unsaturated continuum with one dominant porosity and accounting for the intrinsic compressibility of the solid. The form of (65) resembles that presented by Skempton (1961) for unsaturated porous media, except that Skempton used Bishop's (1959) and Bishop and Blight's (1963) parameter instead of the saturation indicators S^α. If $B = 1$, (65) reduces to

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \sum_{\alpha = l, g} S^\alpha p_\alpha \mathbf{1}, \tag{66}$$

which is the form for the effective stress used by Schrefler (1984) and resembles the Bishop (1959) and Bishop and Blight (1963) stress except for the saturation indicators S^α. For the fully saturated case (no gas), (65) reduces to the form

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + Bp\mathbf{1}, \tag{67}$$

where p is the liquid pore pressure. Nur and Byerlee (1971) showed the above equation to be theoretically exact for fully saturated elastic porous media. Finally, with $B = 1$ Eq. (67) reduces to

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + p\mathbf{1}. \tag{68}$$

The above 'effective stress equation' is, of course, due to Terzaghi (1943) and considered by many to mark the beginning of modern soil mechanics.

The coefficients $G_{\alpha m}$ play the role of Gibbs potentials and are given explicitly by the expression

$$G_{\alpha m} = c^{\alpha m} - \frac{\mathrm{d}S^{\alpha m}}{\mathrm{d}t} \psi^m (1 - \phi^s) - S^{\alpha m} \frac{\mathrm{d}\psi^m}{\mathrm{d}t} (1 - \phi^s) - \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(\phi^{\alpha m}) - \frac{\phi^{\alpha m}}{K_{\alpha m}} \left[\frac{\mathrm{d}p_{\alpha m}}{\mathrm{d}t} + \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(p_{\alpha m}) \right]. \tag{69}$$

We now demonstrate the implications of these potentials to energy balance. Consider a liquid–gas mixture with $\alpha = l, g$, and denote the local degree of liquid saturation in submixture m by S^m (it follows that the local degree of gas saturation is $1 - S^m$). Then

$$-\sum_{m=1}^{2} \sum_{\alpha = l, g} \frac{\mathrm{d}S^{\alpha m}}{\mathrm{d}t} \psi^m (1 - \phi^s) p_{\alpha m} = \sum_{m=1}^{2} \psi^m (1 - \phi^s) s^m \frac{\mathrm{d}S^m}{\mathrm{d}t}, \tag{70}$$

where

$$s^m = (p_g - p_l)_m \tag{71}$$

is the local matric suction stress for submixture m. Save for the coefficient $(1 - \phi^s)$, we thus identify a conjugate pair $\langle S^m, s^m \rangle$ for each submixture. This implies that a constitutive law in the form of a local liquid-retention curve is required for each submixture. An equation of this kind has been experimentally established for incompressible aggregates in Carminati et al. (2007). Note, however, that the energy balance equation only requires the local liquid retention curves and not the global liquid retention curve for the overall mixture.

Next, consider the same unsaturated mixture with double porosity and let $\widetilde{\psi}$ denote the pore fraction ψ^m of the micropores (it follows that the pore fraction for the macropores is $1 - \widetilde{\psi}$). Then

$$-\sum_{m=1}^{2} \sum_{\alpha = l, g} S^{\alpha m} \frac{\mathrm{d}\psi^m}{\mathrm{d}t} (1 - \phi^s) p_{\alpha m} = (1 - \phi^s) \pi \frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}t}, \tag{72}$$

where

$$\pi = \overline{p}_{\mathrm{macro}} - \overline{p}_{\mathrm{micro}} \tag{73}$$

is the difference between the mean pore pressures in the macropores and micropores, and $\overline{p}_{(\cdot)}$ is the mean pore pressure as defined in (64). The energy balance equation thus suggests that the constitutive evolution of the pore fraction $\widetilde{\psi}$ should be linked to the mean pore pressure difference π. It must be noted that even though the micropore changes are relatively insignificant, the macropores are strongly affected by the mechanical loading and could change with plastic deformation (Koliji et al., 2009). This implies that the micropore fraction $\widetilde{\psi}$ is also expected to vary with plastic deformation due to changes in the total volume of the pores. Thus, we have identified another conjugate pair $\langle \widetilde{\psi}, \pi \rangle$ for the subject unsaturated mixture with double porosity.

Finally, the summation

$$-\sum_{m=1}^{2} \sum_{\alpha = l, g} \frac{\phi^{\alpha m}}{K_{\alpha m}} \left[\frac{\mathrm{d}p_{\alpha m}}{\mathrm{d}t} + \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(p_{\alpha m}) \right] p_{\alpha m} = -\sum_{m=1}^{2} \sum_{\alpha = l, g} \frac{\phi^{\alpha m}}{K_{\alpha m}} \frac{\mathrm{d}^{\alpha m} p_{\alpha m}}{\mathrm{d}t} p_{\alpha m} = \sum_{m=1}^{2} \sum_{\alpha = l, g} \phi^{\alpha m} \frac{\mathrm{d}^{\alpha m} \theta_{\alpha m}}{\mathrm{d}t} p_{\alpha m} \tag{74}$$

in which

$$\frac{\mathrm{d}^{\alpha m}\theta_{\alpha m}}{\mathrm{d}t} = -\frac{1}{K_{\alpha m}}\frac{\mathrm{d}^{\alpha m}p_{\alpha m}}{\mathrm{d}t} \tag{75}$$

is the material time derivative of the intrinsic volumetric strain rate $\theta_{\alpha m}$ for fluid α in mixture m, represents the energy required to compress the constituent fluids. The conjugate pairs $\langle p_{\alpha m}, \theta_{\alpha m}\rangle$ are, of course, seen to be linked by the intrinsic elastic bulk moduli $K_{\alpha m}$.

In summary, the rate of change of internal energy density identifying all relevant conjugate pairs for a double-porosity unsaturated continuum is given by

$$\rho \dot{e} = \overline{\boldsymbol{\sigma}} : \boldsymbol{d} + \sum_{m=1}^{2} \psi^m (1-\phi^s) s^m \frac{\mathrm{d}S^m}{\mathrm{d}t} + (1-\phi^s)\pi \frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}t} + \sum_{m=1}^{2}\sum_{\alpha=l,g} \phi^{\alpha m} \frac{\mathrm{d}^{\alpha m}\theta_{\alpha m}}{\mathrm{d}t} p_{\alpha m} - \sum_{m=1}^{2}\sum_{\alpha=l,g} \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(\phi^{\alpha m})p_{\alpha m}$$

$$+ \sum_{m=1}^{2}\sum_{\alpha=l,g} c^{\alpha m}\left(p_{\alpha m} + \frac{1}{2}\boldsymbol{v}_{\alpha m}\cdot\boldsymbol{v}_{\alpha m}\right) + r - \mathrm{div}(\boldsymbol{q}). \tag{76}$$

5. The second law and maximum plastic dissipation

Consider a free energy density Ψ per unit current volume V of an unsaturated solid–fluid mixture with double porosity. For the sake of simplicity we only consider mechanical power in the following discussion. Also, we ignore mass exchanges among the fluids in the mixtures and take the solid deformation to be infinitesimal. The Clausius–Duhem inequality yields the local dissipation function

$$\mathscr{D} = \boldsymbol{\sigma}^s : \dot{\boldsymbol{\varepsilon}} + \sum_{m=1}^{2}\sum_{\alpha=l,g} \boldsymbol{\sigma}^{\alpha m} : \dot{\boldsymbol{\varepsilon}}_{\alpha m} - \dot{\Psi} \ge 0, \tag{77}$$

where $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\nabla}^s \boldsymbol{v}$ and $\dot{\boldsymbol{\varepsilon}}_{\alpha m} = \boldsymbol{\nabla}^s \boldsymbol{v}_{\alpha m}$ are the infinitesimal strain rate tensors. Note that we have replaced $\boldsymbol{d}$ by $\dot{\boldsymbol{\varepsilon}}$ and $\boldsymbol{d}_{\alpha m}$ by $\dot{\boldsymbol{\varepsilon}}_{\alpha m}$. Equivalently, the dissipation inequality may be written in terms of the effective stress tensor $\overline{\boldsymbol{\sigma}}$ as

$$\mathscr{D} = \overline{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} + \sum_{m=1}^{2} \psi^m (1-\phi^s) s^m \frac{\mathrm{d}S^m}{\mathrm{d}t} + (1-\phi^s)\pi \frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}t} + \sum_{m=1}^{2}\sum_{\alpha=l,g} \phi^{\alpha m} \frac{\mathrm{d}^{\alpha m}\theta_{\alpha m}}{\mathrm{d}t} p_{\alpha m}$$

$$- \sum_{m=1}^{2}\sum_{\alpha=l,g} \widetilde{\boldsymbol{v}}_{\alpha m} \cdot \mathrm{grad}(\phi^{\alpha m})p_{\alpha m} - \dot{\Psi} \ge 0. \tag{78}$$

Following standard plasticity theory (Simo and Hughes, 1998), we assume an additive decomposition of the strain rate tensor for the solid matrix into elastic and plastic parts,

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p. \tag{79}$$

Now, consider the following expression for the free energy function:

$$\Psi = \Psi(\boldsymbol{\varepsilon}^e, \theta_{\alpha m}, \widetilde{\boldsymbol{u}}_{\alpha m}, \boldsymbol{\xi}), \tag{80}$$

where $\widetilde{\boldsymbol{u}}_{\alpha m}$ is defined such that $\dot{\widetilde{\boldsymbol{u}}}_{\alpha m} = \widetilde{\boldsymbol{v}}_{\alpha m}$ for $m = 1, 2$ and $\alpha = l, g$; and $\boldsymbol{\xi}$ is a vector of strain-like plastic internal variables. This form of the free energy is motivated by the mathematical structure of the local dissipation function and allows extraction of the relevant constitutive expressions using the standard Coleman arguments. The free energy function is itself an extensive variable, and its rate of change is given by

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \sum_{m=1}^{2}\sum_{\alpha=l,g} \frac{\partial \Psi}{\partial \theta_{\alpha m}} \frac{\mathrm{d}^{\alpha m}\theta_{\alpha m}}{\mathrm{d}t} + \sum_{m=1}^{2}\sum_{\alpha=l,g} \frac{\partial \Psi}{\partial \widetilde{\boldsymbol{u}}_{\alpha m}} \cdot \widetilde{\boldsymbol{v}}_{\alpha m} - \boldsymbol{\chi} \cdot \dot{\boldsymbol{\xi}}, \tag{81}$$

where $\boldsymbol{\chi} = -\partial \Psi/\partial\boldsymbol{\xi}$. Substituting into (78) and using the standard Coleman argument yields the constitutive equations

$$\overline{\boldsymbol{\sigma}} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e}, \quad \phi^{\alpha m}p_{\alpha m} = \frac{\partial \Psi}{\partial \theta_{\alpha m}}, \quad \mathrm{grad}(\phi^{\alpha m})p_{\alpha m} = \frac{\partial \Psi}{\partial \widetilde{\boldsymbol{u}}_{\alpha m}}, \tag{82}$$

and the reduced dissipation inequality

$$\mathscr{D}^p = \overline{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^p + (1-\phi^s)\sum_{m=1}^{2} \psi^m s^m \dot{S}^m + (1-\phi^s)\pi\dot{\widetilde{\psi}} + \boldsymbol{\chi}\cdot\dot{\boldsymbol{\xi}} \ge 0, \tag{83}$$

where $\dot{S}^m = \mathrm{d}S^m/\mathrm{d}t$ and $\dot{\widetilde{\psi}} = \mathrm{d}\widetilde{\psi}/\mathrm{d}t$.

The principle of maximum plastic dissipation is central to the variational formulation of theory of plasticity and can be extended to the unsaturated porous continua with double porosity, at least in principle, as follows. Consider the reduced

dissipation inequality given in (83) and assume a yield function of the form

$$F(\overline{\boldsymbol{\sigma}}, s^m, \pi, \boldsymbol{\chi}) \leq 0. \tag{84}$$

The functional relation is dictated by the form of $\mathscr{D}^p$ and implies that the yield function now depends not only the effective stress $\overline{\boldsymbol{\sigma}}$ and the suction stresses s^m ($m = 1, 2$), but also on the mean pore pressure differential π and (of course) the stress-like plastic internal variable $\boldsymbol{\chi}$. In the fully saturated regime the functional dependence of F on the suction stresses disappears, but note that the functional dependence of F on the pore pressure differential π remains in effect as a result of the double-porosity formulation. We should also note that $(1 - \phi^s) > 0$ and $\psi^m > 0$, which are critical for the following results.

The principle of maximum plastic dissipation implies the following. The plastic flow rule is associative in the sense

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial F}{\partial \overline{\boldsymbol{\sigma}}}, \tag{85}$$

where $\dot{\lambda} > 0$ is a non-negative plastic multiplier. The local degrees of liquid saturation in the two-porosity mixture follows the relation

$$\dot{S}^m = \dot{\lambda} \frac{\partial F}{\partial s^m}, \quad m = 1, 2. \tag{86}$$

The micropore fraction $\widetilde{\psi}$ evolves *plastically* according to the flow rule

$$\dot{\widetilde{\psi}} = \dot{\lambda} \frac{\partial F}{\partial \pi}. \tag{87}$$

The strain-like plastic internal variables $\boldsymbol{\xi}$ follow an associative hardening of the form

$$\dot{\boldsymbol{\xi}} = \dot{\lambda} \frac{\partial F}{\partial \boldsymbol{\chi}}. \tag{88}$$

A standard result is also facilitated by the loading–unloading (Kuhn–Tucker) conditions

$$\dot{\lambda} \geq 0, \quad F(\overline{\boldsymbol{\sigma}}, s^m, \pi, \boldsymbol{\chi}) \leq 0, \quad \dot{\lambda} F(\overline{\boldsymbol{\sigma}}, s^m, \pi, \boldsymbol{\chi}) = 0. \tag{89}$$

Finally, maximum plastic dissipation implies that $F(\overline{\boldsymbol{\sigma}}, s^m, \pi, \boldsymbol{\chi}) \leq 0$ is a convex function. See Simo and Hughes (1998) for further details on this aspect.

As noted in Borja (2004, 2006), the principle of maximum plastic dissipation is a mathematical convenience as it preserves the variational symmetry of the problem, but is hardly justified by experimental evidence especially in soils and rocks. An associative plastic flow for $\dot{\boldsymbol{\varepsilon}}^p$ is seldom supported by experimental observations, and an associative hardening form for $\dot{\boldsymbol{\xi}}$ is never used in reality. In soils, well-established relations exist between the degree of saturation and suction stress, such as the Van Genuchten and Brooks–Corey relations (Bear, 1972; Brooks and Corey, 1966; van Genuchten, 1980); however, they are far from the associative form implied by the maximum plastic dissipation principle. A new term that arises for the first time in the energy balance equation suggests a constitutive relation between the micropore fraction $\widetilde{\psi}$ and the mean pore pressure difference π. No relation currently exists between these two state variables, but the flow rule described in (87) could provide some insight into possible constitutive relations existing between these two variables.

6. Closure

We have used the continuum principles of thermodynamics to derive an expression for the effective stress tensor in multi-phase porous media exhibiting two porosity scales. We emphasize that our approach to constitutive modeling of unsaturated soils is based on the use of this effective stress as the only stress state variable and on direct use of suction in the constitutive relations. Other alternative approaches exist in which different pairs of stress state variables are chosen instead of a single effective stress, such as the net stress and matric suction in the Barcelona Basic Model (Alonso et al., 1990). Detailed discussion of the mechanical constitutive modeling of unsaturated soils and the existing approaches are beyond the goals of the present paper.

The single stress state variable Cauchy effective stress tensor referred to in this paper has the form $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + B\overline{p}\mathbf{1}$, where $\boldsymbol{\sigma}$ is the total Cauchy stress tensor, B is the Biot coefficient, and $\overline{p}$ is the mean fluid pressure weighted according to the local degrees of saturation and pore fractions. An interesting aspect of this definition is that under special conditions this effective stress tensor reduces to some of the more recognizable forms, including the most well-known Terzaghi (1943) effective stress tensor applicable for fully saturated incompressible flows in a porous continuum with one dominant porosity. In addition, we have also identified other emerging energy-conjugate pairs relevant for constitutive modeling of double-porosity unsaturated continua, including the local suction versus degree of saturation pair and the pore volume fraction versus weighted pore pressure difference pair. Finally, we have used the second law of thermodynamics to infer that for loading in the inelastic regime, the yield function for porous continua with two porosity scales is expected to depend not only on the effective stress and the local suction stresses but also on the mean pore pressure difference arising from the double-porosity formulation. The latter result could be useful for developing constitutive relations for such continua, as well as the relevant laboratory testing programs leading to such constitutive relations.

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