# Optimization of Linear Systems

by

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# 1 Introduction

The optimization of linear systems will be considered, subject to restrictions on the form of the control. If $x$ is the state vector and $u$ the control vector the system can be described by

$$\dot{x} = Ax + Bu \quad , \quad x(0) = x_0 \tag{1.1}$$

We are generally interested in controlling an output vector of the form

$$y = Cx \tag{1.2}$$

and it is convenient to measure the performance by a quadratic cost function

$$J = \int_0^t \left(y^T Q y + u^T R u\right) ds \tag{1.3}$$

We seek to minimize $J$. A penalty on $u$ is included to limit its magnitude which might otherwise approach infinity at a minimum of $J$. It is often desirable to use a feedback control of the form

$$u = Dx \tag{1.4}$$

Then

$$\dot{x} = Fx \quad , \quad x(0) = 0 \tag{1.5a}$$

$$\dot{J} = x^T S x \quad , \quad J(0) = 0 \tag{1.5b}$$

where

$$F = A + BD \tag{1.6a}$$

$$S = C^T Q C + D^T R D \tag{1.6b}$$

$D$ may be constant or may be allowed to vary with time. For the sake of engineering simplicity we may also wish to restrict the form of $D$. If no feedback is allowed from the $j^{\text{th}}$ state variable to the $i^{\text{th}}$ control, then

$$D_{ij} = 0$$

# 2 Properties of the transition matrix

If the system satisfies (1.5a), the since the principle of superposition may be applied to solutions for different initial conditions,

$$x(t) = \phi(t,s)\, x(s) \tag{2.1}$$

where for all $t, s, \tau$

$$\phi(t,t) = I \quad , \quad \phi(t,\tau)\,\phi(\tau,s) = \phi(t,s) \tag{2.2}$$

and for arbitrary $x(s)$

$$\frac{d}{dt}\phi(t,s)\, x(s) = F(t)\,\phi(t,s)\, x(s)$$

whence

$$\frac{d}{dt}\phi(t,s) = F(t)\,\phi(t,s) \tag{2.3}$$

Also

$$\frac{d}{ds}x(t) = 0 = \left[\frac{d}{ds}\phi(t,s)\right] x(s) + \phi(t,s)\, F(s)\, x(s)$$

whence

$$\frac{d}{ds}\phi(t,s) = -\phi(t,s)\, F(s) \tag{2.4}$$

if $F$ is constant $\phi$ depends only in the difference $t - s$.

Then (2.3) and (2.4) yield

$$F\phi = \phi F \tag{2.5}$$

If there is a forcing function $y(t)$ such that

$$\dot{x} = Fx + y$$

then it is easy to verify by differentiation that the solution is

$$x(t) = \phi(t,s)\, x(s) + \int_s^t \phi(t,\tau)\, y(\tau)\, d\tau \tag{2.6a}$$

In the case of a backward integration it is more convenient to write

$$x(s) = \phi(s,t)\, x(t) - \int_s^t \phi(s,\tau)\, y(\tau)\, d\tau \tag{2.6b}$$

# 3 Gradient of a function with respect to time-varying and fixed parameters

It is convenient to give a parallel treatment of optimization with respect to time-varying and fixed parameters by introducing the concept of the gradient in function space for time-varying parameters. Consider a linear space of vector functions on the interval $(0, t)$ for which the inner product is defined as

$$\langle x, y \rangle = \int_0^t \sum_i x_i y_i ds$$

If $f$ depends on the function $v$, and a small variation $\delta v$ in $v$ causes $f$ to change by

$$\delta f = \left\langle \frac{\partial f}{\partial v}, \delta v \right\rangle$$

in the sense that

$$\lim_{e \to 0} \frac{f(v + \epsilon h) - f(v)}{\epsilon} = \left\langle \frac{\partial f}{\partial v} \right\rangle$$

then $\frac{\partial f}{\partial v}$ is called the gradient (weak derivation) of $f$ with respect to $v$. If we wish to minimize a function $J$ of the final state $x(t)$,

where

$$\dot{x}_i = f_i(x, v, t) \tag{3.1}$$

and $v$ is a time-varying vector parameter, then for a small change $\delta v$ in $v$,

$$\delta \dot{x}_i = \sum_k \frac{\partial f_i}{\partial x_k} \delta x_k + \sum_j \frac{\partial f_i}{\partial v_j} \delta v_j \quad , \quad \delta x_i(0) = 0 \tag{3.2a}$$

$$\delta J = \sum_i \frac{\partial J}{\partial x_i} \delta x_i(t) \tag{3.2b}$$

We introduce a set of 'costate' functions $\psi$, satisfying the 'adjoint' equations

$$\dot{\psi}_i = -\sum_k \frac{\partial f_k}{\partial x_i} \psi_k \quad , \quad \psi_i(t) = \frac{\partial J}{\partial x_i}$$

Then

$$\frac{d}{dt}\left(\sum_i \psi_i \delta x_i\right) = \sum_j \psi_i \frac{\partial f_i}{\partial v_j}\ \delta v_j$$

$$\delta J = \int_0^t \sum_i \sum_j \psi_i \frac{\partial f_i}{\partial v_j}\ \delta v_j ds = \langle G, \delta v \rangle$$

Where

$$G_j\left(s\right) = \sum_i \psi_i \frac{\partial f_i}{\partial v_j} \tag{3.3}$$

$G$ may thus be identifined as the gradient in function space. Evidently if $J$ reaches a minimum it is necessary that $G$ should vanish throughout the intercal: otherwise one could find a $\delta v$ such that $\delta J < 0$.

If $v$ is a fixed vector the development is similar.

Denote $\frac{\partial x_i}{\partial v_j}$ by $\sigma_{ij}$. Then

$$\dot{\sigma_{ij}} = \sum_k \frac{\partial f_i}{\partial x_k}\sigma_{ij} + \frac{\partial f_i}{\partial v_j} \qquad , \quad \sigma_{ij}\left(0\right) = 0$$

$$\frac{\partial J}{\partial v_j} = \sum_i \frac{\partial J}{\partial x_i}\sigma_{ij}\left(t\right)$$

Again introducing the costate variable $\psi$ :

$$\frac{d}{dt}\left(\sum_i \psi_i \sigma_{ij}\right) = \sum_i \psi_i \frac{\partial f_i}{\partial v_j}$$

and denoting the gradient by $G$ ,

$$G_j = \frac{\partial J}{\partial v_j} = \int_0^t \sum_i \psi_i \frac{\partial f_i}{\partial v_j} ds \tag{3.4}$$

For a fixed interval the gradient with respect to a fixed parameter is thus the integral of the gradient with respect to the same parameter when it is allowed to vary with time.

If the cost function is an integral

$$J = \int_0^t h\left(x, v, s\right) ds$$

then (3.2b) is replaced by

$$\delta \dot{J} = \sum_i \frac{\partial h}{\partial x_i} \delta x_i + \sum_j \frac{\partial h}{\partial v_j} \delta v_j \tag{3.5}$$

It is convenient to identify $J$ with an additional variable $x_{n+1}$ satisfying

$$\dot{x}_{n+1} = h\left(x, v, t\right)$$

Since $x_{n+1}$ does not appear in any of the $f_i$

$$\dot{\psi}_{n+1} = 0, \quad \psi_{n+1}\left(t\right) = \frac{\partial J}{\partial x_{n+1}} = 1$$

whence

$$\psi_{n+1} = 1$$

Also

$$\psi_i\left(t\right) = \frac{\partial J}{\partial x_i} = 0 \qquad , \quad i = 1, n$$

Thus the gradient with respect to a time-varying parameter is

$$G_j\left(s\right) = \sum_i \psi_i \frac{\partial f_i}{\partial v_j} + \frac{\partial h}{\partial v_j} \tag{3.6}$$

where

$$\dot{\psi}_i = -\sum_i \frac{\partial f_k}{\partial x_i} \psi_k - \frac{\partial h}{\partial x_i} \qquad , \quad \psi_i\left(t\right) = 0 \tag{3.7}$$

and the gradient with respect to a fixed parameter is

$$G_j = \int_0^t \left[ \sum_i \psi_i \frac{\partial f_i}{\partial v_j} + \frac{\partial h}{\partial v_j} \right] ds$$

## 4 Evaluation of the cost function

If (1.5) holds then according to (2.1)

$$J(t) - J(s) = \int_s^t x^T(\tau) S(\tau) x(\tau) d\tau \quad (4.1)$$

$$= x^T(s) P(t,s) x(s) \quad (4.2)$$

where

$$P(t,s) = \int_s^t \phi^T(\tau,s) S(\tau) \phi(\tau,s) d\tau \quad (4.3)$$

This may be differentiated, using the properties of the transition matrix expressed in (2.2) and (2.4), to give

$$\frac{d}{ds}P(t,s) = -S(s) - F^T(s) P(t,s) - P(t,s) F(s) \quad , \quad P(t,t) = 0 \quad (4.4)$$

Denote the outer product $xx^T$ by $X$, and denote the trace $\sum_i A_{ii}$ of a square matrix $A$ by $\mathrm{Tr}(A)$. Note that

$$\mathrm{Tr}\left(A^T\right) = \mathrm{Tr}(A)$$

and that as long as $AB$ is square

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

even if $A$ and $B$ are not square.

Then (1.5b) and (4.1) may be written as

$$\dot{J} = \mathrm{Tr}(SX) \quad (4.5)$$

$$J(t) - J(s) = \mathrm{Tr}\left[P(t,s) X(s)\right] \quad (4.6)$$

where $P$ is determined from (4.3) and

$$\dot{X} = FX + XF^T \quad , \quad X(0) = X_0 \quad (4.7)$$

If $A, B, C, D, Q$ and $R$ are constant so that $F$ and $S$ are constant, $P(t,s)$ depends only on $t-s$ and may be written as $P(t-s)$, so that

$$\dot{P} = S + F^T P + PF \quad , \quad P(0) = 0$$

For a constant system an alternative expression for the cost is

$$J = \mathrm{Tr}\,(SW)$$

where

$$W = \int_0^t X(s)\,ds$$

and (4.6) may be integrated to give

$$\dot{W} = X_0 + FW + WF^T \quad , \quad W(0) = 0 \tag{4.8}$$

## 5 Gradient with respect to feedback coefficients

The variational equation corresponding to (1.5) are

$$\delta\dot{x} = F\delta x + B\delta D x \tag{5.1a}$$

$$\delta\dot{J} = 2x^T S\delta x + 2x^T D^T R\delta D x \tag{5.1b}$$

The adjoint equations (3.7) become

$$\dot{\psi} = -F^T\psi \qquad , \quad \psi(t) = 0 \tag{5.2}$$

The gradient in function space with respect to $D_{qr}$ is thus

$$G_{qr}(s) = \sum_i \psi_i(s) B_{iq}(s) x_r(s) + \sum_i \sum_j x_i(s) D_{ji}(s) R_{jq}(s) x_r(s)$$

or using matrix notation and denoting the outer product $xx^T$ by $X$,

$$G(s) = B^T(s)\psi(s) x^T(s) + 2R(s) D(s) X(s) \tag{5.3}$$

The gradient with respect to fixed gains is therefore

$$G = \int_0^t \left(B^T(s)\psi(s) x^T(s) + 2R(s) D X(s)\right) ds \tag{5.4}$$

Let $\zeta(t,s)$ be the transition matrix of the adjoint equations. Since

$$\psi(t) = 0$$

when (2.6b) is applied to (5.1a) it follows that

$$\psi(s) = 2\int_s^t \zeta(s,t) S(\tau) x(\tau) d\tau$$

where

$$\frac{d}{ds}\zeta(s,t) = -F^T(s)\zeta(s,t) \qquad , \quad \zeta(t,t) = I$$

But if $\phi$ is the transition matrix for $x$, then according to (2.4)

$$\frac{d}{ds}\zeta(s,t) = -F^T(s)\,\zeta(s,t) \qquad , \quad \zeta(t,t) = I$$

so $\zeta(s,t)$ can be identified as $\phi^T(s,t)$ and

$$\begin{aligned} \psi(s) &= 2\int_s^t \phi^T(\tau,s)\,S(\tau)\,x(\tau)\,d\tau \\ &= 2\left[\int_s^t \phi^T(\tau,s)\,S(\tau)\,\phi(\tau,s)\,d\tau\right] x(s) \end{aligned}$$

By comparison with (4.2) it can be seen that

$$\psi(s) = 2P(t,s)\,x(s) \tag{5.5}$$

where $P$ is the kernel of the quadratic form for the cost.

Thus the gradient with respect to time-varying gains is

$$G(s) = 2\left[B^T P(t,s) + RD(s)\right] X(s) \tag{5.6}$$

and the gradient with respect to fixed gains is

$$G = 2\int_0^t \left[B^T P(t,s) + RD\right] X(s)\,ds \tag{5.7}$$

If $D$ is allowed to vary with time and there is no restriction on its form then the gradient vanishes regardless of the state when

$$D(s) = -R^{-1}(s)\,B^T(s)\,P(t,s) \tag{5.8}$$

This is a natural optimal solution which does not depend on the initial condition and subsequent path. Substituting (5.4) and (1.6) in (4.3), the equation for $P$ when $D$ is optimal is

$$-\frac{d}{ds}P(t,s) = C^T QC + A^T P + PA - PBR^{-1}B^T P \qquad , P(t,t) = 0 \tag{5.9}$$

This is the well known matrix Riccati equation, the properties of which have been thoroughly explored. Its integration yields jointly the optimal $P$ and the optimal $D$.

If $A, B, C, Q$, and $R$ are constant and $D$ is restricted to be constant then

$$P = P(t - s)$$

In this case if $t \to \infty$ and the system is stable $P$ approaches a limiting value $P_\infty$ so that the gradient vanishes for all $X$ if

$$D = -R^{-1} B^T P_\infty$$

The integrand is then everywhere zero, so this is also the optimal solution for an infinite internal when $D$ is allowed to vary with time. For a finite interval, on the other hand, $P$ is not constant and $B^T P + RD$ cannot be made to vanish throughout the interval by choice of a fixed $D$, so the optimal fixed $D$ depends on $X$, and therefore on the initial state. One can then fine the minimum of $J$ by using one of the gradient or conjugate gradient techniques for functions of a finite number of variables. The gradient can be evaluated from (5.2) and (5.4) or (4.3) and (5.7).

# 6 Gradient with respect to the controls

It is interesting to compare the gradient with respect to the feedback matrix of a closed loop system with the gradient with respect to the control vector $u$ of the corresponding open loop system, (1.1) and (1.5b) yield the variational equations

$$\delta\dot{x} = A\delta x + B\delta u \qquad , \quad \delta x\,(0) = 0$$
$$\delta\dot{J} = 2x^T V\delta x + 2u^T P\delta u \qquad , \quad \delta J\,(0) = 0$$

where

$$V = C^T Q C$$

The adjoint equations (3.7) now become

$$\dot{\psi} = -A^T\psi \qquad , \quad \psi\,(t) = 0$$

and the gradient in function space is

$$g_j\,(s) = \sum_i \psi_i\,(s)\,B_{ij}\,(s) + 2\sum_k u_k\,(s)\,R_{kj}\,(s)$$

or in vector notation

$$g\,(s) = B^T\,(s)\,\psi\,(s) + 2R\,(s)\,u\,(s)$$

Also we can try to satisfy

$$g\,(s) = 0$$

by setting

$$u\,(s) = D\,(s)\,x\,(s)$$

Then $\psi\,(s)$ is given by (5.5) and it can be seen that the gradient does in fact vanish if $D$ is given by (5.8). The optimal feedback solution with time-varying gains is thus also the optimal solution for a free choice of $u$.

# 7 Gradient in terms of the outer product

The gradient may alternatively be deduced from the equation written in terms of the outer product $X$.

(4.4) and (4.6) yield the variation equations

$$\delta\dot{X} = F\delta X + \delta X F^T + B\delta D X + X\delta D^T B^T \quad , \quad \delta X(0) = 0 \tag{7.1a}$$

$$\delta\dot{J} = \mathrm{Tr}\,(S\delta X) + 2\mathrm{Tr}\left(D^T R\delta D X\right) \quad , \quad \delta J(0) = 0 \tag{7.1b}$$

Let $P$ be a symmetric matrix satisfying

$$\dot{P} = -F^T P - PF - S \quad , \quad P(t) = 0 \tag{7.2}$$

Then after cancelling terms, remembering that

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$$

it can be seen that

$$\frac{d}{dt} = \mathrm{Tr}\,(P\delta X) = 2\mathrm{Tr}\left(B^T P X \delta D^T\right) - \mathrm{Tr}\,(S\delta X)$$

Thus

$$\frac{d}{dt}\left[\delta J + \mathrm{Tr}\,(P\delta X)\right] = \mathrm{Tr}\left(G\delta D^T\right)$$

where

$$G = 2\left(B^T P + RD\right) X \tag{7.3}$$

and since $\mathrm{Tr}\,(P\delta X)$ vanishes at both boundaries

$$\delta J(t) = \int_0^t \mathrm{Tr}\left(G\delta D^T\right) ds = \int_0^t \sum_i \sum_j G_{ij}\delta D_{ij} ds \tag{7.4}$$

Thus $G_{ij}$ is the gradient with respect to $D_{ij}$. By comparing (4.3) and (7.2) $P$ may be identified with the kernal $P(t,s)$ of the quadratic form for the cost. This kernal is thus seen to be precisely the costate variable for the outer product.

# 8 The statistical case

Let

$$\dot{x} = Ax + Bu + Gv \tag{8.1}$$

where $v$ is a white noise vector with zero mean and correlation matrix $V(t)\,\delta(t-s)$. Let $\bar{x}$ be the mean of $x$, and let $X$ denote the expectation $E\left(xx^T\right)$. The covariance matrix $E\left\{(x-\bar{x})(x-\bar{x})^T\right\}$, the mean $\bar{x}$, and $X$ are related by the equation

$$X = E\left\{(x-\bar{x})(x-\bar{x})^T\right\} + \bar{x}\bar{x}^T$$

If

$$u = Dx$$

then $\bar{x}$ and $X$ now satisfy the seperate equations

$$\dot{\bar{x}} = F\bar{x} \tag{8.2}$$

$$\dot{X} = FX + XF^T + W \tag{8.3}$$

where $F$ is defined by (1.6a) and

$$W = GVG^T \tag{8.4}$$

Also if for the performance index we now take the expectation

$$J = E\left\{\int_0^t \left(y^TQy + u^TRu\right) ds\right\} = E\left\{\int_0^t x^TSxds\right\}$$

where $S$ is defined by (1.6b) then

$$\dot{J} = \mathrm{Tr}\,(SX)$$

It can easily be verified by differentiation that

$$X(t) = \phi(t,s)\,X(s)\,\phi^T(t,s) + \int_s^t \phi(t,\tau)\,W(\tau)\,\phi^T(t,\tau)\,d\tau$$

Thus

$$\begin{aligned}\dot{J} &= \mathrm{Tr}\left[S(t)\,\phi(t,s)\,X(s)\,\phi^T(t,s)\right] + \mathrm{Tr}\left[S(t)\int_s^t \phi(t,\tau)\,W(\tau)\,\phi^T(t,\tau)\,d\tau\right] \\ &= \mathrm{Tr}\left[\phi^T(t,s)\,S(t)\,\phi(t,s)\,X(s)\right] + \mathrm{Tr}\left[\int_s^t \phi^T(t,\tau)\,S(t)\,\phi(t,\tau)\,W(\tau)\,d\tau\right]\end{aligned}$$

and

$$\begin{aligned}J(t) - J(s) &= \mathrm{Tr}\left[\int_s^t \phi(\tau,s)\,S(\tau)\,\phi(\tau,s)\,X(\tau)\,d\tau\right] \\ &\quad + \mathrm{Tr}\left[\int_s^t d\tau \int_s^t \phi(\tau,\rho)\,S(\tau)\,\phi(\tau,\rho)\,W(\rho)\,d\rho\right]\end{aligned}$$

The first term is $\mathrm{Tr}\left[P(t,s)\,X(s)\right]$ where $P$ has been defined in (4.2), and the second term is

$$\mathrm{Tr}\left[\int_s^t W(\rho)\,d\rho \int_\rho^t \phi^T(\tau,\rho)\,S(\tau)\,\phi(\tau,\rho)\,d\tau\right] = \int_s^t \mathrm{Tr}\left[W(\rho)\,P(t,\rho)\right] d\rho$$

Thus

$$J(t) - J(s) = \mathrm{Tr}\left[P(t,s)\,X(s)\right] + Y(t,s)$$

where

$$\frac{d}{ds}Y(t,s) = -\mathrm{Tr}\left[P(t,s)\,W(s)\right] \quad , \quad Y(t,s) = 0$$

The variational equations

$$\begin{aligned}\delta\dot{X} &= F\delta X + \delta X F^T + B\delta D X + X\delta D^T B^T \\ \delta\dot{J} &= \mathrm{Tr}\left(S\delta X\right) + 2\mathrm{Tr}\left(D^T R\delta D X\right)\end{aligned}$$

are identical to (7.1) with the new definition of $X$.

Thus in the absence of any restriction of the feedback paths the optimal time-varying gains are unchanged by the presence of white noise and may be determined by integrating the matrix Riccati equation (5.9) Ths cost is increased by the term $Y(t,s)$.