Measurement Noise versus Process Noise in Ionosphere Estimation for WAAS

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ABSTRACT

One of the parameters driving the performance of the Wide Area Augmentation System (WAAS) is the Grid Ionospheric Vertical Error (GIVE). The GIVE bounds the estimation error of the ionospheric delay at each Ionospheric Grid Point (IGP). The GIVE is generated such that a user interpolating both the vertical ionospheric delays at the IGP and the GIVE, is protected. The GIVE is a function of several parameters: the geometry of the measurements, the measurement noise, and the state of the ionosphere, which yields the process noise. It is very important to distinguish carefully between measurement noise and process noise, as they have a different behavior in the generation of the confidence bound. The measurement noise contribution to the confidence bound tends to zero as the number of measurements increases. The process noise, characterized in the current WAAS algorithm by a standard deviation around the planar trend in nominal conditions, indicates a lower bound for the confidence bound. For each satellite, the measurement noise is well characterized as a function of the elevation and tracking time. However, the process noise, which has a large variability due to the possibility of storms or mild irregularities, is measured or tested only through observations that have measurement noise. For this reason, the observability of the state of the ionosphere is impaired by the measurement noise. Although this is not a problem for the current level of service for WAAS, it could become an issue as we try to increase availability by decreasing the conservatism of the nominal state of the ionosphere.

In this study, we develop a set of formulas to evaluate the Probability of Hazardously Misleading Information (PHMI) for two possible algorithms. This analysis takes into account the loss of observability of the state of the ionosphere (which determines the process noise) due to the measurement noise. It can be applied to any vertical ionospheric delay model that has a deterministic trend and a random gaussian component. We will also see in what cases it is essential to distinguish between measurement noise and process noise. This analysis will help maintain integrity for further improvements to the WAAS ionospheric algorithms.

INTRODUCTION

It is well known that the large ionosphere variability over time and space together with the –necessarily- irregular sampling of the ionosphere has caused the WAAS ionosphere confidence bounds or Grid Ionospheric Vertical Errors (GIVE) to be very large [1], [2]. The GIVE calculation needs to take into account the geometry of the measurements, the noise affecting each of those measurements –the measurement noise- and the state of the ionosphere –which determines the process noise. While the measurement noise is well known at all times [3], the process noise is unknown in real time, but can be inferred from the measurements (which are affected by measurement noise). The goal of this paper is to understand the effect of the measurement noise on the uncertainty of the ionosphere state, and, as a consequence, on the probability of hazardously misleading information (PHMI) [4].

Despite its variability, the ionosphere can be well described by a very simple model. This model states that, locally, the ionosphere follows a planar trend [1]. Once the trend is removed, the residuals can be modeled by a random gaussian field –the process noise. One can either assume a constant covariance, or more accurately, a covariance that depends on distance [5]. The most common covariance structure is called the nominal ionosphere. Because the ionosphere does not always follow the nominal model, a chi-square test statistic is computed on the available measurements to check that they are compatible with the assumed nominal model [1]. Even if the measurements pass the test, the confidence bound needs to be inflated by a factor labeled R_{irreg}, to take into account the possibility that noise is impeding our ability to detect disturbed ionospheric conditions. The statistic can be used in several ways. In this work we will focus on two possible ways of computing this inflation. The first one is currently used in the WAAS ionospheric correction algorithm. In this option, the chi-square statistic is only used to check whether the measurements are compatible with the model. The second one, a proposed enhancement of the current algorithm, uses the chi-square statistic explicitly to correct the confidence bound [6]. This algorithm is called Real Time R_{irreg}. In both cases we need to compute the PHMI in order to evaluate the integrity of the system.

This problem can only be approached using models for the random processes and for the ionospheric structure –both in quiet and storm conditions. In the first part, we will introduce the models and the assumptions we need to make. Then we will derive formulas to evaluate the PHMI for the two algorithms mentioned above. For both algorithms, we will study the dependency of the PHMI on measurement noise.

DESCRIPTION OF THE PROBLEM

It has been shown that within the thin shell model, the ionosphere is well described by a planar trend and a random gaussian field:

$$I(x) = a_0 + a_1 x^{(1)} + a_2 x^{(2)} + r(x)$$

where x is the location. The covariance function of r is called C However, this covariance structure is usually unknown.

At each time frame we have m ionospheric measurements, collected at the reference stations with dual frequency receivers. These measurements are corrupted by the measurement noise n, whose mean should be zero (the biases are removed using an off line least square process):

$$\tilde{I}(x) = a_0 + a_1 x^{(1)} + a_2 x^{(2)} + r(x) + n(x)$$

We call N the covariance matrix of n(x). It is usually a diagonal matrix since measurement noise is uncorrelated from one receiver to another.

In the two algorithms considered here the user computes the vertical delay correction for each of the satellites in view by forming a linear estimate of the measurements [5]:

$$\hat{I}(x) = \sum_{i=1}^{m} l_i \tilde{I}(x_i)$$

In this work it is not relevant how the weights are computed. The estimation variance is, for each of the delays:

$$E\left(\left(I(x) - \hat{I}(x)\right)^2\right) = E\left(\left(I(x) - \sum_{i=1}^{m} l_i I(x_i)\right)^2\right)$$
$$= C(x,x) - 2\,l^T C(x,x_i) + l^T\left(C(x_i,x_j) + N(x_i,x_j)\right) l$$

We can divide this expression in two terms: the terms due to process noise, C, and the terms due to measurement noise:

$$s^2_{process} = C(x,x) - 2\,l^T C(x,x_i) + l^T C(x_i,x_j)\, l$$
$$s^2_{meas} = l^T N(x_i,x_j)\, l$$
$$s^2_{true} = E\left(\left(I(x) - \hat{I}(x)\right)^2\right) = s^2_{process} + s^2_{meas}$$

The term s_{meas}, which depends on measurement noise, can be computed (or we can at least find an overbound of it). However, $s_{process}$ depends on C, which we do not know. We do know that the ionosphere most of the time is well described by a nominal covariance C_{nom}. The estimation variance assuming a nominal covariance is:

$$s^2_{nom} = s^2_{process,nom.} + s^2_{meas}$$

where:

$$s^2_{process,nom.} = C_{nom}(x,x) - 2\,l^T C_{nom}(x,x_i)$$
$$+ l^T C_{nom}(x_i,x_j)\, l$$

We see that the difference between the true variance estimation and the computed estimation variance can be very large, depending on the state of the ionosphere. There fore we need to correct for this possible difference.

At this point, we need to make additional assumptions. A practical and very good approximation is to assume that true covariance is a multiple of the nominal covariance [7]:

$$C_{true} = u^2 C_{nom.}$$

This characterization captures most of the effects of a disturbed ionosphere, if we assume that the ionosphere is stationary. In this paper, we do not treat deviations from stationarity (for more information on how they are dealt with in WAAS, please refer to [2]). Now, all the uncertainty concerning ionospheric behavior is summarized in the unknown parameter u. To summarize, we have:

$$s_{nom}^{2} = s_{process,nom.}^{2} + s_{meas}^{2}$$
$$s_{true}^{2} = u^{2} s_{process,nom}^{2} + s_{meas}^{2}$$

The question now is how to best use the available real time measurements to correct the difference between these two and fulfill the PHMI requirements. The two algorithms examined here make use of the chi-square statistic of the measurements to compute the inflation factor R_{irreg}. In this work, we choose to multiply also the term s_{meas} because this is how it is done in the current system. The confidence bound is:

$$s_{estimated}^{2} = R_{irreg}^{2}\left(s_{process,nom}^{2} + s_{meas}^{2} \right)$$

It has to be such that, when a correction is sent and flagged as usable, the probability of an actual error is $5.33 s_{estimated}$ is below 10^{-7} (the factor 5.33 corresponds to the 10^{-7} quantile in a gaussian distribution):

$$P\left(\left| I_{real}(x) - \hat{I}_{estimated}(x) \right| > K s_{estimated}^{2} \right) \le 10^{-7}$$

This is what constitutes the PHMI requirement [4].

STORM DETECTOR IN THE CURRENT WAAS ALGORITHM

We now analyze the WAAS storm detector. First, a chi-square distributed quadratic form on the measurements is computed. At this point there are two possible outcomes. If the statistic is above a pre-defined threshold, the measurements are declared not to be compatible with the nominal model and the confidence bounds set to their maximum value. If the statistic is below the threshold the measurements are assumed to be compatible with the nominal model. However, we need to account for the cases where a disturbed ionosphere results on a low chi-square statistic. This is done by inflating the estimation variance by the factor Rirreg. In this algorithm, R_{irreg} is independent of the chi-square statistic.

The first step is to form a chi-square distributed variable from the measurements. This is only a possibility if we know the underlying covariance. Suppose the underlying covariance is the nominal covariance. From the m measurements we can get m-3 residuals that are independent unit gaussian under the nominal model:

$$y = \Gamma I_{meas}$$

Here I_{meas} is the vector of vertical ionospheric delay measurements (See appendix for an expression for G). The sum of the components of y is chi-square distributed under the nominal model. If the true model is not the nominal model, y is still 0 mean but its covariance is no longer the identity. The covariance is instead:

$$E\left(y^{T} y\right) = u^{2} \Gamma C_{nom}\left(x_{i}, x_{j}\right)\Gamma^{T} + \Gamma N\left(x_{i}, x_{j}\right)\Gamma^{T}$$
$$= u^{2} C' + N' = S(u)$$

When u is different from unity, the sum of the components of y will not necessarily be chi-square distributed. There is a case where it will be chi-square distributed, which corresponds to N' bein a multiple of the unity:

$$N' = m I_{m-3}$$

Since we know that we have:

$$C' + N' = I_{m-3}$$

We need to have:

$$C' = (1-m) I_{m-3} \text{ and } m \in [0,1]$$

The parameter µ can be thought of as the proportion of the covariance matrix coming from the measurement noise. We have in this case:

$$E\left(y^{T} y\right) = \left(u^{2}(1-m) + m\right) I_{n-3}$$

This result means that:

$$\frac{y^{T} y}{u^{2}(1-m) + m}$$

is chi-square distributed. We see already in this equation the effect of measurement noise on the chi-square distribution: as µ gets closer to one, the distribution is less dependent on the parameter u. As a consequence, it will be more difficult to distinguish storm conditions from quiet conditions as measurement noise increases.

We now go back to the general case and derive an expression for the PHMI for the storm detector. First of all, we condition the PHMI on the ionosphere state:

$$PHMI = \int_{u=0}^{u=+\infty} P\left(HMI \mid u\right) P(u) du$$

Since we do not want to specify any a priori distribution of u, we are going to study *P(HMI|u)*. This term will be the product of the probability of the chi-square statistic being below the threshold and the probability of having an error 5.33 times larger than the broadcast confidence bound:

$$P(HMI \mid u) = P\left(\left|I - \hat{I}\right| > KR_{irreg}\sqrt{\sigma^2_{process,nom.} + \sigma^2_{meas}} \mid u\right) P\left(y^T y < T \mid u\right) du$$

Here, there are two tunable parameters that can be changed: R_{irreg} and the threshold T. Because we are assuming gaussian random variables, the first factor of this product is:

$$P\left(\left|I - \hat{I}\right| > KR_{irreg}\sqrt{\sigma^2_{process,nom.} + \sigma^2_{meas}}\right) = 2Q\left(KR_{irreg}\sqrt{\frac{\sigma^2_{process,nom.} + \sigma^2_{meas}}{u^2\sigma^2_{process,nom.} + \sigma^2_{meas}}}\right)$$

As it was pointed out above, the term $P(y^T y<T|u)$ is in general not chi-square distributed. Instead, it is a weighted sum of chi-square variables. Although there is no simple analytical expression for such a distribution, an easy approximation can be found in [8].

Figure 1 shows the behavior of the PHMI for a given R_{irreg} and a given threshold T but different levels of noise. We considered a situation with 30 measurements. The ionosphere covariance is $\sigma_{decorr}^2 I$, with σ_{decorr} =.35 m. The measurement noise is assumed to be the same for all measurements. What is very noticeable is the degradation of the PHMI for large values of u as measurement noise increases.

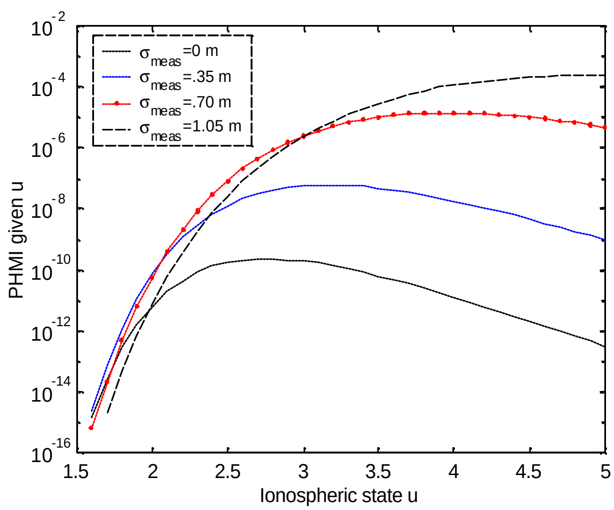

Figure 1. *PHMI as a function of the state of the ionosphere for different measurement noise.*

REAL TIME R_{irreg}

Another way of taking into account the ionosphere behavior is by making R_{irreg} depend on the actual measurements. A reasonable algorithm consists on making R_{irreg}^2 a quadratic form of the measurements:

$$R^2_{irreg} = y^T R y$$

If we assume that we do not make use of the threshold, the PHMI (given the ionosphere state u) is:

$$P(HMI \mid u) = \int_y Q\left(\sqrt{y^T M(u) y}\right) p(y \mid u) p(u) dy$$

where:

$$M(u) = K^2 \frac{\sigma^2_{process,nom.} + \sigma^2_{meas}}{u^2\sigma^2_{process,nom.} + \sigma^2_{meas}} R$$

It would be impossible to try to evaluate this integral numerically, since it has n-3 dimensions (usually above 30). But it turns out that this m-3-dimensional integral can be transformed in a univariate integral, using Craig's formula for the Q-function [9]. The result is:

$$P(HMI \mid u) = \frac{2}{\pi}\int_{\phi=0}^{\frac{\pi}{2}} \left|\frac{M(u)\left(u^2C' + N'\right)}{\sin^2(\phi)} + I_{m-3}\right|^{-\frac{1}{2}} d\phi$$

See the appendix for the details of the proof. Where we have:

$$\frac{M(u)\left(u^2C' + N'\right)}{\sin^2(\phi)} + I_{m-3} = K^2 \frac{\sigma^2_{process,nom.} + \sigma^2_{meas}}{u^2\sigma^2_{process,nom.} + \sigma^2_{meas}} \frac{R\left(u^2C' + N'\right)}{\sin^2(\phi)} + I_{m-3}$$

This integral can be easily computed. Figure 2 shows the results for several parameters. The parameters used are the same as in Figure 1. Again we see how the PHMI increases as the measurement noise increases. It is less dramatic than in the storm detector. However, each curve is monotone (it gets worse as u increases), which is not the case in the storm detector. This could be fixed by adding a threshold.

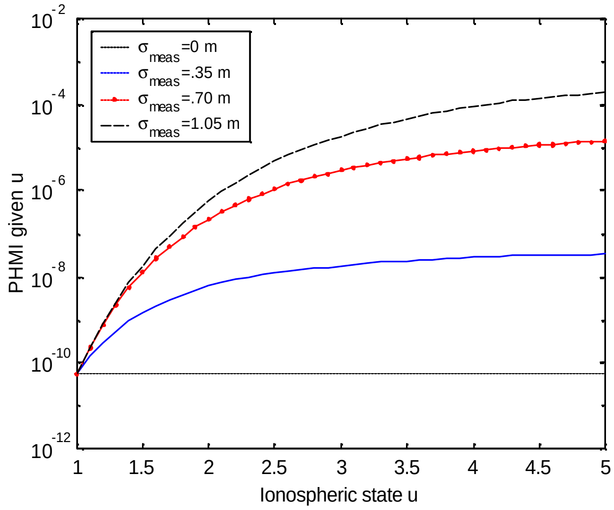

Figure 2. *PHMI as a function of the state of the ionosphere for different measurement noise for the Real Time R_{irreg} algorithm.*

For the noiseless case, the formula becomes:

$$P(HMI \mid u) = \int_{f=0}^{\frac{p}{2}} \left| K^2 \frac{RC'}{\sin^2(f)} + I_{m-3} \right|^{-\frac{1}{2}} df$$

which does not depend on the ionosphere state u. From this formula it is possible to show that the optimal R is a multiple of the identity. Therefore, in the noiseless case, the optimal real time R_{irreg} is a multiple of the chi-square statistic.

CONCLUSION

The large range of ionospheric behavior requires WAAS to monitor the state of the ionosphere before computing the correction and the confidence bound. A good way of monitoring the ionosphere is by using the chi-square statistic. However, it is very important to evaluate the impact of measurement noise on tests based on the chi-square statistic. In this work, we have developed two calculations of the PHMI, one for the current WAAS storm detector and the other for a proposed enhancement of it. Both calculations show that measurement noise has to be taken into account carefully and the confidence bounds need to be adjusted accordingly. Otherwise, the PHMI increases by several orders of magnitude. This analysis will be helpful in particular as conservativeness in the confidence bounds is reduced.

APPENDIX

Obtaining n-3 independent unit gaussian residuals from m measurements.

This is a standard procedure. It is however useful to show here how they can be obtained. The first step is to filter the trend for all measurements. We want to have:

$$\Gamma G a = 0$$

for any vector a. Here G is an n by 3 matrix:

$$G = \begin{bmatrix} 1 & x_1^{(east)} & x_1^{(north)} \\ \vdots & \vdots & \vdots \\ 1 & x_n^{(east)} & x_n^{(north)} \end{bmatrix}$$

We therefore need:

$$\Gamma G = 0$$

The matrix F defined by:

$$F = I - G\left(G^T W G\right)^{-1} G^T W$$

where:

$$W = (C + N)^{-1}$$

has such a property. We have:

$$FI_{meas}(x) = F\left(a_0 + a_1 x^{(1)} + a_2 x^{(2)} + r(x) + n(x)\right)$$
$$= F\left(r(x) + n(x)\right)$$

where $I_{meas}(x)$ designates the vector of measurements and $n(x)$ is the measurement noise.
The resulting vector is now zero mean and gaussian. We only need to diagonalize it. The covariance of FI_{meas} is:

$$E\left(FI_{meas} I_{meas}{}^T F^T\right) = F$$

Since F is rank n-3 (the null space has dimension 3), we will only be able to extract n-3 independent unit gaussian. We need to find a matrix H such that:

$$E\left(HFI_{meas} I_{meas}{}^T F^T H^T\right) = HFF^T H^T = HFH^T = I_{m-3}$$

We will then take:

$$\Gamma = HF$$

Let us see now what H we should use. Let us define $\tilde{H} = HW^{-\frac{1}{2}}$. We have:

$$\tilde{H}\left(I - W^{\frac{1}{2}} G \left(G^T W G \right)^{-1} G^T W^{\frac{1}{2}} \right) \tilde{H}^T = I_{m-3}$$

Let us now define $P = I - W^{\frac{1}{2}} G \left(G^T W G \right)^{-1} G^T W^{\frac{1}{2}}$. P is an orthogonal projection of rank n-3. There exists U orthogonal of size n by n such that:

$$U^T P U = \begin{bmatrix} I_{m-3} & 0 \\ 0 & 0 \end{bmatrix}$$

Let us now write:

$$U = \begin{bmatrix} \tilde{U} & \bar{U} \end{bmatrix}$$

where $\tilde{U}$ is n by m-3, and $\bar{U}$ is n by 3. We have the following equations:

$$\tilde{U}^T P \tilde{U} = I_{n-3}$$
$$\bar{U}^T P \bar{U} = 0$$
$$\bar{U}^T P \tilde{U} = 0$$

From the first equation we see that if we take $\tilde{H} = \tilde{U}^T$ then Γ fulfills the two conditions. $\Gamma$is defined by:

$$\Gamma = \tilde{U}^T \left(W^{\frac{1}{2}} - W^{\frac{1}{2}} G \left(G^T W G \right)^{-1} G^T W \right)$$

Derivation of P(HMI|u) for real time R_{irreg}.

We use Craig's formula for Q(z), where the bounds on the integral do not depend on z:

$$Q(z) = \int_z^{+\infty} \frac{1}{\sqrt{2p}} e^{-\frac{x^2}{2}} dx = \frac{1}{p} \int_{f=0}^{\frac{p}{2}} e^{-\frac{z^2}{2\sin(f)^2}} df$$

We have:

$$P(HMI \mid u) = \int_y 2Q\left(\sqrt{y^T M(u)\, y} \right) p(y \mid u)\, dy$$

$$= \frac{2}{p} \int_y \int_{f=0}^{\frac{p}{2}} e^{-\frac{y^T M(u) y}{2\sin(f)^2}} df \frac{1}{(2p)^{\frac{m-3}{2}} \left| S(u) \right|} e^{-\frac{y^T S(u)^{-1} y}{2}} dy$$

we now switch the order of the integration:

$$P(HMI \mid u) =$$

$$\frac{2}{p} \int_{f=0}^{\frac{p}{2}} \int_y \frac{1}{(2p)^{\frac{m-3}{2}} \left| S(u) \right|^{\frac{1}{2}}} e^{-\frac{y^T M(u) y}{2\sin(f)^2} - \frac{y^T S(u)^{-1} y}{2}} dy df$$

Now we can integrate over y:

$$\int_y e^{-\frac{y^T M(u) y}{2\sin(f)^2} - \frac{y^T S(u)^{-1} y}{2}} dy = (2p)^{\frac{m-3}{2}} \left| \frac{M(u)}{\sin(f)^2} + S(u)^{-1} \right|^{-\frac{1}{2}}$$

finally, we have:

$$P(HMI \mid u) = \frac{2}{p} \int_{f=0}^{\frac{p}{2}} \left| \frac{M(u)}{\sin(f)^2} S(u) + I_{m-3} \right|^{-\frac{1}{2}} df$$

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REFERENCES

[1] Walter T. *et al*, "Robust Detection of Ionospheric Irregularities," in Navigation *Summer 2001, Vol. 48, No. 2.*

[2] Sparks L., *et al*, "The WAAS Ionospheric Threat Model," in proceedings of the Beacon Satellite Symposium, Boston, MA, June 2001.

[3] Shallberg K., Shloss P., Altshuler E., Tahmazyan L., "WAAS Measurement Processing, Reducing the Effects

of Multipath," in proceedings of ION GPS 2001, Salt Lake City, UT, September 2001.

[4] Schempp T.R., Peck S.R., Fries R.M., "PHMI equation WAAS Algorithm Contribution to Hazardously Misleading Information (HMI)," in proceedings of ION GPS 2001, Salt Lake City, UT, September 2001.

[5] Blanch J." An Ionosphere Estimation Algorithm for WAAS Based on Kriging," in proceedings of ION GPS 2002, Portland, OR, September 2002.

[6] Cormier D. "Real Time R_{irreg}" Raytheon internal communication.

[7] Hansen, A., Walter T., Blanch, J., and Enge P., "Ionospheric Correlation Analysis for WAAS: Quiet and Stormy," in proceedings of ION GPS, Salt Lake City, UT, September 2000.

[8] Gabler S, Wolff C. "A quick and easy approximation to the distribution of a sum of weighted chi-square variables." Statistische Hefte 1987; 28:317-25.

[9] Craig J. "New, simple and exact result for calculating the probability of error for two-dimensional signal constellations," Proceedings of Milcom 1991.