No Exist of Odd Perfect Number

Zengyong Liang

Abstract: whether there has exist of odd perfect numbers is a well- known problem in number theory . This paper proved that no prefect number exist in any cases.

Key words: odd perfect number; function $\sigma(n)$;function k

1 Introduction

A perfect number is a number n such that $\sigma(n) = 2n$. In other words a number is perfect number if it is the sum of its divisors other than itself . We now only known a few of perfect numbers which are even numbers[1][2][3] . It seems probable that there are no odd perfect numbers , bat this has not been proved. It is unknown whether there are any odd perfect numbers, though various results have been obtained. Carl Pomerance has presented a heuristic argument which suggests that no odd perfect numbers exist[5].

Any odd perfect number N must satisfy the following conditions:

- $N > 10^{200}$, which it must have at least 8 different prime factors and that its largest prime factor must be greater than 100110 [1].
- $N > 101500$, result published in 2012[4].
- N is of the form

$$N = q^{\alpha} p_1^{2e_1} ... p_k^{2e_k}$$

  where:
  $q, p_1, ..., p_k$ are distinct primes (Euler).
  $q \equiv \alpha \equiv 1 (mod 4)$ (Euler).
  The smallest prime factor of N is less than (2k + 8) / 3 [4].
- The largest prime factor of N is greater than 10^8 [4].
- The second largest prime factor is greater than 10^4, and the third largest prime factor is greater than 100 [4].
- N has at least three prime factors. There is another argument [5].
- $N > 10^{200}$ and with at least 15 different prime factors [5].
- If N is odd perfect and $\omega(N) < k$, then $N < 2^{4^k}$ [6].

1.1 Function $\sigma(N)$

Definition: Function $\sigma(n)$ is the sum of combination of its divisors [1] .
Let $N = q_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_n}$. Using mathematical induction is very easy to prove $\sigma(N)$, which expression of the product below:

$$\sigma(N) = (1 + q_1 + ... + q_1^{\beta_1})(1 + q_2 + ... + q_2^{\beta_2})...(1 + q_i + ... + q_n^{\beta_n}) \quad (1)$$

where q_n is a prime factor ,which $q_1 < q_2 < ... < q_n$.

If N is a prefect number that must be

$$\sigma(N) = 2N \quad (2)$$

2 Coefficient Function k_x

2.1 Function k_x

We obtain

$\sigma(N) = (1 + q_1 + ... + q_1^{\beta_1})(1 + q_2 + ... + q_2^{\beta_2})...(1 + q_i + ... + q_n^{\beta_n}) = 2q_1^{\beta_1} q_2^{\beta_2} ... q_n^{\beta_n}$

by (1)and (2). then

$$\frac{1 + q_1 + ... + q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1 + q_2 + ... + q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1 + q_n + ... + q_n^{\beta_n}}{q_n^{\beta_n}} = 2 \qquad (3)$$

Let $k_x = \frac{1+q_x+...+q_x^{\beta_x}}{q_x^{\beta_x}}$, so

$$k_1 k_2 ... k_n = 2 \qquad (4)$$

The k_x is called as the coefficient function of $\sigma(N)$. Clearly, for any N ,if the equation(4)is true, so $\sigma(N) = 2N$,the N must be a prefect number.

2.2 The Values of k_x

Theorem 1. . $q_x^{\beta_x} + q_x^{\beta_x - 1} + ... + q_x + 1 < (1 + \frac{1}{q_x - 1}) q_x^{\beta_x}$

Proof. By algebraic formula we obtain

$$q_x^{\beta_x} - 1 = (q_x - 1)(q_x^{\beta_x - 1} + q_x^{\beta_x - 2} + ... + q_x + 1)$$

$$q_x^{\beta_x - 1} + q_x^{\beta_x - 2} + ... + q_x + 1 = \frac{q_x^{\beta_x} - 1}{q_x - 1} < \frac{q_x^{\beta_x}}{q_x - 1}$$

so

$$q_x^{\beta_x} + (q_x^{\beta_x - 1} + ... + q_x + 1) < (1 + \frac{1}{q_x - 1}) q_x^{\beta_x}$$

□

Theorem 2. . For any odd q_x, there have $1 < k_x < 1.5$ .As q_x is larger,the values of k_x is less.

Proof. Let q_x is odd ,we obtain

$$q_x^{\beta_x} + q_x^{\beta_x - 1} + ... + q_x + 1 < (1 + \frac{1}{q_x - 1}) q_x^{\beta_x}$$

by Theorem 1.

i.e.

$$k_x = \frac{q_x^{\beta_x} + q_x^{\beta_x - 1} + ... + q_x + 1}{q_x^{\beta_x}} < 1 + \frac{1}{q_x - 1}$$

It is clear that $k_x > 1$.

As q_x is larger, the $\frac{1}{q_x - 1}$ is less. If q_x is odd,when $q_x = 3$ is least odd prime,then $\frac{1}{q_x - 1}$ is maximal,and value of k_x is also maximal . The maximal value of k_x is written max(k_x),then

$$max\ (k_x) = k_3 < (1 + \frac{1}{q_x - 1}) = 1 + \frac{1}{3 - 1} = 1.5$$

but $q_x \to \infty$, $\frac{1}{q_x - 1}$ is $\to$minimal,and value of k_x is $\to$minimal .The minimal value of k_x is written min(k_x),then

$$min\ (k_x) > \lim_{q \to \infty} (1 + \frac{1}{q_x - 1}) = 1$$

Hence , if q_x is odd, $1 < k_x < 1.5$. □

2.3 The Table of Values of k_n

Table 1. . A fell values of k_n

n	p_n	$k(1+p_n)$	$k(1+p_n+p_n^2)$	$k(1+p_n+p_n^2+p_n^3)$	max $k_n <$
1	2	1.5	1.75	1.875	2
2	3	1.3333	1.4444	1.4814	1.5
3	5	1.2	1.24	1.248	1.25
4	7	1.1428	1.1632	1.1661	1.167
5	11	1.0909	1.0991	1.0999	1.1
6	13	1.0769	1.0828	1.0832	1.084

This shows that k_n is a monotone decreasing functions with p_n. The larger p_n , the smaller value of k_n .

3 Proof of proposition

Theorem 3. . Let k_y be a decimal , c be a integer .If to make ck_y is integer ,there have must be the following conditions:

1)if the bottom digit of decimal of k_y is 5 , then c must be even number;for example, $k_y = 1.25, c = 4,\ 1.25 \times 4 = 5$.

2)if the bottom digit of decimal of k_y is a even number, then bottom digit of the c must be 5, for example, $k_y = 1.2, c = 15,\ 1.2 \times 15 = 18$.

Proof. Because there has only a product of 5 and even number can carry to make the decimal k_y change to integer,so this theorem is true.

Theorem 4. . If N is odd ,$N = q_1^{\beta_1} q_2^{\beta_2} ... q_n^{\beta_n}$, then $k_1 k_2 ... k_n \neq 2$.

Proof. Suppose the N is a prefect number, then

$$k_1 k_2 ... k_n = \frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_n+...+q_n^{\beta_n}}{q_n^{\beta_n}} = 2$$

by equations of (3)and (4).

Set $1 + q_n + ... + q_n^{\beta_n} = 2b_n$,

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{n-1}+...+q_{n-1}^{\beta_{n-1}}}{q_{n-1}^{\beta_{n-1}}} \times \frac{2b_n}{q_n^{\beta_n}} = 2 \quad (5)$$

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{n-1}+...+q_{n-1}^{\beta_{n-1}}}{q_{n-1}^{\beta_{n-1}}} \times \frac{b_n}{q_n^{\beta_n}} = 1 \quad (6)$$

so it needs $\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{n-1}+...+q_{n-1}^{\beta_{n-1}}}{q_{n-1}^{\beta_{n-1}}} \times b_n = c$

where c is an integer, and $c = q_n^{\beta_n}$.

I.e., $k_1 k_2 ... k_{n-1} b_n = a$.

Because :1) the right hand side of equation (5) is 2 , then the left hand side must be only containing one factor 2 ,and all denominators are odd, then all numerators must be odd.

2) we known all k_x are decimals , then right hand side of equation (6) can not is a integer by 1) and the Theorem (3).

So

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{n-1}+...+q_{n-1}^{\beta_{n-1}}}{q_{n-1}^{\beta_{n-1}}} \times \frac{b_n}{q_n^{\beta_n}} \neq 1$$

and

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{n-1}+...+q_{n-1}^{\beta_{n-1}}}{q_{n-1}^{\beta_{n-1}}} \times \frac{2b_n}{q_n^{\beta_n}} \neq 2$$

Hence the $k_1k_2...k_n \neq 2$. □

Proposition . *If* N is odd,and $N = q_1^{\beta_1}q_2^{\beta_2}...q_n^{\beta_n}$, then it is impossible that N is an odd perfect number .

Proof. Let N be odd, and $N = q_1^{\beta_1}q_2^{\beta_2}...q_n^{\beta_n}$, then:

1)when $n = 2$, $n = q_1^{\beta_1}q_2^{\beta_2}$,since the least q_x are 3 and 5 , their values are largest which less 1.5 and 1.25 ,then $k_1'k_2' < 1.5 \times 1.25 = 1.875$. So, i.e.,when $n \leq 2$ the $k_1k_2 < 2$, by the Theorem 2.

2)when $n \geq 3$,we can prove also that $k_1k_2...k_n \neq 2$, by the Theorem 4.

Wherefore , if N is odd, the $k_1k_2...k_n \neq 2$. So $\sigma(N) \neq 2N$,and the N is not a prefect number by definition. □

4 Conclusion

Now, we can say that may no odd perfect number . From above various aspects have been proved that can not has exist of odd perfect number . The perfect number is only a characteristic of even number .

References

[1] G.H.Harly E.M.Wright,*An Introduction to the Theorem of Numbers* , Posts and telecom press, Beijing, 2007,19-256.

[2] Zhou Xiaorao, Yu Mo, *Interesting Famous Quistions of Number Theorem* , Hunan university press, Changsha, 2012,115-116.

[3] (U.A.S)Du De Li, (Zhou Zhongliang translated)*Basic Number Theory*, Harbin institute of technology press , Harbin, 2011,47-50.

[4] From Wikipedia, the free encyclopedia , *Perfect Number* ; available at http://en.wikipedia.org/wiki/Perfect number .

[5] San Zun ,*The Knowledge and Question of Elementary Number Theory*, Harbin institute of technology press , Harbin, 2011,54-55.

[6] Paul Pollack , *Finiteness Theorems for Perfect Numbers and Their Kin* , Amer. Math. Monthly 119 (2012),670-681 ; available at http://www.maa.org/.