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Guaranteed error bounds for model reduction of linear time-varying systems

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Abstract

New techniques are presented for the model reduction of linear time-varying and linear periodically-varying systems, including the formulation and proof of guaranteed upper bounds for the error. The commonly used method of balanced truncation for linear time-invariant systems is generalized to the time-varying case with explicit error bounds that are derived based on generalizations of the 'twice-the-sum-of-the-tail' formula. The development of these reduction results for time-varying systems relies on a new operator framework for analysis of linear time-varying systems, presented in [4], in combination with the model reduction methods for uncertain systems developed in [3].

1. Introduction

In this paper, new techniques are developed for the model reduction of linear time-varying (LTV) systems. Explicit bounds are derived for the error achieved when balanced truncation methods are applied to such systems.

The method of balanced truncation has previously been proposed for model reduction of linear time-varying and linear periodically-varying systems by several authors [15, 14, 10, 13]. The work in these papers generalizes the standard time-invariant results by utilizing time-varying versions of the standard controllability and observability gramians, computed via solution of Riccati recursions. However, such recursions are in general difficult to solve computationally. More notably, the previous work does not address the problem of finding an error bound for the reduced model.

In this paper we propose a new approach to this problem. A recently proposed operator framework for analysis of time-varying and periodically-varying systems, developed in [4], is used extensively. This framework allows for a generalization of the notion of balancing, formulated in terms of structured solutions to linear matrix inequalities (LMIs), which was presented in [3] for the model reduction of uncertain systems represented in the linear fractional transformation (LFT) framework. This LMI approach provides a means for simple computation of the required *generalized gramians* and hence also the required balanced and truncated systems.

The main contribution of this paper is the development of guaranteed bounds on the error, given in the induced 2-norm, between the truncated and the original system. For general time-varying systems, a new proof is presented for the error bound. For periodic systems, we show that the system may be viewed as an equivalent time-invariant system with structured linear fractional uncertainty; hence the methods for model reduction of uncertain systems in [3] can be directly applied.

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2. Preliminaries

The notation used throughout this paper will be the same as that developed in [4]. We will be analyzing linear time-varying systems in discrete time, described in state space notation as

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k \\ z_k &= C_k x_k + D_k w_k \end{aligned} \tag{1}$$

for $w \in \ell_2$. As in that reference, the system can be described by block diagonal operators $A = \operatorname{diag}(A_0, A_1, A_2, \dots)$, and similarly for B, C, D. The operator mapping w to z is then given by $G = C(I - ZA)^{-1}ZB + D$, where Z is the shift operator on ℓ_2. Thus G is a bounded operator if and only if $1 \notin \operatorname{spec}(ZA)$, and this is equivalent to the standard notion of stability of LTV systems, that is exponential stability. See [4] for the details.

If F, G, R and S are block-diagonal operators, and A is a *partitioned* operator, each of whose elements is a block-diagonal operator, such as

$$A = \begin{bmatrix} F & G \\ R & S \end{bmatrix}$$

then we define the following notation:

$$\left[\!\!\left[\begin{matrix} F & G \\ R & S \end{matrix}\right]\!\!\right] := \operatorname{diag}\left(\begin{bmatrix} F_0 & G_0 \\ R_0 & S_0 \end{bmatrix}, \begin{bmatrix} F_1 & G_1 \\ R_1 & S_1 \end{bmatrix}, \dots\right),$$

which we call the *diagonal realization* of A. Useful properties of these permutations are

(i) $[\![A + B]\!] = [\![A]\!] + [\![B]\!]$

(ii) $[\![AC]\!] = [\![A]\!]\,[\![C]\!]$.

(iii) If $Q = \left[\!\!\left[\begin{matrix} F & G \\ R & S \end{matrix}\right]\!\!\right]$ then $Z^*QZ = \left[\!\!\left[\begin{matrix} Z^*FZ & Z^*GZ \\ Z^*RZ & Z^*SZ \end{matrix}\right]\!\!\right]$

(iv) $\left[\!\!\left[\begin{matrix} \left[\!\left[\begin{matrix} P & Q \\ R & S \end{matrix}\right]\!\right] & \left[\!\left[\begin{matrix} T \\ U \end{matrix}\right]\!\right] \\ [\![V \;\; W]\!] & X \end{matrix}\right]\!\!\right] = \left[\!\!\left[\begin{matrix} P & Q & T \\ R & S & U \\ V & W & X \end{matrix}\right]\!\!\right]$

(v) $A < \beta I$ holds if and only if $[\![A]\!] < \beta I$.

Note that in the above, partitioned operators are always chosen to have compatible dimensions, and the shift operator Z always has the same spatial dimensions as the block diagonal operator it is multiplying.

3. LTV systems and the system function

It was shown in [4] that the induced norm of an LTV system is given by the maximum norm of an operator-valued function over a complex ball. In this context, we are using a bounded sequence $\lambda_k \in \mathbb{C}$ of complex numbers as a notion of *frequency*. Given such a sequence, define $\Lambda := \operatorname{diag}(\lambda_0 I, \lambda_1 I, \dots)$, a block-diagonal operator with the same spatial dimensions as A.

Theorem 1. *Suppose A, B, C and D are block-diagonal operators on ℓ_2, and $1 \notin \operatorname{spec}(ZA)$. Then*

$$\|C(I-ZA)^{-1}ZB+D\| = \sup_{\lambda_k \in \bar{\mathbb{D}}} \|C(I-\Lambda ZA)^{-1}\Lambda ZB + D\|,$$

where Λ depends on λ_k.

We refer to the function $E(\Lambda) := C(I-\Lambda ZA)^{-1}\Lambda ZB+D$ appearing in Theorem 1 as the *system function.* The system function plays an instrumental role in our viewpoint, allowing a particularly simple analysis of the induced ℓ_2 norm of LTV systems.

3.1. Evaluating the ℓ_2 induced norm

Define the set $\mathcal{T}$ which consists of block diagonal operators with the same structure as the operator A. Also define the set $\mathcal{X}$ to be $\mathcal{X} = \{X > 0; X \in \mathcal{T}\}$, the set of strictly positive self-adjoint block-diagonal operators in $\mathcal{T}$.

With this definition we can state the following important result.

Theorem 2. *The following conditions are equivalent*

(i) $\|C(I-ZA)^{-1}ZB+D\| < 1$ *and* $1 \notin \operatorname{spec}(ZA)$;

(ii) There exists $X \in \mathcal{X}$ such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0. \quad (2)$$

(iii) There exists $T \in \mathcal{T}$ such that

$$\left\| \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \right\| < 1 \qquad (3)$$

(iv) There exists $X \in \mathcal{X}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} Z^*XZ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0. \quad (4)$$

(v) There exists $T \in \mathcal{T}$ such that

$$\left\| \begin{bmatrix} Z^*TZ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \right\| < 1 \qquad (5)$$

Formally, parts (i)–(iii) of this result are the same as those for the linear time-invariant case, except that the operators ZA and ZB replace the usual A-matrix and B-matrix, and X is block-diagonal. This is a general property of the formalism we use, and provides a simple way to construct and to understand the relationship between time-invariant and time-varying systems.

Using Theorem 2 and the resulting formalism, a straightforward derivation of the model reduction error bound may be developed that closely follows the proof of the time-invariant case. The model reduction proof utilizes the notions of *equivalent* and *contractive* realizations for LTV systems. In order to extend the results of [3] to the LTV framework presented herein, we require the following.

Proposition 3. *Let*

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} := \begin{bmatrix} Z^*TZ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}. \qquad (6)$$

Then

$$C(I-ZA)^{-1}ZB + D = \hat{C}(I-Z\hat{A})^{-1}Z\hat{B} + \hat{D}$$

That is, these are equivalent realizations for the same system.

Thus we have the useful result that a system G is contractive if and only if there exists some realization M

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

which is contractive; that is, such that $\|M\| < 1$.

4. Model reduction for time-varying systems

We now define and discuss balanced model reduction methods for LTV systems. Guaranteed error bounds are stated and proven for these reduction methods. The following lemma allows us to define the notion of balanced realizations for time-varying systems.

Lemma 4. *The following are equivalent*

(i) $1 \notin \operatorname{spec}(ZA)$

(ii) There exists $Y \in \mathcal{X}$ such that

$$AYA^* - Z^*YZ + BB^* < 0 \qquad (7)$$

(iii) There exists $X \in \mathcal{X}$ such that

$$A^*Z^*XZA - X + C^*C < 0 \qquad (8)$$

Proof. First, note that there exists $X \in \mathcal{X}$ such that $ZAXA^*Z^* - X < 0$ if and only if $1 \notin \operatorname{spec}(ZA)$. This is simply the standard result that there exists a quadratic Lyapunov function for A if and only if the system is exponentially stable. The above inequalities follow immediately from homogeneity and scaling. ■

The notion of *generalized gramians* was introduced in Beck et al [3]. The above operators X and Y are an infinite dimensional generalization of this. They are non-unique versions of the usual notion of the controllability and observability gramians for a linear system that result from solving Lyapunov inequalities rather than Lyapunov equations. The following gives a definition of a balanced realization. This is exactly the generalized definition used for uncertain systems in [3], applied to the time-varying system.

Definition 5. *The linear periodic system realization is described as balanced if there exist $X, Y \in \mathcal{X}$, satisfying inequalities (8) and (7), such that*

$$X = Y = \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where $\Sigma > 0$, is diagonal.

Clearly, if two realizations M and $\hat{M}$ for the same system are related by equation (6), then their generalized gramians transform as $\hat{Y} = TYT^*$ and $\hat{X} = T^{-1*}XT^{-1}$.

For LTV systems, the existence of balanced realizations is guaranteed by Lemma 4, and by equation (6). Neither the balanced realization nor the balanced Σ is unique.

In order to define the truncation of the system, partition A, B and C according to the partitioning of Σ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \qquad (9)$$

Note that the above partitioning is consistent with the block-diagonal structure of A; each block A_i is partitioned into submatrices, and all blocks are partitioned to have the same spatial dimensions. We can now define the truncated system G_r by

$$G_r := C_1(I - ZA_{11})^{-1}ZB_1 + D.$$

The following technical lemmas will be used in the derivation of the error bound.

Lemma 6. *Let*

$$U_{11} = Z^*\Sigma_1^{-\frac{1}{2}}ZA_{12} \qquad U_{12} = \begin{bmatrix} Z^*\Sigma_1^{-\frac{1}{2}}ZA_{11}\Sigma_1^{\frac{1}{2}} & Z^*\Sigma_1^{-\frac{1}{2}}ZB_1 \end{bmatrix}$$
$$U_{22} = \begin{bmatrix} A_{21}\Sigma_1^{\frac{1}{2}} & B_2 \end{bmatrix} \qquad W = A_{22}$$

and

$$Y = \left[\!\left[\begin{matrix} \Sigma_1 & 0 \\ 0 & I \end{matrix}\right]\!\right]$$

for $\Sigma_1 > 0$ block diagonal. Then Y satisfies equation (7) *if and only if*

$$U = \begin{bmatrix} U_{11} & U_{12} \\ W & U_{22} \end{bmatrix}$$

is contractive.

Proof. Equation (7) can be written as

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^* \\ B^* \end{bmatrix} - Z^*YZ < 0.$$

This is equivalent to

$$\left[\!\left[(Z^*YZ)^{-\frac{1}{2}}AY^{\frac{1}{2}} \quad (Z^*YZ)^{-\frac{1}{2}}B \right]\!\right]$$

being contractive, since Q is contractive if and only if $[\![Q]\!]$ is contractive. Since $Z^*Z = I$, this equals

$$\left[\!\left[\begin{matrix} Z^*\Sigma_1^{-\frac{1}{2}}ZA_{11}\Sigma_1^{\frac{1}{2}} & A_{21}\Sigma_1^{\frac{1}{2}} & Z^*\Sigma_1^{-\frac{1}{2}}ZB_1 \\ A_{21}\Sigma_1^{\frac{1}{2}} & A_{22} & B_2 \end{matrix}\right]\!\right].$$

which is a permutation of U. ■

The following result is similar.

Lemma 7. *Let*

$$V_{11} = A_{21}\Sigma_1^{-\frac{1}{2}} \qquad W = A_{22}$$
$$V_{21} = \begin{bmatrix} Z^*\Sigma_1^{\frac{1}{2}}ZA_{11}\Sigma_1^{-\frac{1}{2}} \\ C_1\Sigma_1^{-\frac{1}{2}} \end{bmatrix} \qquad V_{22} = \begin{bmatrix} Z^*\Sigma_1^{\frac{1}{2}}ZA_{12} \\ C_2 \end{bmatrix}$$

and

$$X = \left[\!\left[\begin{matrix} \Sigma_1 & 0 \\ 0 & I \end{matrix}\right]\!\right].$$

Then X satisfies equation (8) *if and only if*

$$V = \begin{bmatrix} V_{11} & W \\ V_{21} & V_{22} \end{bmatrix}$$

is contractive.

Finally, the following lemma is an operator generalization of Lemma 23 from [3]. The proof follows exactly the steps used in [3] and so is omitted.

Lemma 8. *Suppose U and V as defined above are contractive. Then*

$$L := \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}U_{11} & U_{12} \\ \frac{1}{\sqrt{2}}V_{11} & W & \frac{1}{\sqrt{2}}U_{22} \\ V_{21} & \frac{1}{\sqrt{2}}V_{22} & 0 \end{bmatrix}$$

is contractive.

The following is the main result of this section; it provides an explicit error bound for the induced 2-norm of the error between G and G_r.

Theorem 9. *Suppose X and Y satisfy equations* (7) *and* (8)*, with*

$$X = \left[\!\left[\begin{matrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{matrix}\right]\!\right] \qquad Y = \left[\!\left[\begin{matrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{matrix}\right]\!\right].$$

Suppose $\Sigma_2 = \operatorname{diag}(\Sigma_{2_0}, \Sigma_{2_1}, \dots)$, with

$$\Sigma_{2_k} = \operatorname{diag}(\sigma_{k1}I_{k_1}, \sigma_{k2}I_{k_2}, \dots, \sigma_{kp_k}I_{i_{p_k}})$$

Let

$$A = \left[\!\left[\begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix}\right]\!\right] \qquad B = \left[\!\left[\begin{matrix} B_1 \\ B_2 \end{matrix}\right]\!\right] \qquad C = \left[\!\left[C_1 \quad C_2 \right]\!\right]$$

partitioned in the same way as X, and let G_r be the truncated system with realization (A_{11}, B_1, C_1, D). Then

$$\|G - G_r\| < 2\sum_{i=1}^{p_i} \max_k \sigma_{ki}.$$

Proof. By scaling we can without loss of generality consider the case when $\Sigma_2 = I$, in which case realization for $\frac{1}{2}(G - G_r)$ is given by

$$M = \begin{bmatrix} \left[\!\left[\begin{matrix} A_{11} & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{matrix}\right]\!\right] & \left[\!\left[\begin{matrix} \frac{1}{\sqrt{2}}B_1 \\ \frac{1}{\sqrt{2}}B_1 \\ \frac{1}{\sqrt{2}}B_2 \end{matrix}\right]\!\right] \\ \left[\!\left[-\frac{1}{\sqrt{2}}C_1 \quad \frac{1}{\sqrt{2}}C_1 \quad \frac{1}{\sqrt{2}}C_2 \right]\!\right] & 0 \end{bmatrix}.$$

Define

$$T = \left[\!\left[\begin{matrix} -\frac{1}{\sqrt{2}}\Sigma_1^{\frac{1}{2}} & \frac{1}{\sqrt{2}}\Sigma_1^{\frac{1}{2}} & 0 \\ \frac{1}{\sqrt{2}}\Sigma_1^{-\frac{1}{2}} & \frac{1}{\sqrt{2}}\Sigma_1^{-\frac{1}{2}} & 0 \\ 0 & 0 & I \end{matrix}\right]\!\right]$$

Note T is block diagonal. We will show contractiveness of

$$\hat{M} := \begin{bmatrix} Z^*TZ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

and apply Theorem 2. Multiplying gives $[\![\hat{M}]\!] =$

$$\left[\begin{matrix} Z^*\Sigma_1^{\frac{1}{2}}ZA_{11}\Sigma_1^{-\frac{1}{2}} & 0 & \frac{1}{\sqrt{2}}Z^*\Sigma_1^{\frac{1}{2}}ZA_{12} & 0 \\ 0 & Z^*\Sigma_1^{-\frac{1}{2}}ZA_{11}\Sigma_1^{\frac{1}{2}} & \frac{1}{\sqrt{2}}Z^*\Sigma_1^{-\frac{1}{2}}ZA_{12} & Z^*\Sigma_1^{-\frac{1}{2}}ZB_1 \\ \frac{1}{\sqrt{2}}A_{21}\Sigma_1^{-\frac{1}{2}} & \frac{1}{\sqrt{2}}A_{21}\Sigma_1^{\frac{1}{2}} & A_{22} & \frac{1}{\sqrt{2}}B_2 \\ C_1\Sigma_1^{-\frac{1}{2}} & 0 & \frac{1}{\sqrt{2}}C_2 & 0 \end{matrix}\right]$$

which is a permutation of L. Now applying Lemmas 6, 7 and 8 the desired result follows. ■

This theorem considers only a special case of the general linear time-varying model reduction problem; that in which the singular values being truncated are constant over time.

It is possible to enforce this constraint when searching for balanced realizations, and in [11] we show how to exploit this in the general case. It will be shown in the next section that an additive error bound can be derived.

5. Periodic Systems

In this section we show that periodic systems have a special structure which allows the model reduction problem to be reduced to a finite dimensional problem. An operator G is said to be periodic with period q if $Z^qG = GZ^q$, that is it commutes with the q-shift for q some positive integer.

Such system descriptions arise naturally, in particular when analyzing the effects of perturbations on the behavior of a nonlinear system evolving on a closed periodic orbit.

For periodic systems, the result of Theorem 2 can be strengthened to the following.

Theorem 10. *Suppose A, B, C and D are q-periodic operators, and that $X \in \mathcal{X}$ and satisfies (2). Then there exists a q-periodic operator $X_{per} \in \mathcal{X}$ such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X_{per} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X_{per} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{10}$$

This theorem says that a solution exists to the performance inequality if and only if a periodic solution to the LMI exists. This result is proved in [4]; the proof amounts to taking an average of a sequence of solutions to (2) where each is constructed from X by q-shifting, so that

$$X_{per} = \sum_{i=0}^{\infty} Z^{qi*} X Z^{qi}.$$

Since the performance inequality is convex in X, this average solution also satisfies the inequality.

We will make use of the following additional notation. Let P be a q-periodic block-diagonal operator, and define $\tilde{P}$ to be the truncation of P, namely $\tilde{P} := \operatorname{diag}(P_0, \dots, P_{q-1})$. Also define the cyclic shift matrix $\tilde{Z}$, for $q \geq 2$, by

$$\tilde{Z} = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \quad \text{so that} \quad \tilde{Z}^* \tilde{P} \tilde{Z} = \begin{bmatrix} P_1 & & & 0 \\ & \ddots & & \\ & & P_{q-1} & \\ 0 & & & P_0 \end{bmatrix}.$$

For $q = 1$ set $\tilde{Z} = I$. Note that Z is not diagonal, and $\tilde{Z}$ is not the truncation of Z. Define the truncation of the set $\mathcal{X}$ by

$$\tilde{\mathcal{X}} := \{\tilde{X} : \ X \in \mathcal{X}\}.$$

Using these new definitions we have the following corollary of Theorem 10 and Theorem 2.

Corollary 11. *Suppose A, B, C and D are q-periodic operators. The following conditions are equivalent*

(i) $\|C(I - ZA)^{-1} ZB + D\| < 1$ *and* $1 \notin \operatorname{spec}(ZA)$*;*

(ii) There exists a matrix $\tilde{X} \in \tilde{\mathcal{X}}$ such that

$$\begin{bmatrix} \tilde{Z}\tilde{A} & \tilde{Z}\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^* \begin{bmatrix} \tilde{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{Z}\tilde{A} & \tilde{Z}\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} - \begin{bmatrix} \tilde{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{11}$$

This corollary gives a finite dimensional convex condition for determining the ℓ_2 induced norm of a periodic system of the form (1). This condition can be checked using various convex programming techniques.

Theorem 12. *Suppose that G is a periodic system. Then*

$$\|C(I - ZA)^{-1} ZB + D\| = \sup_{\lambda \in \bar{\mathbb{D}}} \|\tilde{C}(I - \lambda \tilde{Z}\tilde{A})^{-1} \lambda \tilde{Z}\tilde{B} + \tilde{D}\|$$

Further, $1 \notin \operatorname{spec}(ZA)$ if and only if $I - \lambda \tilde{Z}\tilde{A}$ is invertible for all $\lambda \in \bar{\mathbb{D}}$.

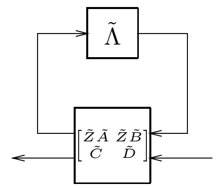

Figure 1: The system to be truncated

The next result is the key result of this section, and is the periodic analogue of Theorem 1. First, let $\{\lambda_0, \lambda_1, \dots, \lambda_{q-1}\}$ be a set of q complex scalars $\lambda_k \in \mathbb{C}$. Define $\tilde{\Lambda} := \operatorname{diag}(\lambda_0 I, \dots, \lambda_{q-1} I)$.

Theorem 13. *Suppose that G is a periodic system. Then*

$$\|C(I - ZA)^{-1} ZB + D\| = \sup_{\lambda_k \in \bar{\mathbb{D}}} \|\tilde{C}(I - \tilde{\Lambda}\tilde{Z}\tilde{A})^{-1} \tilde{\Lambda}\tilde{Z}\tilde{B} + \tilde{D}\|$$

where $\tilde{\Lambda}$ depends on λ_k, $k = 0, \dots, q-1$. Further, $1 \notin \operatorname{spec}(ZA)$ if and only if $I - \tilde{\Lambda}\tilde{Z}\tilde{A}$ is invertible for all $\lambda_k \in \bar{\mathbb{D}}$.

6. Model reduction of periodic systems

Based on the result of Theorem 13, we can view linear periodic systems in the form shown in the block diagram of Figure 1. This corresponds to the usual LFT problem formulation for uncertain systems; the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_0, \dots, \lambda_{q-1})$ can be viewed as a matrix of uncertain complex scalar parameters. For such uncertain systems, model reduction techniques have been developed in [3]. We can therefore directly apply those techniques of uncertain model reduction to the periodic linear system under consideration.

Lemma 14. *The following are equivalent*

(i) $(I - \tilde{\Lambda}\tilde{Z}\tilde{A})$ *is invertible for all* $\lambda_k \in \bar{\mathbb{D}}$

(ii) There exists $\tilde{X} \in \tilde{\mathcal{X}}$ satisfying

$$\tilde{A}^* \tilde{Z}^* \tilde{X} \tilde{Z} \tilde{A} - \tilde{X} + \tilde{C}^* \tilde{C} < 0 \tag{12}$$

(iii) There exists $\tilde{Y} \in \tilde{\mathcal{X}}$ satisfying

$$\tilde{Z} \tilde{A} \tilde{Y} \tilde{A}^* \tilde{Z}^* - \tilde{Y} + \tilde{B} \tilde{B}^* < 0 \tag{13}$$

This result follows immediately from Lemma 9 of Beck et al [3], in which a framework is developed for the model reduction of linear time-invariant systems with an associated uncertainty description. In this section, we show that for periodic time-varying systems, analysis of induced-norm properties has the same form as used for LTI systems with uncertainty. Hence we can apply the results of [3] to periodic LTV systems by simply identifying the appropriate uncertainty structure; in this case the q-periodic LTV system can be viewed as an LTI system with q scalar uncertainty blocks.

Note also that, with the inequalities replaced by equalities in Lemma 14, $\tilde{X}$ and $\tilde{Y}$ are exactly the usual observability and controllability gramians for time-varying systems [1].

The following gives an explicit definition of a balanced realization for periodic LTV systems. This is exactly the generalized definition used for uncertain systems in [3], applied to the periodic system. As shown in [3], such balanced realizations always exist, and they are non-unique.

Definition 15. *The linear periodic system realization is described as balanced if there exist $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$, satisfying inequalities* (12) *and* (13), *such that*

$$\tilde{X} = \tilde{Y} = \Sigma = \operatorname{diag}(\Sigma_0, \Sigma_1, \ldots, \Sigma_{q-1})$$

where $\Sigma_i > 0$, diagonal, with $\dim(\Sigma_i) = \dim(A_i)$.

In order to state the model reduction error bounds for balanced periodic time-varying systems, we partition the Σ_i so that $\Sigma_i = \operatorname{diag}(\Sigma_{1i}, \Sigma_{2i})$. Denote

$$\Sigma_{1i} = \operatorname{diag}(\sigma_{i1} I_{s_{i1}}, \ldots, \sigma_{ik_i}, I_{s_{ik_i}})$$

and

$$\Sigma_{2i} = \operatorname{diag}(\sigma_{i(k_i+1)} I_{s_{i(k_i+1)}}, \ldots, \sigma_{it_i} I_{s_{it_i}})$$

We then truncate both the Σ_{2i} and the corresponding parameter matrices. For example, we have

$$\tilde{A} = \begin{bmatrix} \hat{A}_0 & A_{0_{12}} & & & & & \\ A_{0_{21}} & A_{0_{22}} & & & & & \\ & & \hat{A}_1 & A_{1_{12}} & & & \\ & & A_{1_{21}} & A_{1_{22}} & & & \\ & & & & \ddots & & \\ & & & & & \hat{A}_{q-1} & A_{q-1_{12}} \\ & & & & & A_{q-1_{21}} & A_{q-1_{22}} \end{bmatrix}$$

which we truncate to

$$\tilde{\hat{A}} = \begin{bmatrix} \hat{A}_0 & & & 0 \\ & \hat{A}_1 & & \\ & & \ddots & \\ 0 & & & \hat{A}_{q-1} \end{bmatrix}$$

Similarly $B_i = \begin{bmatrix} \hat{B}_i' & B_{i2}' \end{bmatrix}'$ and $C_i = \begin{bmatrix} \hat{C}_i & C_{i2} \end{bmatrix}$ are truncated to $\hat{B}_i$ and $\hat{C}_i$ respectively, to construct $\tilde{\hat{B}}$ and $\tilde{\hat{C}}$. Define

$$G_r := \hat{C}(I - Z\hat{A})^{-1} Z\hat{B} + \hat{D}$$

where $\hat{A}$ is the periodic block-diagonal operator formed by repeating the blocks of $\tilde{\hat{A}}$. The following is the main result of this section.

Theorem 16. *Suppose G is stable and balanced, and let G_r be the reduced order model formed by truncating G. Then*

$$\|G - G_r\| \le 2 \sum_{i=0}^{q-1} \sum_{j=k_i+1}^{t_i} \sigma_{ij}$$

Outline of Proof. The result follows directly from application of Theorem 15 in [3] to the periodically time-varying system, since truncation of $\tilde{Z}\tilde{A}$ is equivalent to truncating $\tilde{A}$ and multiplying by a shift operator $\tilde{Z}$ of appropriate spatial dimension. See [3] for details. ■

This result gives an explicit error bound for balanced truncation of LTV systems, analagous to the 'twice-the-sum-of-the-tail' formula for LTI systems. It is applicable to any periodic LTV system.

Note that if the system is in fact linear time invariant, so that the q-periodic system simply consists of q identical copies of A, B, C and D, then this result reduces to the standard linear time-invariant result; this is because truncation of multiple identical singular values causes no greater error bound than truncation of one singular value. Note that the above theorem statement does not explicitly include this case, although it follows immediately from [3].

7. Conclusions

In this paper we have derived error bounds for balanced truncation of linear time-varying and linear periodically-varying systems. To our knowledge these are the first error bounds proposed for truncation of time-varying systems. We have developed a generalized notion of balanced realization for time-varying systems, expressed in terms of linear matrix inequalities, allowing standard convex programming techniques to be used in their construction. The bounds developed reduce to those known for the time-invariant case when applied to time-invariant systems.

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