PERGAMON

Automatica 37 (2001) 1909–1922

automatica

www.elsevier.com/locate/automatica

An LMI solution to the robust synthesis problem for multi-rate sampled-data systems☆

Sanjay Lall[a,*,1], Geir Dullerud[b,2]

[a]Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305-4035, USA
[b]Department of Mechanical and Industrial Engineering, University of Illinois, Urbana, IL 61801, USA

Received 15 September 1997; revised form 15 April 2000; received in final form 30 May 2001

Abstract

In this paper we address the asynchronous multi-rate sampled-data H_∞ synthesis problem. Necessary and sufficient conditions are given for the existence of a controller achieving the desired performance, and the problem is shown to be equivalent to a convex optimization problem expressed in the form of linear operator inequalities. In the case where the sample and hold rates are synchronous, these operator inequalities reduce to linear matrix inequalities, for which standard numerical software is available. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Multi-rate control; Sampled-data systems; H_∞ control; Linear matrix inequalities

1. Introduction

In this paper we construct a solution for the multi-rate sampled-data H_∞ synthesis problem, illustrated in Fig. 1. Given a continuous-time system with multiple input and output channels, we would like to control it using a digital controller via multiple sample and hold devices, each of which may be running at a different rate.

The control objective is to minimize the induced norm from the input w to the output z. In this paper, we give necessary and sufficient conditions for the existence of a controller achieving a given performance bound. We show that the question of whether or not there exists such a controller is equivalent to the question of whether or not there exists a feasible solution to a convex optimization problem, which can be expressed as a linear matrix inequality (LMI). If there does exist such a controller, then we give an explicit construction for it, again in terms of LMIs.

The main purpose of this paper is to present a computable solution to this multirate problem, with no unnecessary constraints on the original system data. The two primary methods by which this is achieved are the use of time-varying techniques, and the use of LMIs. Thus, there are no restrictions on the zeros of the original system, nor on its direct feedthrough terms, apart from the fundamental constraint that $D_{21} = 0$, necessary to ensure that the sampling operation is well defined. In order to achieve these results we combine the jump systems framework of Dullerud and Lall (1999a), and the discrete-time time-varying synthesis of Dullerud and Lall (1999b), to provide a computationally explicit solution for multi-rate sampled-data controller synthesis in terms of LMIs. For work closely related to Dullerud and Lall (1999b), see the earlier LTV work by Ball, Gohberg, and Kaashoek (1992), Halanay and Ionescu (1994), and Iglesias (1996). This paper is a longer version of the conference paper (Lall & Dullerud, 1997).

The work that is most closely related to the one presented here is that reported by Sagfors, Toivonen, and Lennartson (1997c, 1998), where a Riccati equation approach is used to solve a similar multi-rate problem. The Riccati method requires the nominal plant to satisfy the rank constraints typically associated with Riccati-based approaches; namely, the requirements that there be a sufficient number of measurement noise sources, that the control weighting be nonsingular, and that the plant

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Per-Olof Gutman under the direction of Editor Tamer Basar.

* Corresponding author.

E-mail address: lall@stanford.edu (S. Lall).

1 Partially supported by AFOSR MURI grant F49620-96-1-0471 and by EPRI-DOD project DAAG55-98-1-3-001.

2 Partially supported by NSF under award ECS-9875244 CAREER.

PII: S0005-1098(01)00167-4

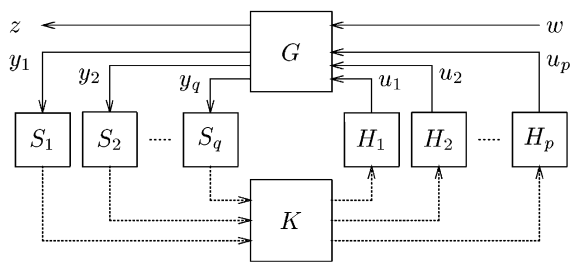

Fig. 1. The multi-rate sampled-data problem.

have no imaginary axis zeros (suitably defined). The controller obtained using the Riccati equation approach satisfies a separation principle.

The development of a theory for control of multi-rate systems originates in Kalman and Bertram (1959), and Kranc (1957). More recent research has considered the sampled-data LQG problem, as in Al-Rahmani and Franklin (1990), Amit (1980), Amit and Powell (1981), Berg, Amit, and Powell (1988), Chen and Francis (1991), Colaneri and De Nicolao (1995), Glasson (1982), and Lennartson (1988) and the parametrization of all stabilizing controllers as in Meyer (1990), and Ravi, Khargonekar, Minto, and Nett (1990).

For the sampled-data H_∞ synthesis problem in both multi-rate and single-rate forms, various approaches have been used. Solutions have been provided for the single rate case in Bamieh and Pearson (1992), Başar (1991), Başar and Bernhard (1995), Bernhard (1991), Chen and Francis (1995, 1996), Hara and Kabamba (1990), Kabamba and Hara (1993), Sun, Nagpal, and Khargonekar (1993), Tadmor (1992), Toivonen (1992, 1995), Toivonen and Sagfors (1997), and Yamamoto (1990). A complete solution to the asynchronous *dual-rate* case is given in Sagfors and Toivonen (1997), and Toivonen (1995). For general periodic multi-rate systems, solutions are given in Chen and Qiu (1994), Lall and Dullerud (1997), Sagfors, Toivonen, and Lennartson (1997b,c, 1998) and Voulgaris and Bamieh (1993); furthermore, the conditions and constructions in Lall and Dullerud (1997) hold in the nonperiodic case but are infinite dimensional.

The sampled-data approaches used in Araki and Ito (1993), Bamieh, Pearson, Francis, and Tannenbaum (1991), Bamieh and Pearson (1992), Chen and Francis (1995, 1996), Chen and Qiu (1994), Dullerud (1996), Hara and Kabamba (1990), Kabamba and Hara (1993), Sagfors, Toivonen, and Lennartson (1997c), Toivonen (1992), and Yamamoto (1990) make use of *lifting*, to map the continuous signal spaces to discrete signal spaces, and construct an equivalent discrete-time system for which standard synthesis techniques can be used. This lifting is time-invariant, in the sense that the continuous signal is broken into segments of fixed duration.

In particular, the multirate solutions of Chen and Qiu (1994), and Voulgaris and Bamieh (1993) are based on time-invariant lifting, and develop a sequence of transformations to construct a controller satisfying the causality constraint, with the final solution reduced to a structured H_∞ model matching problem. This added structure leads to significant technical challenge in these approaches (Meyer, 1990). Time-invariant lifting is also used in Sagfors et al. (1997c).

The approaches of Başar and Bernhard (1995), Bernhard (1991) are based on game-theory, and focus on the finite-horizon case. The solutions of Lall (1995), Sagfors and Toivonen (1997), Sagfors et al. (1997b), Sun et al. (1993) and Toivonen and Sagfors (1997) make use of a direct approach to H_∞ problems in the time-domain, similar to that used for time-varying H_∞ problems presented in Ravi, Nagpal, and Khargonekar (1991). A comparison of these approaches with the lifting technique is given in Sagfors et al. (1998), Toivonen (1995), and Toivonen and Sagfors (1997). The approach of Sagfors et al. (1997c) is via the solution of full-information and estimation problems. All of these approaches give results in terms of Riccati equations; the solutions of Sagfors and Toivonen (1997) and Sun et al. (1993), Toivonen (1995) are in terms of Riccati differential equations with jumps, and in Sagfors, Toivonen, and Lennartson (1997b,c, 1998), solutions are given in terms of algebraic Riccati equations. In all of these approaches a solution is given for systems where the rank of original system matrices satisfies certain technical constraints.

In the current paper, we assume all disturbances and noise enter as continuous time signals; however, there are alternative possibilities in the formulation of this H_∞ sampled-data problem. In particular, it is possible to formulate the problem with discrete noise entering directly into the measurement equation, as in Sagfors et al. (1997b, 1998), Sagfors and Toivonen (1997), and Lennartson, Lindgarde, Toivonen, and Sagfors (1997). Together with the above rank constraints on direct feedthrough terms, this discrete noise makes certain aspects of the problem non-singular.

When formulating the sampled-data nature of the system, there are various methods possible also. The approach adopted in this paper has been to express the multi-rate sampled-data problem as a synthesis problem for jump systems. A formulation involving linear systems with jumps is also used in Lall (1995), Lennartson et al. (1997), Sagfors et al. (1997b, 1998), Sagfors and Toivonen (1997), Sun et al. (1993), and Toivonen and Sagfors (1997). Similar approaches to those used in this paper for analysis of the sample and hold operators are used in Astrom and Wittenmark (1989), Colaneri and De Nicolao (1995), Sagfors and Toivonen (1997), Sagfors, Toivonen, and Lennartson (1997a,b).

A modification of time-invariant lifting is time-varying lifting, where the segments have a duration which varies with time. Time-varying lifting is used in this paper, and has also been used by Lennartson et al. (1997) to solve the state-feedback multi-rate H_∞ control problem.

In Dullerud and Lall (1999b) mathematical tools were developed which allow discrete-time, *time-varying* systems to be treated in a way very similar to the manipulation of transfer functions for discrete-time time-invariant systems. In this paper, this formulation allows application of the H_∞ equivalence and reduction process of Bamieh and Pearson (1992) to time-varying systems.

Using this method, the multi-rate sampled-data problem is converted to an equivalent discrete-time, time-varying synthesis. Solutions to the linear time-varying H_∞ synthesis problem in terms of Riccati recursions exist (Ravi et al., 1991; Xie, de Souza, & Wang, 1993), and Dullerud and Lall (1999b) provides a convex solution to this discrete problem under minimal assumptions. In combination with the time-varying equivalencing, this leads to the solution of the multi-rate sampled-data problem presented in this paper. One of the main advantages of this approach is that the solution is developed via convex optimization. For synthesis problems with rational sampling rates, the solutions are expressed in terms of finite dimensional linear matrix inequalities, for which standard numerical techniques can be used.

The derivation in this paper does not require time-invariance of the original system, nor periodicity of the sample and hold devices; it is valid for irrational and asynchronous sampling and hold rates as well as rational, synchronous systems. In the case when the resulting discrete time-varying system is periodic, then this results in a computable solution to the multi-rate sample-data synthesis problem. In the asynchronous case, we show the equivalence of this problem to an infinite-dimensional convex optimization problem. More details on the numerical issues are in Section 5.1. Other less common cases such as sampling with delays between sampling different channels, or sampling one channel at odd times and another at even times, are easily handled both computationally and theoretically within this framework. A numerical example of this is given in the paper.

1.1. Overview of the procedure

1.1.1. Conversion to a jump system

The first step is to express the system G combined with the multi-rate sampler and hold, S and H, as a linear time-varying (LTV) system with jumps. In general, a jump system is a system which, for times $t \neq t_k$, satisfies the usual continuous-time state-space equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1}$$

$$y(t) = C(t)x(t), \tag{2}$$

with the proviso that at particular times t_k, the state of the system undergoes a 'jump' according to the rule

$$x(t_k^+) = A_d(t_k)x(t_k) + B_d(t_k)u_d(t_k). \tag{3}$$

Since the hold operators are not stateless, the jump system will have $n + n_u$ states. The interval between successive jump times, $h_k := t_{k+1} - t_k$, need not be constant, although it will be periodic provided that the original sample and hold rates are rationally related. The jump system itself will then also be periodic if the original continuous-time system was time-invariant. Note that the periodicity of the jump system may be longer than the period of the sequence h_k, for example in the case when the sampling devices become activated alternately.

1.1.2. Lifting

The time-varying system with jumps is then *lifted*. The lifting procedure is a map between continuous signal spaces and infinite-dimensional discrete signal spaces. The lifted system is a time-varying discrete-time (DT) system, but it has infinite-dimensional input and output spaces at each time t_k. That is, at each time t_k, w_k is an element of the space $L_2(t_k, t_{k+1}]$. For any given controller K, the lifted system has the same induced norm as the original sampled-data system.

1.1.3. Conversion to the equivalent discrete time system

Given the original system G, and fixed periodic sample and hold devices, a standard time-varying DT system is constructed. For the case when the original multi-rate problem is periodic, this DT system will be periodic also. Any controller which achieves internal stability and a contractive closed loop for this DT system is also feasible for the original sampled-data problem.

1.1.4. Periodic and time-varying H_∞ synthesis

The problem is now the standard one of finding a controller which achieves a given induced-norm performance for the time-varying DT system given above. The solution to this problem is expressed as an LMI. If the system is periodic, then this LMI is finite dimensional, and can be solved using standard techniques. The LMI generated requires finding j positive definite $n \times n$ matrices, where j is the periodicity of the jump system.

2. Preliminaries

The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Given a Hilbert space E we denote its norm by $\|\cdot\|_E$; for convenience we frequently omit the subscript. If E and F are Hilbert spaces, then we denote the space of bounded linear operators mapping E to F by $\mathscr{L}(E,F)$, and shorten $\mathscr{L}(E,E)$ to $\mathscr{L}(E)$. If X is in $\mathscr{L}(E)$ then we denote the induced norm of X by $\|X\|$. The main Hilbert spaces of interest in this paper are $L_2^m[0,T)$ for $T > 0$, allowing $T = \infty$, defined to be the set of square integrable functions mapping $[0,T)$ to the Euclidean space $\mathbb{R}^m$ with the inner product

$$\langle x, y\rangle_{L_2[0,T)} = \int_0^T \langle x(t), y(t)\rangle_{\mathbb{R}^m}\, dt. \tag{4}$$

The second Hilbert space of interest is formed given an infinite sequence $\{V_k\}$ of Hilbert spaces, and is denoted $\ell_2(\{V_k\})$. It is defined as the subspace of the Hilbert space direct sum $\oplus_{k=0}^{\infty} V_k$ consisting of elements $(x_0, x_1, x_2, \ldots)$, where $x_k \in V_k$, which satisfy

$$\sum_{k=0}^{\infty} \|x_k\|_{V_k}^2 < \infty. \tag{5}$$

The inner product of x, y in $\ell_2(\{V_k\})$ is therefore defined by the infinite sum $\langle x, y\rangle_{\ell_2} := \sum_{k=0}^{\infty} \langle x_k, y_k\rangle_{V_k}$. When all the spaces V_k are equal we use the shorter notation $\ell_2(V)$; thus, for example, $\ell_2(\mathbb{R})$ is the usual space of real valued, square summable sequences. If the sequence $\{V_k\}$ is clear from the context we abbreviate further to ℓ_2.

Finally, we define the linear fractional transformation describing the closed-loop interconnection of systems. Suppose G is a system such that

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \tag{6}$$

and K is a controller connected in feedback such that $u = \ Ky$. Then the closed-loop map is denoted by $z = F_\ell(G, K)w$, where $F_\ell(G, K) := G_{11} + G_{22}(I - KG_{22})^{-1} KG_{21}$. Throughout the paper we make use of the standard notion of positive definiteness, and make use of the following proposition, which can be found in any standard text on matrix analysis, such as Horn and Johnson (1990).

Proposition 1. *Suppose X and Y are self-adjoint operators on two Hilbert spaces, W is an operator between these spaces, and $\beta \geqslant 0$. Then*

$$\begin{pmatrix} X & W \\ W^* & Y \end{pmatrix} < -\beta I, \tag{7}$$

if and only if $Y < -\beta I$ and $X - WY^{-1}W^* < -\beta P$, where P is the positive operator $(I + WY^{-1}(I + \beta Y^{-1})^{-1} Y^{-1} W^*)$.

Since the systems we will be considering are in general time-varying, we make use of some additional formalism.

Definition 2. A bounded operator Q mapping $\ell_2(\{V_k\})$ to $\ell_2(\{U_k\})$ is block diagonal if there exists a sequence of operators Q_k in $\mathscr{L}(V_k, U_k)$ such that, for all w, z, if $z = Qw$ then $z_k = Q_k w_k$. Then Q has the representation $Q = \mathrm{diag}(Q_0, Q_1, Q_2, \ldots)$.

Further, if $P_k \in \mathscr{L}(V_k, U_k)$ is a uniformly bounded sequence of operators we say $P = \mathrm{diag}(P_0, P_1, \ldots)$ is the block-diagonal operator for $\{P_k\}$. Given a linear time-varying DT system of the form

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k, \\ z_k &= C_k x_k + D_k w_k, \end{aligned} \tag{8}$$

the matrix sequences A_k, B_k, C_k and D_k define block diagonal operators, which we here denote by A, B, C, and D.

Let Z be the unilateral shift defined on $\ell_2(X)$, for X some Hilbert space. Then, for $a = (a_0, a_1, a_2, \ldots)$ in $\ell_2(X)$ we have $(Za) = (0, a_0, a_1, a_2, \ldots)$. Then we can rewrite (8) as

$$\begin{aligned} x &= ZAx + ZBw, \\ z &= Cx + Dw. \end{aligned} \tag{9}$$

This system of equations is well defined if the system is exponentially stable, and in that case

$$z = (C(I - ZA)^{-1} ZB + D)w. \tag{10}$$

Many properties of time-varying systems thus become formally identical to the corresponding property of time-invariant systems; for example, exponential stability of this system is equivalent to the condition that $1 \notin \mathrm{spec}(ZA)$.

We can now move on to the main body of the paper.

3. Problem formulation

The synthesis problem we are considering is illustrated in Fig. 1. Here, G is a continuous-time linear time-invariant system defined in the usual state space notation by the differential equations

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \tag{11}$$

with initial condition $x(0) = 0$. In order for the sampling operation to be well defined, we assume $D_{21} = 0$. Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$, $y(t) \in \mathbb{R}^{n_y}$, and $z(t) \in \mathbb{R}^{n_z}$.

3.1. Definition of the multi-rate sample and hold

The sample and hold operators are constructed in a state-space formulation. A similar formulation was used in Astrom and Wittenmark (1989), Colaneri and De Nicolao (1995), Sagfors and Toivonen (1997), Sagfors et al. (1997a, b). At each time t, the input $u(t)$ and output $y(t)$ signals are partitioned into

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_q(t) \end{pmatrix}, \tag{12}$$

where $u_i(t) \in \mathbb{R}^{q_i}$, $y_i(t) \in \mathbb{R}^{p_i}$, and $\sum_{i=1}^{p} p_i = n_y$, $\sum_{i=1}^{q} q_i = n_u$.

Associated with each sampler S_i let there be a sequence of sampling times $\sigma_1^i, \sigma_2^i, \ldots$, such that $0 < \sigma_j^i < \sigma_{j+1}^i$. Then, we define the ideal sampler S_i as the mapping satisfying

$$(S_i y_i)(\sigma_k^i) := y(\sigma_k^i) \tag{13}$$

for each function $y_i: [0, \infty) \to \mathbb{R}^{p_i}$. If the signal y_i is the output of the above system G with $D_{21} = 0$ and $u, w \in L_2$, then $S_i y_i \in \ell_2$.

Similarly associated with each hold H_i let there be a sequence of hold times $\tau_1^i, \tau_2^i, \ldots$, such that $0 < \tau_j^i < \tau_{j+1}^u$. Then, we define the zero order hold H_i as the mapping satisfying

$$(H_i \tilde{u}_i)(t) = \tilde{u}_i(\tau_k^i) \quad \text{for } t \in (\tau_k^i, \tau_{k+1}^i] \tag{14}$$

for every sequence $\tilde{u}_i \in \ell_2$. Clearly $H_i \in \mathscr{L}(\ell_2, L_2)$.

We make the standing assumptions that the differences $\sigma_{k+1}^i - \sigma_k^i$ and $\tau_{k+1}^i - \tau_k^i$ are bounded sequences for each fixed value of i. Define the sequence t_k as the union of the σ and τ sequences, ordered such that $0 < t_k < t_{k+1}$; so that for any i, j, there exists k, l such that $\sigma_j^i = t_k$ and $\tau_j^i = t_l$. Given a discrete signal $\tilde{u} \in \ell_2$, we denote $\tilde{u}(t_k)$ by $\tilde{u}_k$.

At each time t_k define the diagonal matrix $\Gamma_k \in \mathbb{R}^{n_y \times n_y}$ whose ith entry is equal to 1 if the corresponding channel is sampled at time t_k, with all other entries zero. That is, $\Gamma_k := \operatorname{diag}(g_1(k) I_{p_1}, \ldots, g_p(k) I_{p_p})$, where $g_i(k) = 1$ if $\sigma_j^i = t_k$ for some j, and $g_i(k) = 0$ otherwise. If channel i is not sampled at time t_k, then define the output of the sampler $S_i(t_k)$ to be zero. Then we can define the multi-rate sampler $S: y \mapsto \tilde{y}$ by $\tilde{y}_k = \Gamma_k y(t_k)$. In this way, the multi-rate information is specified by the matrix sequence Γ_k, and the multi-rate sampling can simply be viewed as the irregular sampling of one channel, provided that the controller design process takes into account the matrix Γ_k.

Similarly, to characterize the multi-rate hold, define $\Omega_k := \operatorname{diag}(r_1(k) I_{q_1}, \ldots, r_q(k) I_{q_q})$ where $r_i(k) = 1$ if $\tau_j^i = t_k$ for some j, and $r_i(k) = 0$ otherwise. We can define the multi-rate hold operator $H: \tilde{u} \mapsto u$ by the dynamical system

$$\begin{aligned} \xi(t_k^+) &= (I - \Omega_k)\xi(t_k) + \Omega_k \tilde{u}_k, \quad \xi(t_0) = 0, \\ u(t) &= \xi(t). \end{aligned} \tag{15}$$

where $\xi(t_k^+) := \lim_{t \searrow t_k} \xi(t)$ is the limit of $\xi(t)$ as t tends to t_k from above. Clearly ξ is left-continuous. In this system, only those components of $\tilde{u}_k$ which correspond to active hold operators are used at time t_k; the other components are ignored.

Note from the definitions that there is no direct feedthrough from $\tilde{u}_k$ to $\tilde{y}_k$, even if $D_{22} \neq 0$, since the sampling is defined to take place the instant *before* the input signal undergoes a discontinuity.

The multi-rate sampled-data problem can now be stated as follows; we would like to find a discrete controller K mapping $\tilde{y}$ causally to $\tilde{u}$, such that the L_2 induced norm of the closed-loop system $F_l(G, HKS)$ is less than some prespecified level $\gamma > 0$. Without loss of generality we choose $\gamma = 1$.

4. Synthesis procedure

4.1. Construction of the jump system

Consider the continuous system G defined by Eq. (11). Combined with the multi-rate sampler and hold, it can be rewritten as

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}_1 \tilde{x}(t) + \tilde{B}_1 w(t), \\ \tilde{x}(t_k^+) &= \tilde{A}_2(t_k)\tilde{x}(t_k) + \tilde{B}_2(t_k)\tilde{u}(t_k), \\ z(t) &= \tilde{C}_1 \tilde{x}(t) + \tilde{D}_{11} w(t), \\ \tilde{y}(t_k) &= \tilde{C}_2(t_k)\tilde{x}(t_k), \end{aligned} \tag{16}$$

where $\tilde{x}(t_0) = 0$, and

$$\begin{aligned} \tilde{A}_1 &= \begin{bmatrix} A & B_2 \\ 0_{n \times n_u} & 0_{n_u} \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_1 \\ 0_{n_w} \end{bmatrix}, \\ \tilde{C}_1 &= [C_1 \ \ D_{12}], \quad \tilde{D}_{11} = D_{11}, \\ \tilde{A}_2(t_k) &= \begin{bmatrix} I_n & 0 \\ 0 & I - \Omega(t_k) \end{bmatrix}, \quad \tilde{B}_2(t_k) = \begin{bmatrix} 0_{n \times n_u} \\ \Omega(t_k) \end{bmatrix}, \\ \tilde{C}_2(t_k) &= [\Gamma(t_k) C_2 \ \ \Gamma(t_k) D_{22}]. \end{aligned} \tag{17}$$

The state of the continuous system G is related to the state of the jump system by $x(t) = [I_n \ 0]\tilde{x}(t)$. In fact, for the particular case of the multi-rate sampled-data system, $x(t)$ is continuous, although $\tilde{x}(t)$ is only left-continuous. We will write this system as

$$\begin{pmatrix} z \\ \tilde{y} \end{pmatrix} = \tilde{G} \begin{pmatrix} w \\ \tilde{u} \end{pmatrix} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} \begin{pmatrix} w \\ \tilde{u} \end{pmatrix}. \tag{18}$$

It is straightforward to verify that

$$\begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & H \end{bmatrix} \tag{19}$$

and hence that the jump system $\tilde{G}$ is equivalent to the original multi-rate sample-data system mapping $(\tilde{u}, w)$ to $(\tilde{y}, z)$.

The system is represented as the jump system above, with the n_u extra states the states of the hold operators, so that the input signals which are not modified at any particular time are 'stored' by these states in this realization. The methodology presented here assumes that the sampling does not occur at exactly the same time as the hold operation, so that there is an effective delay between measurement and control. This is different from the approach presented in, for example Bamieh and Pearson (1992), where there is no delay.

The purpose of the jump system is to allow the controller synthesis to explicitly take into account the multi-rate nature of the problem. At times t_k at which there is no sampled information available, $\tilde{y}_k = 0$, since the output matrix $\tilde{C}_2(t_k)$ contains $\Gamma(t_k)$. The controller synthesis procedure explicitly takes account of $\Gamma(t_k)$ through $\tilde{C}_2(t_k)$ and hence does not mistake this zero output for an actual measured zero. Similarly, the effect of $\Omega(t_k)$ in this realization is to prevent the controller violating the restrictions imposed by the multi-rate hold; any controller activity at times when the hold operators are not active is not seen by the continuous system. Hence, the closed-loop map $F_l(\tilde{G}, K)$ is exactly the closed-loop map when the controller K is connected via the multi-rate sample and hold to the continuous system G.

If the sampling and hold operators are *periodic*, and their rates are related by integers, then the jump system so defined will also be periodic. In the sequel we shall see that this leads to a solution computable in terms of finite matrices.

Since $\tilde{A}_2$, $\tilde{B}_2$ and $\tilde{C}_2$ are time-varying, we now have the system written as a time-varying system with jumps. The solution to the synthesis problem for such systems is now presented.

4.2. LTV lifting

Define the lifting operator $W\colon L_2[0,\infty) \to \ell_2(\{L_2(0,h_k]\})$ by $\check{x} = (Wx) = (\check{x}_0, \check{x}_1, \check{x}_2, \ldots)$, where $h_k := t_{k+1} - t_k$, and

$$(\check{x}_k)(s) := x(s + h_k) \quad \text{for } s \in (0, h_k]. \tag{20}$$

Clearly from the definition W^{-1} exists and W is an isometric isomorphism. This lifting operation is a slight generalization of that found in Bamieh and Pearson (1992).

We can now lift the system. Let $\hat{z} = Wz$ and $\hat{w} = Ww$. Then

$$\begin{pmatrix} \hat{z} \\ \tilde{y} \end{pmatrix} = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} \begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \hat{w} \\ \tilde{u} \end{pmatrix}. \tag{21}$$

This lifted system has a realization

$$\hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{B}_{1k} \hat{w}_k + \hat{B}_{2k} \tilde{u}_k,$$

$$\hat{z}_k = \hat{C}_{1k} \hat{x}_k + \hat{D}_{11k} \hat{w}_k + \hat{D}_{12k} \tilde{u}_k,$$

$$\tilde{y}_k = \hat{C}_{2k} \hat{x}_k, \tag{22}$$

where $\hat{x}_k = \tilde{x}(t_k)$, and

$$\hat{A}_k\colon \mathbb{R}^{n+n_u} \to \mathbb{R}^{n+n_u},$$

$$\hat{B}_{1k}\colon L_2[0,h_k) \to \mathbb{R}^{n+n_u}, \quad \hat{B}_{2k}\colon \mathbb{R}^{n_u} \to \mathbb{R}^{n+n_u},$$

$$\hat{C}_{1k}\colon \mathbb{R}^{n+n_u} \to L_2[0,h_k), \quad \hat{C}_{2k}\colon \mathbb{R}^{n+n_u} \to \mathbb{R}^{n_y},$$

$$\hat{D}_{11k}\colon L_2[0,h_k) \to L_2[0,h_k), \quad \hat{D}_{12k}\colon \mathbb{R}^{n_u} \to L_2[0,h_k).$$

Specific expressions for these operators can easily be derived from the state space matrices for the jump system; we include them here for completeness.

$$\hat{A}_k = e^{h_k \tilde{A}_1} \tilde{A}_2(t_k),$$

$$\hat{B}_{1k} \hat{w}_k = \int_{t_k}^{t_{k+1}} e^{\tilde{A}_1(\tau - t_k)} \tilde{B}_1 \hat{w}_k(\tau) \, d\tau,$$

$$\hat{B}_{2k} = e^{h_k \tilde{A}_1} \tilde{B}_2(t_k),$$

$$\hat{C}_{2k} = \tilde{C}_2(t_k),$$

$$(\hat{C}_{1k} \hat{x}_k)(t) = \tilde{C}_1 e^{\tilde{A}_1(t - t_k)} \tilde{A}_2(t_k) \hat{x}_k,$$

$$(\hat{D}_{11k} \hat{w}_k)(t) = \tilde{D}_{11} \hat{w}_k(t) + \tilde{C}_1 \int_{t_k}^{t} e^{\tilde{A}_1(\tau - t_k)} \tilde{B}_1 \hat{w}_k(\tau) \, d\tau,$$

$$(\hat{D}_{12k} \tilde{u}_k)(t) = \tilde{C}_1 e^{\tilde{A}_1(t - t_k)} \tilde{B}_2(t_k) \tilde{u}_k. \tag{23}$$

Here $t \in (t_k, t_{k+1}]$. Since W is an isometric isomorphism, with any controller in place the induced norm of the lifted system (22) is the same as that of the original system.

4.3. The equivalent discrete system

Having the system in lifted form does not enable us immediately to perform the synthesis, however. The next step is to construct a standard DT system, with finite dimensional input and output spaces for z and w, for which we can apply standard synthesis techniques.

In order to do this, note that the lifted system has the same state dimension $n + n_u$ as the jump system. As is well known in the time-invariant case, this implies that the operators $\hat{B}_1$ and $\hat{C}_1$ have finite rank. If $\hat{D}_{11}$ were zero, then we could simply replace these operators with finite matrices, and the synthesis problem would be unchanged. We now construct the equivalent DT system for the time-varying case.

Suppose that the system is being controlled by a DT controller K characterized by

$$\xi_{k+1} = A_k^K \xi_k + B_k^K y_k,$$

$$u_k = C_k^K \xi_k + D_k^K y_k. \tag{24}$$

Connecting this controller with the realization of the lifted system (22), the closed-loop system $M := WF_l(G, HKS)W^{-1}$ is exactly described by

$$\zeta_{k+1} = \begin{bmatrix} \hat{A}_k + \hat{B}_{2k}D_k^K\hat{C}_k & \hat{B}_{2k}C_k^K \\ B_k^K\hat{C}_{2k} & A_k^K \end{bmatrix}\zeta_k + \begin{bmatrix} \hat{B}_{1k} \\ 0 \end{bmatrix}\hat{w}_k, \qquad (25)$$

$$\hat{z}_k = [\hat{C}_{1k}\ \hat{D}_{12k}]\begin{bmatrix} I & 0 \\ D_k^K\hat{C}_{2k} & C_k^K \end{bmatrix}\zeta_k + \hat{D}_{11k}\hat{w}_k.$$

We can now define new operators and rewrite (25) as

$$\zeta_{k+1} = \hat{A}_k^0\zeta_k + \hat{B}_k^0\hat{w}_k,$$
$$\hat{z}_k = \hat{C}_k^0\zeta_k + \hat{D}_k^0\hat{w}_k. \qquad (26)$$

Making use of the block-diagonal description of this system, we have

$$\zeta = Z\hat{A}^0\zeta + Z\hat{B}^0\hat{w},$$
$$\hat{z} = \hat{C}^0\zeta + \hat{D}^0\hat{w}. \qquad (27)$$

The following is an operator version of a well-known result.

Theorem 3. *The system M defined by Eq.* (26) *is exponentially stable, and satisfies* $\|M\| < 1$, *if and only if there exists a block diagonal operator* $X > 0$ *such that*

$$\begin{bmatrix} -Z^*X^{-1}Z & \hat{A}^0 & \hat{B}^0 & 0 \\ \hat{A}^{0*} & -X & 0 & \hat{C}^{0*} \\ \hat{B}^{0*} & 0 & -I & \hat{D}^{0*} \\ 0 & \hat{C}^0 & \hat{D}^0 & -I \end{bmatrix} < 0, \qquad (28)$$

where the block structure of X is the same as that of $\hat{A}^0$.

A proof of this result for the case when the input and output spaces are finite-dimensional can be found in Dullerud and Lall (1999b). The case in which the system is a lifted continuous-time system, and hence the input and output spaces are infinite-dimensional, can be proved in a similar way. Using this result, we can now prove the following theorem.

Theorem 4. *Suppose that* $\|\hat{D}^0\| < 1$, *and that* $\bar{C}$ *and* $\bar{B}$ *are block-diagonal operators satisfying*

$$(\bar{C}^*\bar{C})_k = (\hat{C}^{0*}(I - \hat{D}^0\hat{D}^{0*})^{-1}\hat{C}^0)_k,$$
$$(\bar{B}\bar{B}^*)_k = (\hat{B}^0(I - \hat{D}^{0*}\hat{D}^0)^{-1}\hat{B}^{0*})_k. \qquad (29)$$

Define $\bar{A} := \hat{A}^0 + \hat{B}^0(I - \hat{D}^{0*}\hat{D}^0)^{-1}\hat{D}^{0*}\hat{C}^0$. *Then the following conditions are equivalent*:

(i) $1 \notin \mathrm{spec}(Z\hat{A}^0)$ *and* $\|\hat{C}^0(I - Z\hat{A}^0)^{-1}Z\hat{B}^0 + \hat{D}^0\| < 1$
(ii) $1 \notin \mathrm{spec}(Z\bar{A})$ *and* $\|\bar{C}(I - Z\bar{A})^{-1}Z\bar{B}\| < 1$.

Proof. The proof makes use of repeated application of the Schur complement formula. Applying this to Eq. (28), we have that condition (i) above holds if and only if there exists a block diagonal $X > 0$ such that

$$\begin{bmatrix} -Z^*X^{-1}Z & \hat{A}^0 \\ \hat{A}^{0*} & -X \end{bmatrix} - \begin{bmatrix} \hat{B}^0 & 0 \\ 0 & \hat{C}^{0*} \end{bmatrix}\begin{bmatrix} -I & \hat{D}^{0*} \\ \hat{D}^0 & -I \end{bmatrix}^{-1}\begin{bmatrix} \hat{B}^{0*} & 0 \\ 0 & \hat{C}^0 \end{bmatrix} < 0. \qquad (30)$$

By directly expanding the matrix inverse, it is clear that this is tantamount to

$$\begin{bmatrix} -Z^*X^{-1}Z & \hat{A}^0 \\ \hat{A}^{0*} & -X \end{bmatrix} + \begin{bmatrix} \hat{B}^0(I - \hat{D}^{0*}\hat{D}^0)^{-1}\hat{B}^{0*} & \hat{B}^0(I - \hat{D}^{0*}\hat{D}^0)^{-1}\hat{D}^{0*}\hat{C}^0 \\ \hat{C}^{0*}\hat{D}^0(I - \hat{D}^{0*}\hat{D}^0)^{-1}\hat{B}^{0*} & \hat{C}^{0*}(I - \hat{D}^0\hat{D}^{0*})^{-1}\hat{C}^0 \end{bmatrix} < 0, \qquad (31)$$

which we can immediately rewrite as

$$\begin{bmatrix} -Z^*X^{-1}Z + \bar{B}\bar{B}^* & \bar{A} \\ \bar{A}^* & -X + \bar{C}^*\bar{C} \end{bmatrix} < 0. \qquad (32)$$

Reversing the Schur complement formula, this holds if and only if

$$\begin{bmatrix} -Z^*X^{-1}Z & \bar{A} & \bar{B} & 0 \\ \bar{A}^* & -X & 0 & \bar{C}^* \\ \bar{B}^* & 0 & -I & 0 \\ 0 & \bar{C} & 0 & -I \end{bmatrix} < 0, \qquad (33)$$

which holds if and only if condition (ii) above holds. □

The following result is an immediate consequence of this. It states that a given controller achieves a closed-loop induced norm bound of less than 1 for this system if and only if it achieves a closed-loop induced norm bound of less than 1 for a standard DT system, constructed in the hypotheses.

Theorem 5. *Suppose* $\|\hat{D}_{11k}\|_{L_2 \to L_2} < \gamma$ *holds for all* $k \geqslant 0$, *where* $\gamma < 1$. *Then let* $\bar{G}$ *be a purely DT system with state space realization given by*

$$x_{k+1} = \bar{A}_k x_k + \bar{B}_{1k}w_k + \bar{B}_{2k}u_k,$$
$$z_k = \bar{C}_{1k}x_k + \bar{D}_{12k}u_k, \qquad (34)$$
$$y_k = \bar{C}_{2k}x_k,$$

where

$$\bar{A}_k = \hat{A}_k + \hat{B}_{1k}\hat{D}_{11k}(I - \hat{D}_{11k}\hat{D}_{11k}^*)^{-1}\hat{C}_{1k}, \qquad (35)$$

$$\bar{B}_{2k} = \hat{B}_{2k} + \hat{B}_{1k}\hat{D}_{11k}(I - \hat{D}_{11k}\hat{D}_{11k}^*)^{-1}\hat{D}_{12k}, \qquad (36)$$

$$\bar{C}_{2k} = \hat{C}_{2k}, \qquad (37)$$

$$\hat{L}_k = (I - \hat{D}_{11k}\hat{D}_{11k}^*)^{-1}, \qquad (38)$$

and $\bar{B}_1$,$\bar{C}_1$,$\bar{D}_{12}$ *satisfy*

$$\bar{B}_{1k}\bar{B}_{1k}^* = \hat{B}_{1k}(I - \hat{D}_{11k}^*\hat{D}_{11k})^{-1}\hat{B}_{1k}^*, \tag{39}$$

$$\begin{bmatrix} \bar{C}_{1k}^* \\ \bar{D}_{12k}^* \end{bmatrix}[\bar{C}_{1k}\ \bar{D}_{12k}] = \begin{bmatrix} \hat{C}_{1k}^* \\ \hat{D}_{12k}^* \end{bmatrix}\hat{L}_k[\hat{C}_{1k}\ \hat{D}_{12k}]. \tag{40}$$

Then the following are equivalent:

(i) *The controller K stabilizes the jump system* $\tilde{G}$ *and the closed-loop system satisfies* $\|F_l(\tilde{G}, K)\|_{L_2 \to L_2} < 1$

(ii) *The controller K stabilizes the DT system* $\bar{G}$ *and the closed-loop system satisfies* $\|F_l(\bar{G}, K)\|_{\ell_2 \to \ell_2} < 1$.

Note that in the above theorem, $\bar{B}_1$, $\bar{C}_1$, and $\bar{D}_{12}$ are not uniquely defined, but they can always be constructed using simple matrix factorizations. Also it is clear that the hypothesis is reasonable; if the norm $\|\hat{D}_{11k}\|_{L_2 \to L_2}$ is not uniformly less than 1 then no controller achieving the desired performance will exist. The latter, so-called compression norm, can be computed using a variety of techniques (Dullerud, 1999; Flamm & Mitter, 1987; Foias, Tannenbaum, & Zames, 1987; Zhou & Khargonekar, 1987). An immediate corollary is the following.

Corollary 6. *Let* $\bar{G}$ *be defined as in Theorem* 5. *Then the following are equivalent*:

(i) *The controller K stabilizes the DT system* $\bar{G}$, *and the closed-loop system satisfies* $\|F_l(\bar{G}, K)\|_{\ell_2 \to \ell_2} < 1$.

(ii) *The controller K stabilizes the multi-rate sampled-data system G and the closed-loop system satisfies* $\|F_l(G, HKS)\|_{L_2 \to L_2} < 1$.

Formulae for the equivalent DT system. We now proceed to give formulae for the above DT system. Clearly, all we need to do is calculate the composition of the relevant operator expressions for the lifted system to construct the finite dimensional matrices

$$\hat{B}_{1k}(I - \hat{D}_{11k}^*\hat{D}_{11k})^{-1}\hat{B}_{1k}^*, \tag{41}$$

$$\begin{bmatrix} \hat{C}_{1k}^* \\ \hat{D}_{12k}^* \end{bmatrix}(I - \hat{D}_{11k}\hat{D}_{11k}^*)^{-1}[\hat{C}_{1k}\ \hat{D}_{12k}], \tag{42}$$

$$\hat{B}_{1k}\hat{D}_{11k}(I - \hat{D}_{11k}\hat{D}_{11k}^*)^{-1}\hat{C}_{1k}[\hat{C}_{1k}\ \hat{D}_{12k}]. \tag{43}$$

This task is straightforward and can be performed using similar methods to those used in the single-rate sampled-data case in Bamieh and Pearson (1992), Cantoni and Glover (1996), Hara and Kabamba (1990), Chen and Francis (1996), and Toivonen (1993). Since the derivation is standard we omit it here, and simply state the formulae.

Define the Hamiltonian matrix

$$E = \begin{bmatrix} -\tilde{A}_1^* - \tilde{C}_1^*\tilde{D}_{11}\tilde{M}\tilde{B}_1^* & -\tilde{C}_1^*\tilde{L}\tilde{C}_1 \\ \tilde{B}_1\tilde{M}\tilde{B}_1^* & \tilde{A}_1 + \tilde{B}_1\tilde{M}\tilde{D}_{11}^*\tilde{C}_1 \end{bmatrix}, \tag{44}$$

where $\tilde{L} = (I - \tilde{D}_{11}\tilde{D}_{11}^*)^{-1}$ and $\tilde{M} = (I - \tilde{D}_{11}^*\tilde{D}_{11})^{-1}$. Let $Q_k = e^{h_k E}$, and partition Q_k as

$$Q_k = \begin{bmatrix} Q_{11k} & Q_{12k} \\ Q_{21k} & Q_{22k} \end{bmatrix}, \tag{45}$$

where Q_{11k} has the same dimension as $\tilde{A}_1$. Then

$$\bar{B}_{1k}\bar{B}_{1k}^* = Q_{21k}Q_{11k}^{-1}, \tag{46}$$

$$\begin{bmatrix} \bar{C}_{1k}^* \\ \bar{D}_{12k}^* \end{bmatrix}[\bar{C}_{1k}\ \bar{D}_{12k}] = -\begin{bmatrix} \tilde{A}_2(t_k)^* \\ \tilde{B}_2(t_k)^* \end{bmatrix}Q_{11k}^{-1}Q_{12k}[\tilde{A}_2(t_k)\quad \tilde{B}_2(t_k)], \tag{47}$$

$$\bar{A}_k = Q_{11k}^{-1'}\tilde{A}_2(t_k), \tag{48}$$

$$\bar{B}_{2k} = Q_{11k}^{-1'}\tilde{B}_2(t_k), \tag{49}$$

$$\bar{C}_{2k} = \tilde{C}_2(t_k). \tag{50}$$

The final DT system is obtained by symmetric factorization of expressions (46) and (47); all that remains is to synthesize a controller for this discrete-time system.

5. Discrete LTV synthesis

We now wish to synthesize a controller for the discrete system $\bar{G}$ in (34). This system will be *periodic*, if the multi-rate sample and hold operators are periodic and have rational rates. However, in the general case this will not be the case, and as a result both the jump system $\tilde{G}$ and the DT system $\bar{G}$ will be aperiodically time-varying. We therefore turn to the problem of synthesizing a controller for this time-varying system.

Making use of Definition 2, the matrix sequences $\bar{A}_k$, $\bar{B}_{1k}$, $\bar{B}_{2k}$, $\bar{C}_{1k}$, $\bar{C}_{2k}$ and $\bar{D}_{22k}$ in (34) define block-diagonal operators, which we will denote by $\bar{A}$, $\bar{B}_1$, $\bar{B}_2$, $\bar{C}_1$, $\bar{C}_2$ and $\bar{D}_{22}$. Then we can rewrite (34) as

$$x = Z\bar{A}x + Z\bar{B}_1 w + Z\bar{B}_2 u,$$

$$z = \bar{C}_1 x + \bar{D}_{12}u,$$

$$y = \bar{C}_2 x. \tag{51}$$

Further, for each k, define N_k^R to be an orthonormal basis for the null space of $[\bar{B}_{2k}^*\ \bar{D}_{12k}^*]$ and N_k^S to be an orthonormal basis for the null space of $[\bar{C}_{2k}\ \bar{D}_{21k}]$. Let $\bar{N}^R = \text{diag}\{N_0^R, N_1^R, \dots\}$ and $\bar{N}^S = \text{diag}\{N_0^S, N_1^S, \dots\}$. The following result is taken from Dullerud and Lall (1999b).

Theorem 7. *The following are equivalent*:

(i) *There exists a controller K that stabilizes the DT system* $\bar{G}$, *and the closed-loop system satisfies* $\|F_l(\bar{G}, K)\|_{\ell_2 \to \ell_2} < 1$.

(ii) *There exist block diagonal operators* $\bar{R}, \bar{S} \in \mathscr{L}(\ell_2)$, $\bar{R} > 0$, $\bar{S} > 0$, *such that*

$$\begin{bmatrix} \bar{N}^R & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \bar{A}\bar{R}\bar{A}^* - Z^*\bar{R}Z & \bar{A}\bar{R}\bar{C}_1^* & \bar{B}_1 \\ \bar{C}_1\bar{R}\bar{A}^* & \bar{C}_1^*\bar{R}\bar{C}_1 - I & 0 \\ \bar{B}_1^* & 0 & -I \end{bmatrix} \begin{bmatrix} \bar{N}^R & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \bar{N}^S & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \bar{A}^*Z^*\bar{S}Z\bar{A} - \bar{S} & \bar{A}^*Z^*\bar{S}Z\bar{B}_1 & \bar{C}_1^* \\ \bar{B}_1^*Z^*\bar{S}Z\bar{A} & \bar{B}_1^*Z^*\bar{S}Z\bar{B}_1 - I & 0 \\ \bar{C}_1 & 0 & -I \end{bmatrix} \begin{bmatrix} \bar{N}^S & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \bar{R} & I \\ I & \bar{S} \geqslant 0 \end{bmatrix}.$$

The above result is a characterization of the existence conditions for LTV synthesis for DT systems in terms of affine operator inequalities. It is equivalent to the usual characterization in terms of a DT Riccati recursion over infinite time, such as can be found in Xie et al. (1993). However, the above formulation is convex, and as such holds the possibility for analysis of almost periodic sampled-data systems. Indeed, it is reasonable to assume that given a solution to the above operator inequalities, small perturbations to the sampling times (but not the sampling *rate*) will not cause that solution to be infeasible. Hence, this gives some justification to the standard treatment that sample and hold rates are exact.

The following corollary provides a general solution to the multi-rate sampled-data problem.

Corollary 8. *There exists a discrete multi-rate sampled-data controller K for the LTI system G such that the closed-loop system is stable and satisfies* $\|F_\ell(G, HKS)\|_{L_2 \to L_2} < 1$ *if and only if there exists block diagonal operators* $\bar{R}$ *and* $\bar{S}$ *satisfying the linear matrix inequalities of Theorem* 7(ii).

5.1. Synthesis for periodic systems

In general, the above operator inequalities for the general time-varying problem are difficult to compute, with little currently known about numerical methods for the infinite dimensional case. As far as the authors are aware, there are only numerical techniques currently available for the case when the sample and hold operators are periodic, and their rates are related by rational numbers. In this case, jump system (16) will also be periodic. This implies that Γ_k, Ω_k, and h_k must all be periodic. Let the periodicity be j, so that $h_{k+j} = h_k$, $\tilde{A}_2(t_k) = \tilde{A}_2(t_{k+j})$, $\tilde{B}_2(t_k) = \tilde{B}_2(t_{k+j})$, and $\tilde{C}_2(t_k) = \tilde{C}_2(t_{k+j})$ for all $k \geqslant 0$.

For the case when the system matrices are all periodic, then it is shown in Dullerud and Lall (1999b) that the $\bar{R}$ and $\bar{S}$ operators can always be chosen to have periodic diagonal blocks. In this case, the above problem becomes the simple matrix inequality as follows. Define the block diagonal matrices

$$\begin{aligned} \underline{A} &= \operatorname{diag}\{\bar{A}_0, \ldots, \bar{A}_{j-1}\}, \\ \underline{B}_1 &= \operatorname{diag}\{\bar{B}_{1,0}, \ldots, \bar{B}_{1,j-1}\}, \\ \underline{B}_2 &= \operatorname{diag}\{\bar{B}_{2,0}, \ldots, \bar{B}_{2,j-1}\}, \\ \underline{C}_1 &= \operatorname{diag}\{\bar{C}_{1,0}, \ldots, \bar{C}_{1,j-1}\}, \\ \underline{C}_2 &= \operatorname{diag}\{\bar{C}_{2,0}, \ldots, \bar{C}_{2,j-1}\}, \\ \underline{D}_{12} &= \operatorname{diag}\{\bar{D}_{12,0}, \ldots, \bar{D}_{12,j-1}\}, \\ \underline{N}^R &= \operatorname{diag}\{N_0^R, \ldots, N_{j-1}^R\}, \\ \underline{N}^S &= \operatorname{diag}\{N_0^S, \ldots, N_{j-1}^S\} \end{aligned} \tag{52}$$

and the cyclic shift matrix

$$\tilde{Z} = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}. \tag{53}$$

Theorem 9. *Suppose the DT system* $\bar{G}$ *is periodic, with period j. Then there exists a controller K such that the closed-loop system is stable and satisfies* $\|F_\ell(\bar{G}, K)\|_{\ell_2 \to \ell_2} < 1$ if and only if there exist block-diagonal matrices $R > 0$ and $S > 0$ satisfying

$$\begin{bmatrix} \underline{N}^R & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \underline{A}R\underline{A}^* - \tilde{Z}^*R\tilde{Z} & \underline{A}R\underline{C}_1^* & \underline{B}_1 \\ \underline{C}_1R\underline{A}^* & \underline{C}_1^*R\underline{C}_1 - I & 0 \\ \underline{B}_1^* & 0 & -I \end{bmatrix} \begin{bmatrix} \underline{N}^R & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \underline{N}^S & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \underline{A}^*\tilde{Z}^*S\tilde{Z}\underline{A} - S & \underline{A}^*\tilde{Z}^*S\tilde{Z}\underline{B}_1 & \underline{C}_1^* \\ \underline{B}_1^*\tilde{Z}^*S\tilde{Z}\underline{A} & \underline{B}_1^*\tilde{Z}^*S\tilde{Z}\underline{B}_1 - I & 0 \\ \underline{C}_1 & 0 & -I \end{bmatrix} \begin{bmatrix} \underline{N}^S & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geqslant 0.$$

The following corollary is the main result of this paper.

Corollary 10. *Suppose that the sequences* h_k, Γ_k *and* Ω_k *characterizing the multi-rate sample and hold devices* *are periodic in k. Then there exists a discrete multi-rate sampled-data controller K for the LTI system G such that the closed-loop system is stable and satisfies* $\|F_l(G, HKS)\|_{L_2 \to L_2} < 1$ *if and only if there exist block diagonal matrices* $R > 0$ *and* $S > 0$ *satisfying the linear matrix inequalities of Theorem* 9.

These conditions are linear matrix inequalities, and hence the constraints they express on R and S are convex. Standard convex optimization techniques and software can be used to check for existence of feasible solutions.

Thus we have reduced the question of whether there exists a multi-rate sampled-data controller for the LTI system G to a set of finite dimensional linear matrix inequalities. All that remains is to construct the controller.

6. Controller construction

Once we have solved the above linear matrix inequalities for R and S, it remains to construct the discrete controller. The controller is a j-periodic DT controller with state dimension $n + n_u$. It is specified as

$$x_{k+1} = A_k^d x_k + B_k^d y_k,$$
$$u_k = C_k^d x_k + D_k^d y_k. \tag{54}$$

This controller has the usual DT state space form. Note that, when implementing this controller, at any time k at which there is no sampled measurement available on a given channel that channel should simply be set to zero. In this way the measured signal y_k is exactly that produced by the jump system for which the controller achieves the desired induced-norm. Similarly the controller produces a control signal u_k at all times k, and if a hold channel is not active at that time then it should simply ignore the signal.

In order to construct the controller, a convex optimization in the form of a linear matrix inequality must be solved. Define the matrices J_k by

$$J_k = \begin{bmatrix} A_k^d & B_k^d \\ C_k^d & D_k^d \end{bmatrix}. \tag{55}$$

The following procedure is used to construct the LMI; it is an extension of the methods used for time-invariant systems in Gahinet and Apkarian (1994) to the time-varying case.

The first step is to construct nonsingular matrices M_k, N_k such that $M_k N_k^* = I - R_k S_k$ where R_k and S_k are the kth block of the block-diagonal matrices R and S respectively. Now construct the matrices $\{X_0, \ldots, X_{j-1}\}$, where $X_k \in \mathbb{R}^{2(n+n_u) \times 2(n+n_u)}$ is the unique solution of

$$\begin{bmatrix} S_k & I \\ N_k^* & 0 \end{bmatrix} = X_k \begin{bmatrix} I & R_k \\ 0 & M_k^* \end{bmatrix}. \tag{56}$$

The controller is given by finding a solution J_k to the linear matrix inequality

$$H_k + Q_k^* J_k^* P_k + P_k^* J_k Q_k < 0, \tag{57}$$

where

$$H_k = \begin{bmatrix} -X_{k+1}^{-1} & A_k^0 & B_k^0 & 0 \\ A_k^{0^*} & -X_k & 0 & C_k^{0^*} \\ B_k^{0^*} & 0 & -I & 0 \\ 0 & C_k^0 & 0 & -I \end{bmatrix}, \tag{58}$$

$$P_k = \left[\begin{bmatrix} 0 & I \\ \bar{B}_{2k}^* & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{D}_{12k}^* \end{bmatrix} \right], \tag{59}$$

$$Q_k = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I_n \\ \bar{C}_{2k} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{D}_{21k} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] \tag{60}$$

and

$$A_k^0 = \begin{bmatrix} \bar{A}_k & 0 \\ 0 & 0_{n+n_u} \end{bmatrix}, \qquad B_k^0 = \begin{bmatrix} \bar{B}_{1k} \\ 0 \end{bmatrix} \tag{61}$$

$$C_k^0 = [\bar{C}_{1k} \quad 0]. \tag{62}$$

Since this controller achieves the desired performance for the DT system $\bar{G}$, when connected to the multi-rate sampled-data system G it achieves the induced norm performance that $\|F_l(G, HKS)\| < 1$ as required.

7. Example synthesis

The numerical procedure is as follows; first, for a given sample and hold rate, the jump system is constructed. Then the standard binary search over candidate γ values for performance is performed, with a test for controller existence for each value of γ. For each candidate γ, the equivalent DT system is constructed, and the matrix inequalities of Theorem 9 tested for feasibility. If they are feasible, then γ is decreased and the binary iteration continues until the desired tolerance is achieved.

As an example, we consider the following arbitrarily chosen system

$$A = \begin{bmatrix} -0.5485 & 1.0812 \\ 0.3041 & -2.6803 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1.3908 & -1.1711 \\ 0.0364 & 0.5731 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1.3572 & -1.7605 \\ 0.3329 & 0.0048 \end{bmatrix}, \quad C_1 = [0.3359 \; 0.6503],$$

$$C_2 = [-0.6097 \; 0.2265], \quad D_{11} = [1.2005 \; 0.3263],$$

$$D_{12} = [0.8595 \; -0.5162], \quad D_{22} = [-0.0406 \; 0.3559].$$

The optimal H_∞ norm achievable using a continuous controller for this system is 1.2441.

Our first synthesis for this system is a simple multi-rate problem, with actuation rates half of the sensing rates.

Table 1
Achieved induced norm for multi-rate syntheses

	Sample period	Hold period (channel 1)	Hold period (channel 2)	γ
1	0.75	1.5	1.5	1.5616
2	0.75	1.5	0.75	1.4225
3	0.75	0.75	1.5	1.4196
4	0.75	0.75	0.75	1.4148
5	0.75	1.5	1.5 (offset)	1.4240

Sampling rates are chosen sufficiently slow that the sampled-data performance is significantly less than the continuous performance. Using a sample period of 0.75 and a hold period of 1.5, the optimal achievable closed-loop induced norm is 1.5616. The compression norm calculation of $\|\hat{D}_{11k}\|_{L_2\to L_2}$ was carried out using the method in Dullerud (1999); this is only required for $k=1$, the longest time interval between jumps.

As shown in Table 1 decreasing the hold interval on either channel leads to better performance. Experiments 2 and 3 have one actuator channel twice the speed of the other. Experiment 5 is different from Experiment 1 in that the two actuator channels are offset from each other by half the actuation period, as shown in Fig. 2. In Experiment 5, the Ω and Γ matrices are given by

$$\Omega_1=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \Omega_2=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \Gamma_1=1,\quad \Gamma_2=1.$$

This leads to an improved performance over the case when the two channels are actuated at the same time with a period of 1.5. In fact, it can be shown that the induced-norm must be greater than that achievable by doubling either actuator rate, although it need not be less than the synchronous case.

For this system, where the state dimension is 2, the hold period is twice the sample period, and there are 2 inputs, the resulting matrix inequalities consist of an optimization over two 8×8 block diagonal matrices. For a processor with a specfp of about 30, these computations required about 100 s overall using standard tools such as MATLAB and the LMI Control toolbox (Gahinet, Nemirovski, Laub, & Chilali, 1995).

Further details of this specific example are provided in the following subsection. However, at this point we provide some general comments about the computational effort required to perform synthesis using Theorem 9 and the controller construction of Section 6.

In Theorem 9, the two variables are R and S, which are symmetric block diagonal matrices. Each block is $(n+n_u)\times(n+n_u)$ in dimension, making the total number of decision variables $(n+n_u)(n+n_u+1)j$, where j is the system period. These decision variables are subject to the three LMIs given in the theorem. Since the block-diagonal matrices $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are sparse, the LMIs are

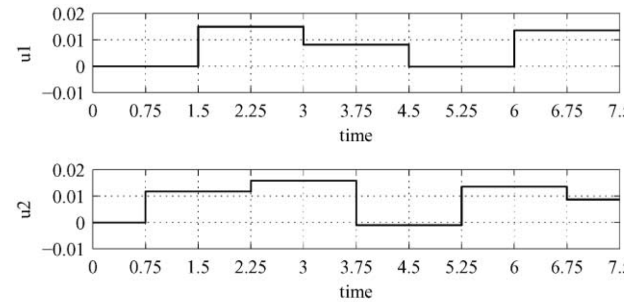

Fig. 2. Input signals for Experiment 5.

equivalent to the following j coupled LMIs

$$\begin{bmatrix}N_k^R&0\\0&I\end{bmatrix}^*\begin{bmatrix}\bar{A}_kR_k\bar{A}_k'-R_{k+1}&\bar{A}_kR_k\bar{C}_{1,k}'&\bar{B}_{1,k}\\\bar{C}_{1,k}R_k\bar{A}_k'&\bar{C}_{1,k}R_k\bar{C}_{1,k}'-I&\bar{D}_{11,k}\\\bar{B}_{1,k}'&\bar{D}_{11,k}^*&-I\end{bmatrix}\begin{bmatrix}N_k^R&0\\0&I\end{bmatrix}<0,$$

$$\begin{bmatrix}N_k^S&0\\0&I\end{bmatrix}^*\begin{bmatrix}\bar{A}_k'S_{k+1}\bar{A}_k-S_k&\bar{A}_k'S_{k+1}\bar{B}_{1,k}&\bar{C}_{1,k}\\\bar{B}_{1,k}'S_{k+1}\bar{A}_k&\bar{B}_{1,k}^*S_{k+1}\bar{B}_{1,k}-I&\bar{D}_{11,k}^*\\\bar{C}_{1,k}&\bar{D}_{11,k}&-I\end{bmatrix}\begin{bmatrix}N_k^S&0\\0&I\end{bmatrix}<0,$$

$$\begin{bmatrix}R_k&I\\I&S_k\end{bmatrix}\geqslant 0,$$

where $0\leqslant k\leqslant j-1$. Thus, for long periods we can expect that computations will be expensive if general purpose LMI software is used, as both the number of decision variables and the number of constraints grows linearly in the period j. Directly implementing this will lead to a greater than linear (but polynomial) growth rate with respect to the period j; see Nesterov and Nemirovskii (1994) for details on analysis of computational cost of current general purpose semidefinite programming algorithms. The above LMIs have a very specific recursive structure, so specialized routines for exploiting this may be possible. It is also important to note, once R and S satisfying the above inequalities are found, that from (57) we see the computational effort required to explicitly construct an admissible controller K grows linearly with the system period.

In Sagfors et al. (1997c, 1998) a multirate synthesis problem is solved in terms of two algebraic Riccati equations. Although this solution is restricted to systems with rank constraints, the number of synthesis conditions one has to check grows linearly with respect to the period.

These conditions can be computed via a matrix recursion in explicit form.

The earlier multi-rate approaches in Chen and Qiu (1994), and Voulgaris and Bamieh (1993) use lifting techniques to convert the multirate synthesis problem, to a constrained model matching problem. The constraint is that the D-matrix of the lifted compensator must be lower block-triangular (causal). In Chen and Qiu (1994) the results of Glover, Limebeer, Doyle, Kasenally, and Safonov (1991) are first used to parametrize all solutions to the *unconstrained* model matching problem. Then, using nest algebra theory, explicit matrix conditions are provided for determining whether an element of this parametrization exists, which satisfies the required causality constraint. A different approach is used in Voulgaris and Bamieh (1993), and the model matching problem is solved by appealing directly to Voulgaris, Dahleh, and Valavani (1991), where the related discrete time problem is solved in terms of a finite dimensional convex programming problem, plus a standard Nehari problem.

7.1. Numerical details for Experiment 3

In Experiment 3, the sampling period is 0.75, the first actuator channel has hold period of 0.75, and the second actuator channel has hold period of 1.5. Actuator inputs when the system is driven by noise are shown in Fig. 3.

Both holds occur synchronized with the sampler. Applying the construction of Section 3, the Ω and Γ matrices are

$$\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_1 = 1, \quad \Gamma_2 = 1.$$

The optimal achievable norm, γ_0, was found to be 1.4196. A controller was synthesized for a desired $\gamma = 1.4204$, and this controller achieved a closed-loop norm of 1.4199. The induced norm of the closed-loop was computed by converting the final closed-loop system to a jump system and using an analysis procedure based on the methods in this paper.

The synthesis results in a state space controller whose parameters are

$$A_0^d = \begin{bmatrix} -0.0830 & 0.0231 & 0.1210 & 0.4738 \\ -0.3620 & 0.1027 & 0.3781 & 1.0694 \\ -0.0983 & 0.0285 & 0.0629 & 0.0245 \\ 0.5824 & -0.1671 & -0.4892 & -0.9246 \end{bmatrix},$$

$$A_1^d = \begin{bmatrix} -0.0405 & -0.0032 & 0.0692 & 0.0773 \\ -0.5653 & 0.1154 & 0.4599 & 0.3554 \\ -0.0207 & -0.0134 & 0.0829 & 0.0928 \\ -0.3498 & 0.0016 & 0.5494 & 0.5356 \end{bmatrix},$$

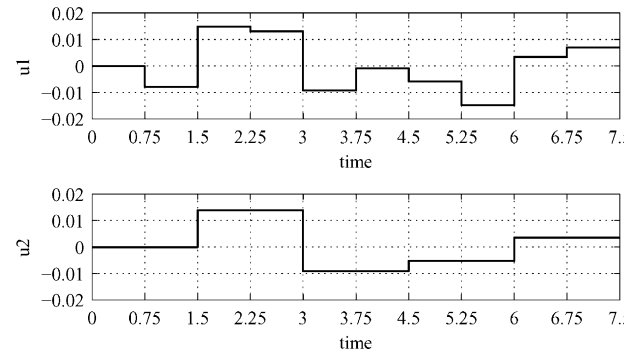

Fig. 3. Input signals for Experiment 3.

$$C_1^d = \begin{bmatrix} 134.2302 & -38.2747 & -128.8920 & -320.9355 \\ 136.9993 & -39.1042 & -128.9204 & -309.9957 \end{bmatrix},$$

$$C_2^d = \begin{bmatrix} -63.4246 & -3.7096 & 115.8516 & 115.1837 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1^d = \begin{bmatrix} -3.3042 \\ -6.0095 \\ 0.6089 \\ 2.9592 \end{bmatrix} \times 10^{-3}, \quad B_2^d = \begin{bmatrix} -2.9397 \\ -7.0285 \\ 0.3593 \\ -0.4840 \end{bmatrix} \times 10^{-3},$$

$$D_1^d = \begin{bmatrix} 1.5910 \\ 1.4759 \end{bmatrix}, \quad D_2^d = \begin{bmatrix} 0.5894 \\ 0 \end{bmatrix}.$$

The Matlab code which was used to perform these computations has been developed in the form of a toolbox for solving general H_∞ sampled-data problems, and is available from the web site, http://www.stanford.edu/~lall.

8. Conclusions

We have given an explicit method for H_∞ synthesis for multi-rate sampled-data systems. The techniques require only the solution of finite-dimensional LMIs. The conditions for existence of a controller achieving the desired performance are necessary and sufficient, and can be computed simply by testing the feasibility of two LMIs, using standard numerical methods. Furthermore, these conditions put no unnecessary restrictions on the realization of the system to be controlled.

The techniques are directly applicable to a wide class of multi-rate sampled-data problems; indeed, any problem where the jump system is periodic may be immediately solved. This covers the class of systems in which the sample and hold devices are synchronous and rational, for which previous synthesis techniques have been developed, and also covers other systems, such as those in which different sample channels are running at the same rates, but are offset by some fixed, not necessarily rational, time from each other.

The main problem solved in this paper is the multi-rate sampled-data synthesis problem. However, any problem which can be cast into the jump system framework can also be solved in exactly the same way; this includes the calculation of the induced-norm for multi-rate sampled-data problems, and sampled-data analysis and synthesis problems where the system G is composed of several discrete and continuous components connected via multiple sample and hold-devices. In this paper we have also concentrated on the case when G is linear time-invariant. In the time-varying case only the formulae (46)–(50) require modification; the methods and results remain unchanged.

References

Al-Rahmani, H., & Franklin, G. F. (1990). A new optimal multirate control of linear periodic and time-invariant systems. *IEEE Transactions on Automatic Control*, *35*(4), 406–415.

Amit, N. (1980). *Optimal control of multirate digital control systems*. Ph.D. thesis, Stanford, Dept. Aero. Astro., Report 523.

Amit, N., & Powell, J. D. (1981). Optimal digital control of multirate systems. *AIAA Guidance and Control Conference*.

Araki, M., & Ito, Y. (1993). Frequency response of sampled-data systems 1: Open-loop considerations. *Proceedings of the IFC World Congress*, 259–262.

Astrom, K. J., & Wittenmark, B. (1989). *Adaptive control*. Reading, MA: Addison-Wesley.

Ball, J. A., Gohberg, I., & Kaashoek, M. A. (1992). Nevanlinna–Pick interpolation for time-varying input–output maps: The discrete case. In I. Gohberg (Ed.), *Operator theory: Advances and applications*, Vol. 56, (pp. 1–51) Basel: Birkäuser.

Bamieh, B., Pearson, J. B., Francis, B. A., & Tannenbaum, A. (1991). A lifting technique for linear periodic systems with applications to sampled-data control. *Systems and Control Letters*, *17*, 79–88.

Bamieh, B. A., & Pearson, J. B. (1992). A general framework for linear periodic systems with applications to H-infinity sampled-data control. *IEEE Transactions on Automatic Control*, *37*(4), 418–435.

Başar, T. (1991). Optimum H-infinity designs under sampled state measurements. *Systems and Control Letters*, *16*, 399–409.

Başar, T., & Bernhard, P. (1995). *H-infinity optimal control and related minimax design problems. A dynamic game approach*. (2nd ed). Basel: Birkhauser.

Berg, M. C., Amit, N., & Powell, J. D. (1988). Multirate digital control system design. *IEEE Transactions on Automatic Control*, *33*, 1139–1150.

Bernhard, P. (1991). Application of the min–max certainty equivalence principle to the sampled data output feedback H-infinity control problem. *Systems and Control Letters*, *16*, 229–234.

Cantoni, M., & Glover, K. (1996). A design framework for continuous time systems under sampled-data control. *Proceedings of the 35th conference on decision and control* (pp. 458–463).

Chen, T., & Francis, B. A. (1991). H_2 optimal sampled-data control. *IEEE Transactions on Automatic Control*, *36*, 387–397.

Chen, T., & Francis, B. A. (1995). *Optimal sampled-data control systems*. Berlin: Springer.

Chen, T., & Francis, B. A. (1996). H-infinity optimal sampled-data control: computation and design. *Automatica*, *32*, 223–228.

Chen, T., & Qiu, L. (1994). H-infinity design of general multirate sampled-data control systems. *Automatica*, *30*(7), 1139–1152.

Colaneri, P., & De Nicolao, G. (1995). Multirate LQG control of continuous-time stochastic systems. *Automatica*, *31*, 1025–1042.

Dullerud, G., & Lall, S. (1999a). Asynchronous hybrid systems with jumps—analysis and synthesis methods. *Systems and Control Letters*, *37*, 61–69.

Dullerud, G., & Lall, S. (1999b). A new approach for analysis and synthesis of time-varying systems. *IEEE Transactions on Automatic Control*, *44*(8), 1486–1497.

Dullerud, G. E. (1996). *Control of uncertain sampled-data systems*. Basel: Birkhauser.

Dullerud, G. E. (1999). Computing the induced norm of a compression operator. *Systems and Control Letters*, *37*, 87–91.

Flamm, D., & Mitter, S. K. (1987). H_∞ sensitivity minimization for delay systems. *Systems and Control Letters*, *9*, 17–24.

Foias, C., Tannenbaum, A., & Zames, G. (1987). On the H_∞-optimal sensitivity problem for systems with delays. *SIAM Journal of Control and Optimization*, *25*, 686–706.

Gahinet, P., & Apkarian, P. (1994). A linear matrix inequality approach to H-infinity control. *International Journal of Robust and Nonlinear Control*, *4*(4), 421–448.

Gahinet, P., Nemirovski, A., Laub, A. J., & Chilali, M. (1995). *The LMI Control Toolbox*. Natick, MA: The Mathworks, Inc.

Glasson, D. P. (1982). A new technique for multirate digital control design and sample rate selection. *AIAA Journal on Guidance, Control and Dynamics*, *5*, 379–382.

Glover, K., Limebeer, D. J. N., Doyle, J. C., Kasenally, E. M., & Safonov, M. G. (1991). A characterization of all solutions to the four block general distance problem. *SIAM Journal of Control and Optimization*, *29*, 283–324.

Halanay, A., & Ionescu, V. (1994). *Time-varying discrete linear systems*. Basel: Birkhäuser.

Hara, S., & Kabamba, P. T. (1990). Worst case analysis and design of sampled-data control systems. *Proceedings of the 29th IEEE CDC*.

Horn, R. A., & Johnson, C. R. (1990). *Matrix analysis*. Cambridge: Cambridge University Press.

Iglesias, P. A. (1996). An entropy formula for time-varying discrete-time control systems. *SIAM Journal of Control and Optimization*, *34*, 1691–1706.

Kabamba, P. T., & Hara, S. (1993). Worst case analysis and design of sampled-data control systems. *IEEE Transactions on Automatic Control*, *38*, 1337–1357.

Kalman, R. E., & Bertram, J. E. (1959). A unified approach to the theory of sampling systems. *Journal of Franklin Institute*, *267*, 405–436.

Kranc, G. M. (1957). Input–output analysis of multirate feedback systems. *IRE Transactions on Automatic Control*, *3*, 21–28.

Lall, S. (1995). *Robust control synthesis in the time domain*. Ph.D. thesis, Cambridge University.

Lall, S., & Dullerud, G. (1997). H-infinity synthesis for multi-rate hybrid systems. *Proceedings of the IEEE Conference on Decision and Control* (pp. 2035–2040).

Lennartson, B. (1988). Periodic solutions of Riccati equations applied to multirate sampling. *International Journal of Control*, *48*, 1025–1042.

Lennartson, B., Lindgarde, O., Toivonen, H., & Sagfors, M. (1997). Mixed continuous-time and discrete-time H-infinity control. *Proceedings of the IEEE Conference on Decision and Control*.

Meyer, D. G. (1990). A parametrization of stabilizing controllers for multirate sampled-data systems. *IEEE Transactions on Automatic Control*, *35*, 429–433.

Nesterov, Y., & Nemirovskii, A. (1994). *Interior-Point Polynomial Algorithms in Convex Programming*. Society for Industrial and Applied Mathematics. Philadelphia: SIAM.

Ravi, R., Khargonekar, P. P., Minto, K., & Nett, C. N. (1990). Controller parametrization for time-varying multirate plants. *IEEE Transactions on Automatic Control*, *35*, 1259–1262.

Ravi, R., Nagpal, K. M., & Khargonekar, P. P. (1991). H-infinity control of linear time varying systems: A state space approach. *SIAM Journal of Control and Optimization*, *29*(6), 1394–1413.

Sagfors, M. F., & Toivonen, H. T. (1997). H-infinity and LQG control of asynchronous sampled-data systems. *Automatica*, *5*, 1663–1668.

Sagfors, M. F., Toivonen, H. T., & Lennartson, B. (1997a). H-infinity control of multirate sample-data systems: A state space approach, Technical Report 97-4, Department of Chemical Engineering, Abo Akademi University, Finland.

Sagfors, M. F., Toivonen, H. T., & Lennartson, B. (1997b). H-infinity control of multirate sampled-data systems: A Riccati equation solution. *Proceedings of the IEEE Conference on Decision and Control*.

Sagfors, M. F., Toivonen, H. T., & Lennartson, B. (1997c). State-space solution to the periodic multirate H-infinity control problem: A lifting approach. *Proceedings of the IEEE Conference on Decision and Control*.

Sagfors, M. F., Toivonen, H. T., & Lennartson, B. (1998). H-infinity control of multirate sampled-data systems: A state-space approach. *Automatica*, *34*(4), 415–428.

Sun, W., Nagpal, K. M., & Khargonekar, P. P. (1993). H-infinity control and filtering for sampled-data systems. *IEEE Transactions on Automatic Control*, *38*, 1162–1175.

Tadmor, G. (1992). H-infinity optimal sampled-data control in continuous-time systems. *International Journal of Control*, *56*, 99–114.

Toivonen, H. T. (1992). Sampled-data control of continuous-time systems with and H-infinity optimality criterion. *Automatica*, *28*, 45–54.

Toivonen, H. T. (1993). Worst-case sampling for sampled-data H-infinity design, *Proceedings of the IEEE Conference on Decision and Control* (pp. 337–342).

Toivonen, H. T. (1995). Digital control with an H-infinity optimality criterion. In C. T. Leondes (Ed.), *Control and Dynamic Systems*, Vol. 71. San Diego: Academic Press. pp. 215–262.

Toivonen, H. T., & Sagfors, M. F. (1997). The sampled-data H-infinity problem: A unified framework for discretization-based methods and riccati equation solution. *International Journal of Control*, *66*, 289–309.

Voulgaris, P. G., & Bamieh, B. (1993). Optimal H-infinity and H-2 control of hybrid multirate systems. *Systems and Control Letters*, *20*, 249–261.

Voulgaris, P. G., Dahleh, M. A., & Valavani, L. S. (1991). H-infinity and H-2 optimal controllers for periodic and multirate systems. *Proceedings of the IEEE Conference on Decision and Control* (pp. 214–216).

Xie, L., de Souza, C. E., & Wang, Y. (1993). Robust control of discrete time uncertain dynamical systems. *Automatica*, *29*(4), 1133–1137.

Yamamoto, Y. (1990). A new approach to sampled-data control systems—a function space method. *Proceedings of the IEEE Conference on Decision and Control* (pp. 1882–1887).

Zhou, K., & Khargonekar, P. P. (1987). Sensitivity minimization for delay systems. *Systems and Control Letters*, *8*, 307–312.

Geir E. Dullerud was born in Oslo, Norway, in 1966. He received the BASc. degree in Engineering Science, in 1988, and the MASc. degree in Electrical Engineering, in 1990, both from the University of Toronto, Canada. In 1994, he received his Ph.D. in Engineering from the University of Cambridge, England. During 1994 and 1995 he was a Research Fellow and Lecturer at the California Institute of Technology, in the Control and Dynamical Systems Department; from 1996 to 1998 he was an assistant professor in Applied Mathematics at the University of Waterloo, Canada. Since July 1998 he has been Assistant Professor of Mechanical Engineering at the University of Illinois, Urbana-Champaign. He has published two books: Control of Uncertain Sampled-data Systems, Birkhauser, 1996, and A Course in Robust Control Theory (with F. Paganini), Texts in Applied Mathematics, Springer, 2000. His areas of current research interest include robust and nonlinear control, complex and hybrid dynamical systems, and networks. He received the National Science Foundation CAREER Award in 1999.

Sanjay Lall received the Ph.D. degree in 1995 from Cambridge University, UK. He was a NATO Research Fellow in the Laboratory for Information and Decision Systems at Massachusetts Institute of Technology from 1995 to 1996. From 1996 until 2000 he was a Research Fellow in the Department of Control and Dynamical Systems at California Institute of Technology. Since October 2000 he has been Assistant Professor of Aeronautics and Astronautics at Stanford University.