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On Global Stability of Internet Congestion Control

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Abstract— **In this paper, we address the question of global asymptotic stability of TCP/AQM congestion control protocols. We analyze a well-known model, whose dynamics were previously shown to be locally stable via analysis of its linearization. We show that in fact the nonlinear dynamics are globally stable, and we explicitly account for the effects of both nonlinearities and time-delays in the dynamics. These results apply to the case of a single link with sources of identical fixed delay, and show that global stability holds under the same conditions that local stability does. The dynamic model analyzed is nonlinear, nonsmooth, and contains a delay, and the proof is based on the theory of integral-quadratic constraints.**

I. Introduction and Prior Work

The analysis of Internet congestion control protocols has received much attention recently. Explicit mathematical modelling of the Internet has allowed analysis of existing protocols from a number of different theoretical perspectives and has generated some suggestions for improvement to current protocols. This work has been motivated by concern about the ability of the current protocols to ensure stability and performance of the Internet as the number of users and amount of bandwidth continues to increase. Although the protocols that have been used in the past have performed remarkably well as the Internet has increased in size, analysis [1] indicates that as capacities and delays increase, instability will become a problem.

Many algorithms have been proposed for Internet congestion control, some of which have been shown to be globally stable in the presence of delay and nonlinearities [2], [3], [4], [5], [6]. In all of these cases, stability has been shown with various restrictions on system parameters and the number of links in the network.

In this paper, we address the question of global asymptotic stability of TCP/AQM congestion control protocols. We analyze a well-known model, whose dynamics were previously shown to be locally stable via analysis of its linearization in [7]. We show that in fact the nonlinear dynamics are *globally* stable, and explicitly account for the effects of both nonlinearities and time-delays in the dynamics. These results hold in the case of a single link with sources of identical fixed delay under the same conditions on the system parameters as used in [7].

In Low and Lapsley [3], it was shown that the dynamics of the Internet with certain control algorithms could be interpreted as a decentralized implementation of the gradient projection algorithm to solve the dual to the network optimization problem, thus showing global convergence to optimality for sufficiently small step size. In Paganini et al. [7] it was shown with a certain class of the pricing functions that a bound of $\alpha < \pi/2$ on a certain parameter α at the source allows a proof of linear stability for arbitrary topology and heterogeneous time delays. Global asymptotic stability with time-delay of the protocols by Paganini et al. was discussed by Wang and Paganini [4] for a nonlinear implementation. In the single-source with a single-link case, this paper gave a proof of global asymptotic stability with time delay for all $\alpha < \alpha_{\max}$, where $\alpha_{\max} = \ln(x_{\max}/c)/\big((x_{\max}/c)-1\big)$, c is the capacity of the link and $x_{\max}$ is a maximum data rate parameter. In this paper, we show that for all $0 < \alpha < \pi\alpha_{\max}/2$, the congestion control algorithm with time-delays is globally asymptotically stable. If $x_{\max} = c$, then $\alpha_{\max} = 1$ and this is the same bound used in [7] for the linearized case.

This paper uses an input-output approach to stability, similar to the work in [6], [4]. The input-output stability result is then used to prove global asymptotic stability. The paper is organized as follows. We first discuss the derivation of the proposed algorithm from the general network optimization problem in Section II-B. In Section II-C, we discuss stability in terms of delay-differential equations and the interconnection of operators and give a generalization of passivity theory in the form of IQCs. In Section III-A, we form the interconnection and in Sections III-B through III-D we use IQC theory to prove stability. We then conclude global asymptotic stability and briefly discuss implementation.

II. Background Material

A. Notation

The space of continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^n$ is denoted by $\mathcal{C}$ with norm $\|\phi\| = \sup_\theta \|\phi(\theta)\|$. We use $\mathcal{C}_\tau$ to denote the space of continuous functions which map $[0,\tau] \to \mathbb{R}^n$ with the same norm. A function $x : \mathbb{R} \to \mathbb{R}$ is ***absolutely continuous*** if for any integer N and any sequence $t_1, \dots, t_N$, we have $\sum_{k=1}^{N-1} |x(t_k) - x(t_{k+1})| \to 0$ whenever $\sum_{k=1}^{N-1} |t_k - t_{k+1}| \to 0$. The closed norm ball of radius β is denoted by $\mathcal{B}(\beta)$.

$L_2(-\infty,\infty)$ is the Hilbert space of Lebesgue measurable complex vector-valued functions $x : \mathbb{R} \to \mathbb{C}^n$ with inner-product $\langle u, v\rangle_2 = \int_{-\infty}^{\infty} u(t)^* v(t) dt$, where $u(t)^*$ denotes the conjugate transpose of $u(t)$. L_2 denotes $L_2[0,\infty) = \{\, x \in L_2(-\infty,\infty) \mid x(t) = 0 \text{ for all } t < 0 \,\}$ and is a Hilbert subspace of $L_2(-\infty,\infty)$. P_T is the truncation operator such that if $y = P_T z$, then $y(t) = z(t)$ for all

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$t \leq T$ and $y(t) = 0$ otherwise. L_{2e} denotes the space of functions such that for any $T > 0$ and $y \in L_2e$, we have $P_T y \in L_2$. The dimensions of the various L_2 spaces used should be clear from context and are not explicitly stated. A causal operator $H : L_{2e} \to L_{2e}$ is bounded if $H(0) = 0$ and if it has finite gain, defined as

$$\|H\| = \sup_{u \in L_2 \neq 0} \frac{\|Hu\|}{\|u\|}$$

$\hat{L}_2$ denotes the Hilbert space of complex vector-valued functions on the imaginary axis, $x : j\mathbb{R} \to \mathbb{C}^n$ with inner-product $\langle \hat{u}, \hat{v} \rangle_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{v}(j\omega) d\omega$. $\hat{L}_\infty$ denotes the Banach space of matrix-valued functions on the imaginary axis, $\hat{G} : j\mathbb{R} \to \mathbb{C}^{m \times n}$ with norm $\|\hat{G}\|_\infty = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \bar{\sigma}\left(\hat{G}(j\omega)\right)$ where $\bar{\sigma}(\hat{G}(j\omega))$ denotes the maximum singular value of $\hat{G}(j\omega)$. $\hat{u}$ denotes the either the Fourier or Laplace transform of u, depending on u. We will also make use of the following specialized set of transfer functions, $\mathcal{A}$, defined to be those transfer functions which are the Laplace transform of functions of the form

$$g(t) = \begin{cases} h(t) + \sum_{i=1}^{N} g_i \delta(t - t_i) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $h \in L_1$, $g_i \in \mathbb{R}$ and $t_i \geq 0$.

B. The Internet Optimization Problem

We view the Internet as an abstract collection of sources and links. The term ***source*** refers to a connection between a user and a single destination. Sources transmit data in packets. The rate at which a source i transmits packets is dictated by the ***round-trip time***, the time taken for the user to receive acknowledgements for transmitted packets, τ_i, as well as by the number of packets which are allowed to be simultaneously unacknowledged (or window size), w_i. In this paper, we assume that packet losses do not affect the source transmission rates. We assume that acknowledgements contribute to delay but do not contribute to congestion at the links. We assume a fixed bit size for all packets and that τ_i is known at least for the purposes of determining data rate. The packet transmission rate, x_i, at source i can be controlled by the window size according to

$$w_i = x_i \tau_i. \tag{1}$$

The term ***link*** refers to a single congested resource such as a router. Packets arriving at a link enter an entrance queue. A link can process packets in the queue at rate capacity c_j. If too many data packets arrive in a given period of time, the size of the queue may grow and some packets may experience a queueing delay while in the queue. In this paper, we assume that the dynamics from this variable queueing delay are negligible and we only model the delay due to the fixed propagation time. Links are also able to feed back information. This can be done either through the ECN bit in the packet header, through packet dropping schemes or through measurement of variations in queueing delay. The value of the congestion indicator at the link is denoted p_j. In most schemes the congestion indicator received at each source is the summation of the indicators of all links in the source's route. This value is denoted q_i.

Sources and links are related by routing tables which specify the route or set of links, J_i through which the packets from source i to a certain destination must pass. The rate of packets received at a link is then the sum of the rates of all sources using that link and is denoted by y_j. The set of users for link j is denoted I_j. Ignoring delay for the moment, we have the following equations.

$$y = Rx, \quad y \leq c, \quad q = R^T p,$$

where

$$R_{ji} = \begin{cases} 1 & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$

1) Optimization Model: The following model for optimizing flow rates in a network was proposed by Kelly et al. [8].

$$\begin{aligned} \text{maximize} \quad & \sum_{i}^{N} U_i(x_i) \\ \text{subject to} \quad & x \geq 0, \quad Rx \leq c \end{aligned}$$

Assume that the U_i are strictly concave non-decreasing functions. Note that, as N increases, the problem becomes progressively more difficult to solve using a centralized algorithm. The dual problem has dual variable $p \in \mathbb{R}^M$, where M is the number of links, and is given by

$$\begin{aligned} \text{minimize} \quad & h(p) \\ \text{subject to} \quad & p \geq 0 \end{aligned}$$

where the dual function h is given by

$$\begin{aligned} h(p) &= \max_{x \geq 0} \sum_i (U_i(x_i)) - p^T(Rx - c) \\ &= \sum_i U_i(x_{\text{opt},i}) - p^T(Rx_{\text{opt}} + c), \end{aligned}$$

where $q = R^T p$ and $x_{\text{opt},i} = U_i'^{-1}(q_i)$.

We would like to construct a dynamical system which converges to the solution of the dual problem. One such system is given by the gradient projection algorithm. In discrete-time, this is

$$p_j(t+1) = \max\{0, p_j(t) - \gamma_j D_j h(p(t)\}, \tag{2}$$

where D_j denotes the partial derivative with respect to the j'th argument. A continuous-time analog is

$$\dot{p}_j(t) = \begin{cases} \gamma_j D_j h(p(t)) & \text{if } p_j(t) > 0 \\ \max\{0, \gamma_j D_j h(p(t))\} & \text{otherwise} \end{cases} \tag{3}$$

Here γ_j denotes a parameter, corresponding to step-size in discrete time.

This algorithm has the remarkable property that it is decentralized, corresponding to the separable structure of the constraints. For network optimization, it is as follows.

$$\dot{p}_j(t) = \begin{cases} \gamma(y_j(t) - c_j) & p_j(t) > 0 \\ \max\{0, \gamma(y_j(t) - c_j)\} & p_j(t) \le 0 \end{cases}$$
$$x_i(t) = U_i'^{-1}(q_i(t)), \quad y(t) = Rx(t), \quad q(t) = R^T p(t)$$

If γ is sufficiently small, the gradient projection algorithm will converge to the solution of the dual problem [3]. Because of convexity of the problem, strong duality implies that x_{opt} will converge to the solution of the primal problem. The gradient projection algorithm can be implemented in the Internet in the following manner. p_j is computed at each of M links. Link j requires only knowledge of y_j to compute this value. $x_{\text{opt},i}$ is computed at each of N sources. Source i requires only knowledge of q_i to compute this value.

2) Stability Properties: We must also consider the delay in transmitting packets from the source to the link and then receiving acknowledgements at the source. The delay from source i to link j is denoted τ_{ij}^f and the delay from link to source is denoted τ_{ij}^b. For any source, the total round trip time is fixed, i.e. $\tau_i = \tau_{ij}^f + \tau_{ij}^b$ for all $j \in J_i$. We express these delays in the frequency domain by replacing the entries of the routing matrix R with delay transfer functions, giving

$$y(s) = R^f(s)x(s), \quad q(s) = R^b(s)^T p(s)$$
$$R_{ji}^f(s) = \begin{cases} e^{-\tau_{ij}^f s} & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$
$$R_{ji}^b(s) = \begin{cases} e^{-\tau_{ij}^b s} & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$

This system has a stable linearization about its positive equilibrium point [7] if $\gamma_j = 1/c_j$ and

$$\frac{d}{dq_i} U_i'^{-1}(q_i) = -\frac{\alpha_i}{M_i \tau_i} U_i'^{-1}(q_i),$$

where M_i is a bound on the number of links in the path of source i and $\alpha_i < \pi/2$. The choice of

$$U_i(x) = \frac{M_i \tau_i}{\alpha_i}\left(1 - \ln \frac{x}{x_{\max,i}}\right),$$

with $x \le x_{\max,i}$ was suggested in [7] as a utility function such that $U_i'^{-1}(q) = x_{\max,i} e^{-\frac{\alpha_i}{M_i \tau_i} q}$, has the necessary derivative. In the case of a single source and a single link, the paper by Wang and Paganini [4] has shown this implementation to be globally stable for $\alpha < \alpha_{\max}$. When $x_{\max} = c$, then $\alpha_{\max} = 1$ and this condition becomes $\alpha < 1$.

C. Stability Analysis using Passivity

Two definitions of stability will be used in this paper. The first, finite-gain L_2 stability, is used to define stability of an interconnection of operators and is a property of input-output behavior. The second defines stability of a delay-differential equation and is a property of the behavior of the state given initial conditions. Consider the following

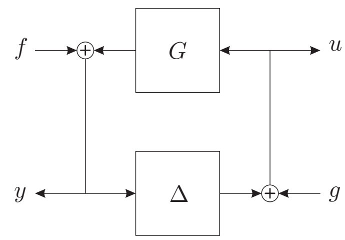

Fig. 1. Interconnection of systems

equations which define an interconnection between G and Δ where $\hat{G} \in \mathcal{A}$ and $\Delta : L_{2e} \to L_{2e}$ is causal and bounded.

$$y = Gu + f$$
$$u = \Delta y + g$$

Definition 1 (Jönsson [9], p71): The interconnection of G and Δ is ***well-posed*** if for every input pair (f, g) with $f, \dot{f} \in L_2$ and $g \in L_{2e}$, there exists a solution $u, y, \dot{y} \in L_{2e}$ and the map $(f, g) \to (u, y)$ is causal. The interconnection is L_2 ***stable*** if it is well-posed and there exists positive constants ρ_1, ρ_2, c such that for all $T > 0$,

$$\int_0^T (|y(t)|^2 + |u(t)|^2)dt \le \rho_1 \|f\|^2 + \rho_2 \|\dot{f}\|^2 + c \int_0^T |g(t)|^2 dt$$

Now consider a delay-differential equation of the following form with the assumption that $f(0, 0) = 0$.

$$\dot{x}(t) = f(x(t), x(t - \tau)) \tag{4}$$

The state of the differential equation at time t is $x_t \in \mathcal{C}_\tau$, where $x_t(\theta) = x(t - \tau + \theta)$ for all $\theta \in [0, \tau]$. $x \in \mathcal{C}$ is said to be a solution of (4) with initial condition $x^{\text{initial}} \in \mathcal{C}_\tau$ if $x(t) = x^{\text{initial}}(t)$ for $t \in [0, \tau]$ and equation (4) holds for all $t \ge \tau$. The following definition assumes existence of a unique solution for any initial conditions $x^{\text{initial}} \in \mathcal{C}_\tau$ and that the solution depends continuously on the initial condition.

Definition 2 (Hale [10]): The equation (4) is said to be ***stable*** if for any $\epsilon > 0$, there is a δ such that $x^{\text{initial}} \in \mathcal{B}(\delta)$ implies that for the solution x with initial condition x^{initial}, one has $x_t \in \mathcal{B}(\epsilon)$ for all $t \ge \tau$. The system is ***asymptotically stable*** if it is stable and every solution tends to 0 as $t \to \infty$.

1) Theory of Integral-Quadratic Constraints: We now move on to address stability of the interconnection of G and Δ. The result is based on analysis of the behavior of the operators with respect to the inputs and outputs.

Definition 3: For an operator, $\rho : X \to X$, the ***graph*** of ρ is the set of points $\Phi(\rho) = \{(x, y) : y = \rho(x), x \in X\}$. The ***inverse graph*** of ρ is the set $\Phi_i(\rho) = \{(x, y) : x = \rho(y), y \in X\}$.

Many theorems concerning the stability of the interconnection of operators can be viewed as separating the graph and inverse graph of two operators. That is, if, for some function $\sigma : X \times X \to \mathbb{R}$, we have $\sigma(x) \geq 0 \quad \forall x \in g_1$ and $\sigma(y) < 0 \quad \forall y \in g_2,\, y \neq 0$, then $g_1 \cap g_2 = 0$ and σ is a separating function, assuming both graphs contain the origin. For example, the small gain theorem can be expressed using $\sigma((x,y)) = \|x\| - k\|y\|$ for any $k > 0$. Similarly, passivity can be expressed using $\sigma((x,y)) = \langle x, y\rangle$. Definition 4 gives a class of separating functions which have been shown to be sufficient to prove L_2 stability.

Definition 4 (Rantzer [11]): The mapping $\sigma : L_2 \to \mathbb{R}$ is ***quadratically continuous*** if for every $\delta > 0$, there exists a η_δ such that the following holds.

$$|\sigma(x_1) - \sigma(x_2)| \leq \eta_\delta \|x_1 - x_2\|^2 + \delta\|x_2\|^2$$

for all $x_1, x_2 \in L_2$

This class includes the small gain and passivity functions. Furthermore, for any bounded linear transformations Π_1, Π_2, the function $\Sigma(w) = \langle \Pi_1 w, \Pi_2 w\rangle$ is quadratically continuous. In this paper we use the following results from Jönsson [9] to prove stability.

Definition 5: Let $\Pi_B : j\mathbb{R} \to \mathbb{C}^{n\times n}$ be a bounded and measurable function that takes Hermitian values and $\lambda \in \mathbb{R}$. We say that Δ satisfies the IQC defined by Π_B, λ, if there exists a positive constant γ such that for all $y, \dot{y} \in L_2$ and $v = \Delta y \in L_2$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix}^* \Pi_B(j\omega) \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix} d\omega + 2\langle v, \lambda\dot{y}\rangle \geq -\gamma|y(0)|^2$$

Theorem 6: Assume that

1) G is a linear causal bounded operator with $s\hat{G}(s), \hat{G}(s) \in \mathcal{A}$
2) For all $\kappa \in [0,1]$, the interconnection of $\kappa\Delta$ and G is well-posed
3) For all $\kappa \in [0,1]$, $\kappa\Delta$ satisfies the IQC defined by Π_B, λ
4) There exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}^* \left(\Pi_B(j\omega) + \begin{bmatrix}0 & \lambda j\omega^*\\ \lambda j\omega & 0\end{bmatrix}\right) \begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix} \leq -\eta I$$

Then the interconnection of G and Δ is L_2 stable.

III. Results

In this section we represent the single source/single link case of the proposed congestion control algorithm as the interconnection of a linear system with delay and a non-linear system without delay. This approach was motivated by Wang[4] and Jönsson[12]. We then show that using the IQC defined by Π_B, $\lambda = \frac{2}{\pi}$, where

$$\Pi_B = \begin{bmatrix}0 & \beta\\ \beta & -\frac{4}{\pi} - 2\end{bmatrix} \tag{5}$$

and $\beta = \alpha/(\alpha_{\max}\tau)$ we can establish L_2 stability of the interconnection for any $\tau \geq 0$, $0 < \alpha < \pi/2\alpha_{\max}$. We also show that L_2 stability of the interconnection implies asymptotic stability of the original delay-differential equation.

A. Preliminary Results

If we consider the problem of a single link and a single source, then from the development in Section II-B we have that $y(t) = x(t - \tau^f)$ and $q(t) = p(t - \tau^b)$. The dynamics can now be summarized as

$$\dot{p}(t) = \begin{cases}\frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau}p(t-\tau)} - 1 & p(t) > 0\\ \max\{0, \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau}p(t-\tau)} - 1\} & p(t) \leq 0\end{cases} \tag{6}$$

$$x(t) = x_{\max} e^{-\frac{\alpha}{\tau}p(t-\tau^b)} \tag{7}$$

Since the dynamics of Equation (6) are decoupled from those of (7) and stability of x follows from that of p, we need only consider stability of Equation (6). Now consider the equilibrium point of Equation (6), $p_0 = \frac{\tau}{\alpha}\ln\frac{x_{\max}}{c}$. As is customary, we change to variable z, where $z(t) = p(t) - p_0$ so that the origin is an equilibrium point. Now we have

$$\dot{z}(t) = \begin{cases} e^{-\frac{\alpha}{\tau}z(t-\tau)} - 1 & z(t) > -p_0\\ \max\{0, e^{-\frac{\alpha}{\tau}z(t-\tau)} - 1\} & z(t) \leq -p_0\end{cases} \tag{8}$$

Implicit in these dynamics is the constraint $z(t) \geq -p_0$. If we assume that any initial condition will satisfy this constraint, we can include the constraint in the dynamics without altering the solutions. For convenience, we define the bounded continuous functions

$$f_1(y) = \min\{e^{\frac{\alpha}{\tau}y} - 1, e^{\frac{\alpha}{\tau}p_0} - 1\}$$
$$f_2(y) = \max\{0, f_1(y)\}$$
$$f_c(x,y) = \begin{cases} f_1(y) & \text{if } x > -p_0\\ f_2(y) & \text{otherwise}\end{cases}$$

where the domains are constrained such that $y \geq -p_0$. These functions are illustrated in Figure 2. We now have

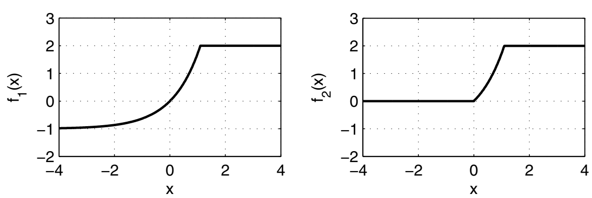

Fig. 2. f_1 and f_2

$$\dot{z}(t) = f_c(z(t), z(t-\tau)) \tag{9}$$

Assume an initial condition which is absolutely continuous. From boundedness and upper semi-continuity of the associated differential inclusion, we can establish via Fillipov [13][p77] the existence of an absolutely continuous solution over a time interval of length τ. This solution can be extended indefinitely by intervals of τ seconds by viewing the previous solution as an initial condition. Uniqueness and continuity of solutions follows from boundedness of the derivatives of f_1 and f_2 and Fillipov[p106]. Thus we have existence and uniqueness of solutions.

1) Separation into subsystems: Equation (9) is a delay-differential equation defined by a nonlinear, discontinuous function. To aid in the analysis, we will construct an interconnection of systems whose L_2 stability implies asymptotic stability of the original differential equation. Define the map G by $w = Gu$ if

$$w(t) = \int_{t-\tau}^{t} u(\theta)d\theta \qquad (10)$$

Define the map Δ by $v = \Delta y$ if for some $\tilde{z}$ such that $\tilde{z}(0) = 0$, we have

$$v(t) = \dot{\tilde{z}}(t), \quad \dot{\tilde{z}}(t) = f_c\big(\tilde{z}(t), y(t) - \tilde{z}(t)\big)$$

If $y, \dot{y} \in L_2$, then y is absolutely continuous (See p. 25 in Jönsson [9]). Thus we have existence and uniqueness as above. If we form the interconnection of G and Δ as defined above with inputs f, g where $g = 0$, then $\tilde{z}(0) = 0$ and

$$\dot{\tilde{z}}(t) = f_c(z(t), f(t) - \tilde{z}(t-\tau))$$

Now let z be the solution of the original delay differential equation (9) with initial condition $z^{\text{initial}} \in \mathcal{C}_\tau$. It can be shown that there exists some choice of $f, \dot{f} \in L_2$ such that for some $t_f > 0$, $\tilde{z}(t + t_f) = z(t)$ for all $t > t_f$. Therefore, if for any inputs $g, f, \dot{f} \in L_2$, we have that $\lim_{t\to\infty} \tilde{z}(t) = 0$ where $\tilde{z}$ is the solution to the equations defined by the interconnection of G and Δ, then Equation (9) is asymptotically stable. For the remainder of this paper, we drop the notation $\tilde{z}$ and simply use z. We note that well-posedness of the interconnection of G and $\kappa\Delta$ for all α, τ, κ follows from the same theorems cited previously for existence and uniqueness of solutions.

B. Δ *satisfies the IQC*

In this section we show that if $\alpha > 0$, then Δ and consequently $\kappa\Delta$ are bounded and satisfy the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0, 1]$. That is, for $\gamma = 4\beta/\pi > 0$, we prove the following for all $y, \dot{y} \in L_2$, $v = \Delta y$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix}^* \begin{bmatrix}0 & \beta\\ \beta & -\frac{4}{\pi} - 2\end{bmatrix} \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \geq -\gamma|y(0)|^2$$

By Parseval's formula, this is equivalent to

$$\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle \geq -\frac{\gamma}{2}|y(0)|^2$$

First notice that f_1, f_2 are sector bounded, i.e. $0 \leq f_i(x)x \leq \beta x^2$ where $\beta = \frac{e^{\frac{\alpha}{\tau}p_0} - 1}{p_0}$, denoted $f_i \in \text{sector}[0, \beta]$. Also notice that

$$f_c(x, y) = \begin{cases} f_1(y) & \text{if } x \geq -p_0 \text{ or } y \geq 0\\ 0 & \text{otherwise}\end{cases}$$

Lemma 7: If $v = \Delta y$ with $y, \dot{y} \in L_2$, then

1) $v \in L_2$ with norm bound $\beta\|y\|$,
2) $\langle v, \beta y - v\rangle \geq 0$

Proof: As a consequence of the above sector bounds, we have

$$f_c(x, y)^2 \leq \beta y f_c(x, y).$$

Let z be the solution of Equation (11), then this implies

$$\dot{z}(t)^2 \leq \beta(y(t) - z(t))\dot{z}(t)$$

Now for any $T \geq 0$, we have

$$\begin{aligned}\|P_T v\|^2 &= \int_0^T v(t)^2 dt = \int_0^T \dot{z}(t)^2 dt\\ &\leq \beta\int_0^T \dot{z}(t)(y(t) - z(t))dt\\ &= \beta\int_0^T \dot{z}(t)y(t)dt - \frac{\beta}{2}(z(T)^2 - z(0)^2)\\ &\leq \beta\langle P_T\dot{z}, y\rangle \qquad (11)\\ &\leq \beta\|P_T\dot{z}\|\|y\| = \beta\|P_T v\|\|y\|\end{aligned}$$

Therefore, $\|P_T\dot{z}\| \leq \beta\|y\|$ for all $T \geq 0$. Thus $v \in L_2$ with norm bounded by $\beta\|y\|$. Statement 2 follows from the line 11 by letting $T \to \infty$. ■

Lemma 8: If $v = \Delta y$ with $y, \dot{y} \in L_2$ and z satisfies Equation (11), then $\lim_{t\to\infty} z(t) = 0$.

Proof: Suppose $T_2 > T_1 > 0$, and let $H = P_{T_2} - P_{T_1}$. Then

$$\begin{aligned}\|Hv\|_2^2 &= \int_{T_1}^{T_2} \dot{z}(t)^2 dt\\ &\leq \beta\int_{T_1}^{T_2} \dot{z}(t)y(t)\,dt - \beta\int_{T_1}^{T_2} \dot{z}(t)z(t)\,dt\\ &= \beta\langle Hv, Hy\rangle - \frac{\beta}{2}(z(T_2)^2 - z(T_1)^2)\\ &\leq \beta\|Hv\|_2\|Hy\|_2 - \frac{\beta}{2}\big(z(T_2)^2 - z(T_1)^2\big)\end{aligned}$$

Hence

$$\begin{aligned}z(T_2)^2 - z(T_1)^2 &\leq 2\|Hv\|_2\|Hy\|_2 - \frac{2}{\beta}\|Hv\|_2^2\\ &\leq 2\|Hv\|_2\|Hy\|_2\end{aligned}$$

By Lemma 7, $v \in L_2$. Since $\|v\|$ and $\|y\|$ exist, we can use the Cauchy criterion to establish that for any $\delta > 0$, there exists a T_δ such that $T_2 > T_1 > T_\delta$ implies $(z(T_2)^2 - z(T_1)^2) < \delta$. It can be shown this implies that for any infinite increasing sequence $\{T_i\}$, $\{z(T_i)^2\}$ is a Cauchy sequence and therefore $z(t)^2$ converges to a limit. Since z is continuous, this implies that $z(t)$ also converges to a limit, z_∞. Since $y, \dot{y} \in L_2$, we have $\lim_{t\to\infty} y(t) = 0$. Thus $\lim_{t\to\infty} \dot{z}(t) = \lim_{t\to\infty} f_c(z(t), y(t) - z(t)) = f_c(z_\infty, -z_\infty)$ since f_c is discontinuous only for $z(t) = -p_0, y(t) \leq z(t)$. Since $\dot{z} \in L_2$, if $\dot{z}$ has a limit, it must be 0 and since $f_c(z_\infty, -z_\infty) = 0$ implies $z_\infty = 0$, we have $z_\infty = 0$. ■

Lemma 9: If $v = \Delta y$ with $y, \dot{y} \in L_2$, then $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$.

Proof: Define the variable $r(t) = y(t) - z(t)$ and the set $M = \{t : z(t) > -p_0 \text{ or } r(t) \geq 0\}$, then

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \langle \dot{z}, \dot{y} - \dot{z}\rangle = \int_0^\infty \dot{z}(t)\dot{r}(t)dt \\ &= \int_M f_1(r(t))\dot{r}(t)dt \leq \beta\|y\|\|\dot{y}\| + \beta^2\|v\|^2\end{aligned}$$

Since $y, \dot{y} \in L_2$, we have that y is absolutely continuous and thus r is absolutely continuous. Since r, z are continuous functions and since by Lemma 8, we have $z(t) \to 0$, we can partition the set M into the countable union of sequential disjoint intervals $\bigcup_i I_i \bigcup I_f$ where $I_i = [T_{a,i}, T_{b,i})$ with $\{T_{a,i}\}, \{T_{b,i}\} \subset \mathbb{R}^+$ and $I_f = [T_{a,f}, \infty)$. To see that the intervals are closed on the left, suppose I_i were open on the left. Then, since $T_{a,i} \notin M$, $z(T_{a,i}) = -p_0$ and $r(T_{a,i}) < 0$. However, since r is continuous, $r(T_{a,i} + \eta) < 0$ for η sufficiently small. Since $r(t) < 0$ implies $\dot{z}(t) \leq 0$, we have that $z(T_{a,i} + \eta) < 0$ and thus $T_{a,i} + \eta \notin M$ for η sufficiently small, which is a contradiction. Thus all the intervals are closed on the left. Similarly, one can show that all the intervals are open on the right. Now, consider time $T_a > 0$, where $T_a \in M$ defines the start of one of the intervals described above. If $z(T_a) > -p_0$, then since z is continuous, $z(T_a - \eta) > -p_0$ for all η sufficiently small. Therefore $T_a - \eta \in M$ for all η sufficiently small. This contradicts the statement that the intervals are disjoint. We thus conclude $z(T_a) = -p_0$ and consequently $r(T_a) \geq 0$ by definition of M. Now suppose $r(T_a) > 0$. Since r is continuous, $r(T_a - \epsilon) > 0$ and consequently $T_a - \epsilon \in M$ for all ϵ sufficiently small, which contradicts the statement that the intervals are disjoint. Therefore we conclude $r(T_a) = 0$ if $T_a \neq 0$. Then

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \sum_i \int_{I_i} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^\infty f_1(r(t))\dot{r}(t)dt \\ &= \sum_i \int_{T_{a,i}}^{T_{b,i}} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^\infty f_1(r(t))\dot{r}(t)dt\end{aligned}$$

We will assume that $T_{a,1} = 0$. If $T_{a,1} \neq 0$, we have $r(T_{a,1}) = 0$ and the proof becomes simpler. Since $f_1(r)$ is continuous in r and $r(t)$ is absolutely continuous in time, by the substitution rule we have

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \sum_i \int_{r(T_{a,i})}^{r(T_{b,i})} f_1(r)dr \\ &= \int_{r(0)}^{r(T_{b,1})} f_1(r)dr + \sum_{i\neq 1}\int_0^{r(T_{b,i})} f_1(r)dr \\ &= \int_{r(0)}^0 f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr\end{aligned}$$

Since $f_1 \in \text{sector}[0, \beta]$, $\int_0^{r(T_{b,i})} f_1(r)dr \geq 0$ for any $r(T_{b,i}) \in \mathbb{R}$. The summation converges since it is bounded, increasing. Furthermore, since $r(0) = y(0) - z(0) = y(0)$ and $|\int_0^y f_1(r)dr| \leq f_1(y)y \leq \beta y^2$ for any y, we have

$$\langle v, \dot{y} - v\rangle = \int_{y(0)}^0 f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr \geq -\beta|y(0)|^2$$

■

To summarize, we have shown that Δ is bounded and for any $y, \dot{y} \in L_2$, $v = \Delta y$, we have that $\langle v, \beta y - v\rangle \geq 0$ and $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$. Therefore, we conclude that Δ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$, since

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^\infty \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix}^* \begin{bmatrix}0 & \beta \\ \beta & -\frac{4}{\pi} - 2\end{bmatrix}\begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \\ \geq -\frac{4\beta}{\pi}|y(0)|^2\end{aligned}$$

We conclude as a consequence that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$, since

$$\begin{aligned}&\frac{2}{\pi}\langle \kappa v, \dot{y} - \kappa v\rangle + \langle \kappa v, \beta y - \kappa v\rangle \\ &\geq \kappa\left(\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle\right) \\ &\geq -\kappa\frac{2\beta}{\pi}|y(0)|^2 \geq -\frac{2\beta}{\pi}|y(0)|^2\end{aligned}$$

C. Properties of G

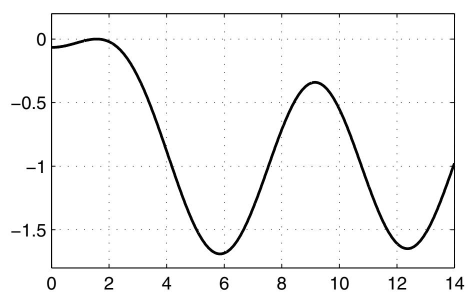

Fig. 3. $\frac{\pi}{2}\frac{\sin(\omega)}{\omega} - \frac{2}{\pi}\cos(\omega) - 1$

Recall that we define the map G as follows. $w = Gu$ if

$$w(t) = \int_{t-\tau}^t u(\theta)d\theta.$$

We first note that G is a linear bounded causal operator with $\hat{G}(j\omega) = \frac{1 - e^{-j\omega\tau}}{j\omega}$. Moreover, $\hat{G} \in \mathcal{A}$ since G can be represented by the convolution operation $w = g * u$ with $g(t) = \text{step}(t) - \text{step}(t - \tau) \in L_1$. Also, $s\hat{G}(s) \in \mathcal{A}$ using $g(t) = \delta(t) - \delta(t - \tau)$. Now, examine the term

$$\begin{aligned}&\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}^*\left(\Pi_B + \begin{bmatrix}0 & \lambda j\omega^* \\ \lambda j\omega & 0\end{bmatrix}\right)\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix} \\ &= \begin{bmatrix}\frac{1-e^{-j\omega\tau}}{j\omega}\\ 1\end{bmatrix}^* \begin{bmatrix}0 & \beta + \frac{2}{\pi}j\omega^* \\ \beta + \frac{2}{\pi}j\omega & -\frac{4}{\pi} - 2\end{bmatrix}\begin{bmatrix}\frac{1-e^{-j\omega\tau}}{j\omega}\\ 1\end{bmatrix} \\ &= 2\text{Real}\left(\beta\frac{1 - e^{-j\omega\tau}}{j\omega} - \frac{2}{\pi}e^{-j\omega\tau} - 1\right) \\ &= 2\left(\beta\tau\frac{\sin(\omega\tau)}{\omega\tau} - \frac{2}{\pi}\cos(\omega\tau) - 1\right)\end{aligned}$$

The plot of $\beta\tau\frac{\sin(\omega)}{\omega} - \frac{2}{\pi}\cos(\omega) - 1$ is given in Figure 3 for $\beta\tau = \pi/2$. If $0 \le \beta\tau < \frac{\pi}{2}$, then one can find some $\eta > 0$ such that $\beta\tau\frac{\sin(\omega\tau)}{\omega\tau} - \frac{2}{\pi}\cos(\omega\tau) - 1 \le -\eta$ for all $\omega \in \mathbb{R}$. Since $\beta\tau = \tau(e^{\frac{\alpha}{\tau}p_0} - 1)/p_0 = \alpha/\alpha_{\max}$, if $\alpha < \pi\alpha_{\max}/2$, we have that $\beta\tau < \pi/2$, and hence if $0 < \alpha < \pi\alpha_{\max}/2$, condition 4 of theorem 6 is satisfied.

D. Stability of the Interconnection

In this section, we summarize with the following theorem.

Theorem 10: Suppose $\alpha \in (0, \pi\alpha_{\max}/2)$. Then the delay-differential equation (6) describing the algorithm proposed by Paganini et al. [7] is asymptotically stable about the equilibrium.

Proof: We have shown that G is a linear causal bounded operator, that $G(s), sG(s) \in \mathcal{A}$, that the interconnection of G and $\kappa\Delta$ is well-posed for all $\kappa \in [0, 1]$, that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0, 1]$ and that for all $\alpha \in (0, \pi\alpha_{\max}/2)$, condition 4 of theorem 6 is satisfied. We can therefore use Theorem 6 to prove finite-gain stability of the interconnection for any $\alpha \in (0, \pi\alpha_{\max}/2)$. Since, as already mentioned, we can choose appropriate inputs $y, \dot{y} \in L_2$ to recreate any initial condition $x^{\text{initial}} \in \mathcal{C}_\tau$, and since for any such inputs, Lemma 8 implies $z(t) \to 0$, we have global asymptotic stability of the algorithm from [7] about the equilibrium for any $\alpha \in (0, \pi\alpha_{\max}/2)$. ■

IV. Implementation

To implement the proposed algorithm in the Internet framework, the window size is adjusted to deliver the required packet rate as given by equation (1). In implementation, the delay is unlikely to be known. In this case, a bound on the expected delay size can be used. Overestimation of the delay will result in an increased stability margin.

Modification of the link can take many forms. Price information from the link must be fed back to the source. Since queues themselves integrate excess rate, price of a congested resource can be computed directly using the queueing delay. However, this approach results in non-empty equilibrium queues and the possibility of unmodeled dynamics due to variable queueing delays. If a link instead uses a virtual capacity to avoid non-empty equilibrium queues, then explicit integration of incoming packets would be required and another mechanism must be used to feed back price information. An example of direct feedback of price information using packet marking is given by ECN. In one of the proposed implementations, packets are randomly marked at each link with probability $1 - \phi^{-p_j(t)}$ for some fixed $\phi > 1$. Thus, assuming no duplications, if ν is the percentage of marked packets received at the source, then the aggregate price can be measured as $q_i(t) = -\frac{\log(1-\nu)}{\log(\phi)}$. This variant is known as random exponential marking.

V. Conclusion

To conclude, for the case of a single source with a single link, we have proven global asymptotic stability of an implementation of the algorithm proposed by Paganini et al. [7]. We note that as a consequence of the use of input-output theory in the proof, we can also conclude asymptotic stability for any price disturbance f where $f, \dot{f} \in L_2$. Interest in stability with respect to disturbances is motivated by viewing internet traffic as dominated by large sustained source transmissions or 'elephants' and viewing small transient sources as noisy 'mice'.

We note that the same stability proof also holds for multiple sources with identical delays. The only change required in the proof is to replace $x_{\max}$ with $\sum_i x_{\max,i}$. We also note that the choice of implementation $U^{-1}(q) = x_{\max,i}e^{-\frac{\alpha}{\tau}q} - 1$ is not unique. If the function U satisfies the continuity, monotonicity and boundedness assumptions of the proofs, one can still conclude global asymptotic stability.

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