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Topology Independent Controller Design for Networked Systems

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Abstract

In this paper we present a method for synthesis of decentralized controllers for systems interconnected on a graph. We develop a synthesis procedure which will achieve a specified level of performance for any graph topology satisfying given degree bounds, independent of the size of the graph. We also give an analysis condition which gives an upper bound on the performance achieved for all such topologies. The methods reduce to computation via semidefinite programming, and the size of the resulting optimization problem does not grow with the size of the graph.

1 Introduction

Many systems of practical interest can be modelled as large collections of interacting subsystems. Examples of such systems include electrical power distribution networks [5], data networks [8], and collections of vehicles travelling in formation [4]. Several practical issues arise when attempting to design controllers for such systems. Implementation of classical control schemes typically requires that each subsystem has access to the states or outputs of each other subsystem. This is often impractical. Most practical control schemes for such systems are *decentralized*. That is, each subsystem uses only local information when making control decisions. Another issue is uncertainty in the subsystem interconnection topology. With electrical networks, failures of transmission lines or individual generators cause changes in the interconnection topology. With data networks, changes in the number of subsystems and the subsystem interconnection topology are common during normal operation. It is not always clear how a system's ability to reject external disturbances is affected by changes in the subsystem interconnection topology or system scale. It may be possible that a collection of interconnected subsystems remain stable as we introduce additional subsystems into the interconnection, but overall system output errors propagate in an undesirable way. This will be shown in an upcoming example.

The issues discussed above will be the focus of this paper. We concentrate on the design of decentralized controllers which are provably scalable. Such controllers guarantee a bound on the overall system H_∞ norm even if we introduce new subsystems or change the subsystem interconnection topology. This means that, for an acceptable upper bound, we do not have to redesign our control laws as the complexity of the overall system increases. Here we only consider systems formed by interconnecting identical subsystems, although the results could readily be extended to accommodate non-identical subsystems.

2 Previous Work

The analysis and control of collections of interconnected systems has been widely studied in the literature. Early work on stability analysis and decentralized control of large-scale interconnected systems is found in [6, 7, 10, 9, 12]. A common theme in many of these works are decompositions which allow a stability analysis for the interconnected system to be performed at a subsystem level.

In [1], analysis conditions and control synthesis procedures for guaranteeing scalable and topology-independent stability are given. When the analysis conditions are met, stability is guaranteed for a collection of interconnected systems for all possible interconnection topologies satisfying some local connectivity bounds.

Stability, however, is not always sufficient for desirable scalable performance of large scale systems. For example, in [4], the notion of *string stability* is introduced for an infinite linear array of subsystems. Roughly speaking, a system is string stable if disturbances diminish as they propagate through the system. Here we would like to obtain similar performance guarantees by finding an upper bound on the overall system norm of a collection of interconnected systems for all possible

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interconnection topologies satisfying local connectivity bounds.

3 Main Result

In this section we present a semidefinite program which produces an upper bound on the H_∞ norm of an interconnected system. This upper bound holds for all interconnection topologies among the subsystems satisfying a certain degree of sparsity in the interconnection structure. We will first present an analysis condition, then extend it to a decentralized control synthesis procedure with which a controller can be synthesized to minimize this upper bound.

3.1 Analysis of Identical Interconnected Subsystems

Here we will consider systems formed by interconnecting a collection of identical subsystems by a directed graph. The interconnection structure is specified by a simple directed graph $G = (V, E)$, with N vertices $V = \{1, \ldots, N\}$ and edge set $E \subset V \times V$. Here *simple* means the graph has no self-loops, that is $(i, i) \notin E$ for all i. We say vertices i and j are *adjacent* if $(i, j) \in E$ or $(j, i) \in E$, and define the *degree* of vertex i as the number of vertices j adjacent to it. In terms of the adjacency matrix, the degree of vertex i is

$$d_i(G) = \sum_{j=1}^{N} (1 - (1 - U_{ji})(1 - U_{ij})).$$

We define

$$d_{\max}(G) = \max_i d_i(G),$$

the maximum degree of any vertex of G.

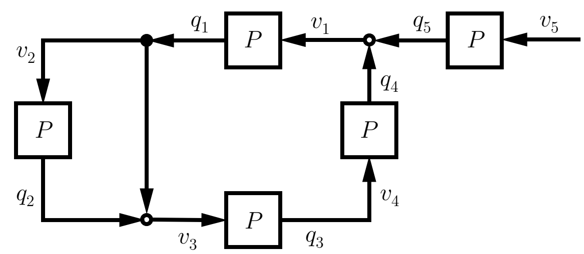
The subsystems are given in terms of state space realizations,

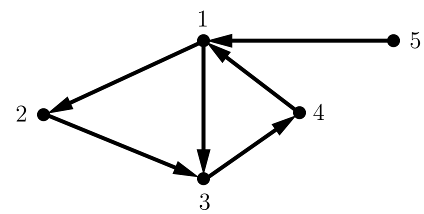
$$\begin{aligned} \dot{x}_i(t) &= A^{\mathrm{s}} x_i(t) + L^{\mathrm{s}} v_i(t) + B^{\mathrm{s}} w_i(t) \\ z_i(t) &= C^{\mathrm{s}} x_i(t) \\ q_i(t) &= F^{\mathrm{s}} x_i(t), \end{aligned} \tag{1}$$

each of which defines a linear map from signals (w_i, v_i) to (z_i, q_i). These systems are interconnected according to

$$v_i(t) = \sum_{j=1}^{N} U_{ij} q_j(t). \tag{2}$$

We interpret inputs w_i as disturbances affecting system i and outputs z_i as error signals to be kept small by a controller. When w_i is zero, we can view each system i as a map from signals v_i to signals q_i. Each system corresponds to vertex in the graph. We interpret edges as signals; all signals entering vertex i are summed to construct the input to system i. Similarly, all signals leaving a vertex are simply copies of the output of system i. This is illustrated in the figure below:





A consequence of interconnecting the systems via the graph G is that the dynamics of the subsystems becomes coupled. For a specific interconnection topology, the dynamics of the interconnected system can be expressed as

$$\begin{aligned} \dot{x}_i(t) &= A^{\mathrm{s}} x_i(t) + \sum_{j=1}^{N} U_{ij} L^{\mathrm{s}} F^{\mathrm{s}} x_j(t) + B^{\mathrm{s}} w_i(t) \\ z_i(t) &= C^{\mathrm{s}} x_i(t) \end{aligned}$$

for all $i = 1, \ldots, N$. We can write the dynamic equations above as

$$\begin{aligned} \dot{x}(t) &= \mathcal{A} x(t) + \mathcal{B} w(t) \\ z(t) &= \mathcal{C} x(t), \end{aligned}$$

where

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{bmatrix} \qquad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_N(t) \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{A} &= (I_N \otimes A^{\mathrm{s}}) + (U \otimes L^{\mathrm{s}} F^{\mathrm{s}}) \\ \mathcal{B} &= I_N \otimes B^{\mathrm{s}} \\ \mathcal{C} &= I_N \otimes C^{\mathrm{s}}. \end{aligned}$$

Here, I_N denotes the $N \times N$ identity matrix. The resulting feedback structure is a map

$$z = T(G) w$$

where the linear system $T(G)$ depends on the graph G. It is the map from the vector of all disturbances to the vector of all errors. The following result gives a bound on the performance of the system, in particular the induced-norm of the map from w to z, which holds for any graph G such that $d_{\max}(G) \leq d$.

Theorem 1. *Suppose there exists $\gamma \in \mathbb{R}$ and $X^s \succ 0$ such that*

$$\begin{bmatrix} A_1X + XA_1^T + BB^T & XC^T \\ CX & -\gamma I \end{bmatrix} \prec 0$$
$$\begin{bmatrix} A_2X + XA_2^T + BB^T & XC^T \\ CX & -\gamma I \end{bmatrix} \prec 0, \tag{3}$$

where

$$A_1 = \begin{bmatrix} A^s & dL^sF^s \\ (dL^sF^s)^T & A^s \end{bmatrix}, \quad A_2 = \begin{bmatrix} A^s & dL^sF^s \\ 0 & A^s \end{bmatrix},$$
$$B = \begin{bmatrix} B^s & 0 \\ 0 & B^s \end{bmatrix} \qquad C = \begin{bmatrix} C^s & 0 \\ 0 & C^s \end{bmatrix}$$

and the semidefinite programming variable X^s enters via

$$X = \begin{bmatrix} X^s & 0 \\ 0 & X^s \end{bmatrix}.$$

Then

$$\|T(G)\| < \sqrt{\gamma}$$

for all G such that $d_{max}(G) \leq d$.

This result gives a semidefinite program which can be used to verify that the interconnected system will be stable and satisfy the above performance bound independent of both the number of systems and the topology, provided the structural constraint on the degree is satisfied. Notice that in this SDP, the variable is the matrix $X^{\mathrm{s}} \in \mathbb{R}^{n\times n}$, and both the number of constraints and the number of variables is independent of the graph.

In order to prove this result, we make use of some additional technical facts. The Kalman-Yakubovich-Popov lemma implies that $\|T(G)\| < \sqrt{\gamma}$ if and only if there exists a matrix $\mathcal{X} \succ 0$ such that $\mathcal{AX} + \mathcal{XA}^T + \mathcal{BB}^T + \frac{1}{\gamma}\mathcal{XC}^T\mathcal{CX} \prec 0$. Since this condition depends on the graph G, an alternative sufficient condition may be obtained if we restrict ourselves to an $\mathcal{X}$ of the form $\mathcal{X} = I_N \otimes X^{\mathrm{s}}$. Any γ such that the inequality $\mathcal{AX} + \mathcal{XA}^T + \mathcal{BB}^T + \frac{1}{\gamma}\mathcal{XC}^T\mathcal{CX} \prec 0$ is feasible gives $\sqrt{\gamma}$ as an upper bound on the system H_∞ norm. This restriction will allow us to determine an upper bound on the norm of the interconnected system which holds for multiple topologies. This restriction will also render the decentralized control problem computationally tractable, as shown in the next section.

With the variable $\mathcal{X}$ restricted as such, the i, j block of the matrix $\mathcal{AX} + \mathcal{XA}^T + \mathcal{BB}^T + \frac{1}{\gamma}\mathcal{XC}^T\mathcal{CX}$ is

$$\begin{aligned} &A^{\mathrm{s}}X^{\mathrm{s}} + X^{\mathrm{s}}(A^{\mathrm{s}})^T + B^{\mathrm{s}}(B^{\mathrm{s}})^T \\ &\qquad + \tfrac{1}{\gamma}X^{\mathrm{s}}(C^{\mathrm{s}})^TC^{\mathrm{s}}X^{\mathrm{s}} \quad \text{for } i = j \\ &U_{ij}L^{\mathrm{s}}F^{\mathrm{s}}X^{\mathrm{s}} + U_{ji}X^{\mathrm{s}}(L^{\mathrm{s}}F^{\mathrm{s}})^T \quad \text{for } i \neq j. \end{aligned}$$

The proof of Theorem 1 will involve relating properties of these blocks to negative definiteness of the matrix $\mathcal{AX} + \mathcal{XA}^T + \mathcal{BB}^T + \frac{1}{\gamma}\mathcal{XC}^T\mathcal{CX}$. The following theorem appears in [1], and is used in the proof of Theorem 1.

Theorem 2. *Let H be a Hermitian matrix partitioned into blocks H_{ij}, where $i, j = 1, \dots, N$. Let m_i be the number of nonzero off-diagonal blocks in row i of H. Suppose, without loss of generality, that each row has at least one nonzero off-diagonal block. If*

$$\begin{bmatrix} \frac{1}{m_i}H_{ii} & H_{ij} \\ H_{ji} & \frac{1}{m_j}H_{jj} \end{bmatrix} \succ 0$$

for all $i, j = 1, \dots, N$, $i \neq j$, then $H \succ 0$.

Proof. Let U be the $N \times N$ matrix such that for $i \neq j$, $U_{ij} = 1$ if $H_{ij} \neq 0$. Otherwise, $U_{ij} = 0$. Note that $U_{ii} = 0$ for all i. For any vector x,

$$\begin{aligned} x^*Hx &= \sum_{i}^{n} x_i^*H_{ii}x_i + \sum_{i=1}^{n}\sum_{j>i}(x_i^*H_{ij}x_j + x_j^*H_{ji}x_i) \\ &= \sum_{i=1}^{n}\sum_{j>i}U_{ij}\left(\frac{1}{m_i}x_i^*H_{ii}x_i + \frac{1}{m_j}x_j^*H_{jj}x_j\right) \\ &\qquad + \sum_{i=1}^{n}\sum_{j>i}U_{ij}(x_i^*H_{ij}x_j + x_j^*H_{ji}x_i) \\ &= \sum_{i=1}^{n}\sum_{j>i}U_{ij}\begin{bmatrix} x_i \\ x_j \end{bmatrix}^* \begin{bmatrix} \frac{1}{m_i}H_{ii} & H_{ij} \\ H_{ji} & \frac{1}{m_j}H_{jj} \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} \end{aligned}$$

Clearly, if

$$\begin{bmatrix} \frac{1}{m_i}H_{ii} & H_{ij} \\ H_{ji} & \frac{1}{m_j}H_{jj} \end{bmatrix} \succ 0$$

for all $i, j = 1, \dots, N$, $i \neq j$, then $x^*Hx > 0$ for all nonzero x, or $H \succ 0$. ∎

We can now apply this result to prove Theorem 1.

Proof of Theorem 1. If the matrix inequalities (3) are satisfied, then

$$A_1X + XA_1^T + BB^T + \frac{1}{\gamma}XC^TCX \prec 0 \tag{4}$$

$$A_2X + XA_2^T + BB^T + \frac{1}{\gamma}XC^TCX \prec 0. \tag{5}$$

In terms of the individual blocks, the above inequalities imply

$$\begin{bmatrix} \frac{1}{d_1}\Phi & \Psi + \Psi^T \\ \Psi + \Psi^T & \frac{1}{d_2}\Phi \end{bmatrix} \prec 0 \quad \text{and} \quad \begin{bmatrix} \frac{1}{d_1}\Phi & \Psi \\ \Psi^T & \frac{1}{d_2}\Phi \end{bmatrix} \prec 0$$

for all $1 \leq d_1, d_2, \leq d$, where Φ and Ψ are given by

$$\begin{aligned} \Phi &= \big(A^{\mathrm{s}}X^{\mathrm{s}} + X^{\mathrm{s}}(A^{\mathrm{s}})^T \\ &\qquad + \frac{1}{\gamma}X^{\mathrm{s}}(C^{\mathrm{s}})^TC^{\mathrm{s}}X^{\mathrm{s}} + B^{\mathrm{s}}(B^{\mathrm{s}})^T\big) \\ \Psi &= L^{\mathrm{s}}F^{\mathrm{s}}X^{\mathrm{s}} \end{aligned}$$

Suppose $d_{\max}(G) \le d$, and let U be the adjacency matrix corresponding to the graph G. Then the above matrix inequalities imply

$$\begin{bmatrix} \frac{1}{d_i}\Phi & U_{ij}\Psi + U_{ji}\Psi^T \\ (U_{ij}\Psi + U_{ji}\Psi^T)^T & \frac{1}{d_j}\Phi \end{bmatrix} \prec 0 \qquad (6)$$

for all $i \neq j$. To show this, consider the four possible cases ($U_{ij} = U_{ji} = 0$), ($U_{ij} = U_{ji} = 1$), ($U_{ij} = 1, U_{ji} = 0$), and ($U_{ij} = 0, U_{ji} = 1$). The matrix inequality (4) clearly implies that (6) holds in the first two cases. The matrix inequality (5) clearly implies that (6) holds in the third case. Also (6) holds in the fourth case since

$$\begin{bmatrix} \frac{1}{d_2}\Phi & \Psi^T \\ \Psi & \frac{1}{d_1}\Phi \end{bmatrix} \prec 0$$

by permuting the blocks in (5). Since (6) holds for all $i \neq j$, this implies

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T + \frac{1}{\gamma}\mathcal{X}\mathcal{C}^T\mathcal{C}\mathcal{X} \prec 0,$$

or equivalently

$$\begin{bmatrix} \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T & \mathcal{X}\mathcal{C}^T \\ \mathcal{C}\mathcal{X} & -\gamma I \end{bmatrix} \prec 0.$$

Also, it is clear that $X \succ 0$ implies $\mathcal{X} \succ 0$. Therefore, the KYP lemma implies

$$\|T(G)\| < \sqrt{\gamma}$$

as desired. ■

Note that we only need the first inequality in Theorem 1 when considering interconnected systems where all of the links are bi-directional, *i.e.* the adjacency matrix U is symmetric. We only need the second inequality when considering interconnected systems where none of the links are bi-directional, *i.e.* $U_{ij} = 1$ if and only if $U_{ji} = 0$.

3.2 Decentralized Control Synthesis

At this point we extend the analysis condition presented in the previous section to a procedure for designing scalable decentralized controllers which guarantee a bound on the system norm. Each subsystem now has a control input u_i:

$$\begin{aligned} \dot{x}_i(t) &= A^{\mathrm{s}}x_i(t) + L^{\mathrm{s}}v_i(t) + H^{\mathrm{s}}u_i(t) + B^{\mathrm{s}}w_i(t) \\ z_i(t) &= C^{\mathrm{s}}x_i(t) + D^{\mathrm{s}}u_i(t) \\ q_i(t) &= F^{\mathrm{s}}x_i(t). \end{aligned}$$

The desired control law determines control inputs for each subsystem using only measurements of the local subsystem state. Due to the restricted form of the matrix $\mathcal{X}$ used in our analysis condition, we can readily extend this condition to a procedure for designing controllers with the desired structure.

The desired control law is a decentralized state feedback control law where each local controller is identical. In other words, we would like to choose a controller of the form $\mathcal{K} = I_N \otimes K^{\mathrm{s}}$ to minimize an upper bound on the H_∞ norm of the system

$$\begin{aligned} \dot{x}(t) &= (\mathcal{A} + \mathcal{H}\mathcal{K})x(t) + \mathcal{B}w(t) \\ z(t) &= (\mathcal{C} + \mathcal{D}\mathcal{K})x(t) \end{aligned}$$

where $\mathcal{H} = I_N \otimes H^{\mathrm{s}}$ and $\mathcal{D} = I_N \otimes D^{\mathrm{s}}$. We can denote the interconnected system under this control law by the map

$$z = T_{cl}(G)w$$

where, again, the linear system $T_{cl}(G)$ depends on the graph G.

When there are no constraints on the structure of $\mathcal{K}$, the semidefinite programming approach to state feedback synthesis involves introducing a variable $\mathcal{Z} = \mathcal{K}\mathcal{X}$ and finding $\mathcal{Z}$ and $\mathcal{X} \succ 0$ such that

$$\begin{bmatrix} \mathcal{A}\mathcal{X}+\mathcal{X}\mathcal{A}^T+\mathcal{H}\mathcal{Z}+\mathcal{Z}^T\mathcal{H}^T+\mathcal{B}\mathcal{B}^T & (\mathcal{C}\mathcal{X}+\mathcal{D}\mathcal{Z})^T \\ \mathcal{C}\mathcal{X}+\mathcal{D}\mathcal{Z} & -\gamma I \end{bmatrix} \prec 0$$

Upon finding such an $\mathcal{X}$ and $\mathcal{Z}$, we can construct a control law as $\mathcal{K} = \mathcal{Z}\mathcal{X}^{-1}$. Existence of a solution to this semidefinite program is equivalent to existence of a controller guaranteeing $\|T_{cl}(G)\| < \sqrt{\gamma}$. However, when the desired controller has special structure, there is no known equivalent semidefinite program. This is because the resulting constraints on $\mathcal{X}$ and $\mathcal{Z}$ are typically non-convex. However, recall that for our analysis condition we are restricting ourselves to an $\mathcal{X}$ of the form $\mathcal{X} = I_N \otimes X^{\mathrm{s}}$. When restricting $\mathcal{X}$ to this form, we can make a change of variables $\mathcal{Z} = \mathcal{K}\mathcal{X}$, where $\mathcal{K}$ is of the desired form if and only if $\mathcal{Z}$ is of the form $\mathcal{Z} = I_N \otimes Z^{\mathrm{s}}$. This provides a computationally tractable condition for synthesis of a decentralized controller. With the variables $\mathcal{X}$ and $\mathcal{Z}$ restricted as such, the i, j block of the matrix

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \mathcal{H}\mathcal{Z} + \mathcal{Z}^T\mathcal{H}^T + \mathcal{B}\mathcal{B}^T + \frac{1}{\gamma}(\mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{Z})^T(\mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{Z})$$

is

$$A^{\mathrm{s}}X^{\mathrm{s}} + X^{\mathrm{s}}(A^{\mathrm{s}})^T + H^{\mathrm{s}}Z^{\mathrm{s}} + (Z^{\mathrm{s}})^T(H^{\mathrm{s}})^T + B^{\mathrm{s}}(B^{\mathrm{s}})^T + \frac{1}{\gamma}(C^{\mathrm{s}}X^{\mathrm{s}} + D^{\mathrm{s}}Z^{\mathrm{s}})^T(C^{\mathrm{s}}X^{\mathrm{s}} + D^{\mathrm{s}}Z^{\mathrm{s}})$$

for $i = j$, and

$$U_{ij}(L^{\mathrm{s}}C^{\mathrm{s}})X^{\mathrm{s}} + U_{ji}X^{\mathrm{s}}(L^{\mathrm{s}}C^{\mathrm{s}})^T$$

for $i \neq j$. We can use this fact to obtain the following synthesis condition:

Theorem 3. *Suppose the there exist γ, Z, and $X^s \succ 0$ such that*

$$\begin{bmatrix} A_1X{+}XA_1^T{+}HZ{+}Z^TH^T{+}BB^T & XC^T{+}Z^TD^T \\ CX{+}DZ & -\gamma I \end{bmatrix} \prec 0$$
$$\begin{bmatrix} A_2X{+}XA_2^T{+}HZ{+}Z^TH^T{+}BB^T & XC^T{+}Z^TD^T \\ CX{+}DZ & -\gamma I \end{bmatrix} \prec 0$$

where

$$A_1 = \begin{bmatrix} A^s & dL^sF^s \\ dL^sF^s & A^s \end{bmatrix} \quad A_2 = \begin{bmatrix} A^s & dL^sF^s \\ 0 & A^s \end{bmatrix}$$
$$B = \begin{bmatrix} B^s & 0 \\ 0 & B^s \end{bmatrix} \quad H = \begin{bmatrix} H^s & 0 \\ 0 & H^s \end{bmatrix}$$
$$C = \begin{bmatrix} C^s & 0 \\ 0 & C^s \end{bmatrix} \quad D = \begin{bmatrix} D^s & 0 \\ 0 & D^s \end{bmatrix}$$

and the semidefinite programming variables X^s and Z^s enter via

$$X = \begin{bmatrix} X^s & 0 \\ 0 & X^s \end{bmatrix} \qquad Z = \begin{bmatrix} Z^s & 0 \\ 0 & Z^s \end{bmatrix}.$$

Then the decentralized control law $u_i(t) = K^s x_i(t) = Z^s(X^s)^{-1}x_i(t)$ will guarantee that

$$\|T_{cl}(G)\| < \sqrt{\gamma}$$

for all G such that $d_{max}(G) \le d$.

Note that we can minimize over γ to find the minimum such upper bound. Proof of this theorem is not given since it is nearly identical to the proof of Theorem 1.

4 Numerical Example

Here we present a numerical example to illustrate the methods of this paper. This example is not necessarily chosen to reflect the dynamics of any particular physical system, but is chosen because it clearly illustrates the key points of this paper. Suppose we have subsystems of the form

$$\begin{aligned} \dot{x}_i(t) &= A^s x_i(t) + L^s v_i(t) + H^s u_i(t) + B^s w_i(t) \\ z_i(t) &= C^s x_i(t) \\ q_i(t) &= F^s x_i(t), \end{aligned}$$

where

$$A^s = \begin{bmatrix} -0.82 & 0.03 & -0.03 \\ 0.07 & -0.88 & -0.02 \\ 0 & 0.07 & -1 \end{bmatrix},$$

$$L^s = \begin{bmatrix} -0.02 & 0.65 \\ 0.29 & -1.84 \\ 1.11 & 0.73 \end{bmatrix},$$

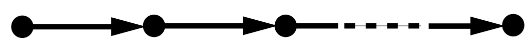
$$H^s = \begin{bmatrix} -2.45 & -0.59 \\ 0.47 & -0.65 \\ 0.12 & -1.08 \end{bmatrix},$$

$$B^s = \begin{bmatrix} 0.81 & 0.33 \\ 0.64 & -0.67 \\ 1.31 & -0.15 \end{bmatrix},$$

$$C^s = \begin{bmatrix} -0.05 & -0.33 & -0.04 \\ 0.38 & -0.5 & -0.17 \end{bmatrix},$$

$$F^s = \begin{bmatrix} -0.19 & 0.84 & -0.72 \\ 0.01 & -0.72 & -0.2 \end{bmatrix}.$$

It can be easily verified that this subsystem is stable. Suppose we form an interconnection of these subsystems by a graph of the form:



Specifically, such graphs are described by adjacency matrices satisfying

$$U_{ij} = \begin{cases} 1 & \text{for } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

The resulting interconnected system is stable since it consists of a loop-free interconnection of stable systems. However, we observe the following behavior as we vary the number of subsystems in the interconnection:

# of subsystems	H_∞norm
5	11.82
10	222.35
20	7.66×10^4
30	2.64×10^7

The system's ability to reject external disturbances rapidly diminishes as we introduce additional subsystems into the interconnection. As one would expect, the analysis conditions of this paper fail to produce a scalable upper bound on the overall system H_∞ norm. However, we can design a decentralized control law which, when applied to the subsystems, guarantees such an upper bound.

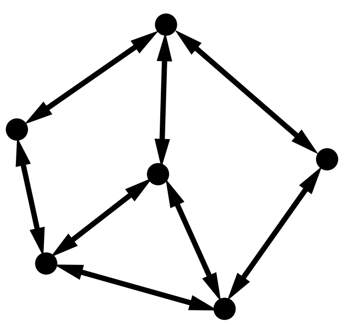
Suppose we restrict ourselves to the set of interconnection topologies such that each subsystem has degree less than or equal to 3 (with this degree bound chosen arbitrarily). We can then apply the synthesis methods of this paper to determine the decentralized state-feedback control law

$$K^{\mathrm{s}} = \begin{bmatrix} -0.0211 & -3.3218 & -2.1114 \\ 0.8245 & 14.2558 & 9.3180 \end{bmatrix}.$$

Under this control law, we can be guaranteed that $\|T_{cl}(G)\| < 1.812$ for all G such that $d_{\max}(G) \leq 3$. For the topologies considered earlier, the controlled system now has:

# of subsystems	H_∞norm
5	0.333
10	0.339
20	0.342
30	0.343

We can also guarantee upper bounds on the system norm for more complex topologies, such as the one shown below:



In this case, the interconnected system has H_∞ norm equal to 1.004.

5 Conclusions

In this paper we addressed the problem of designing scalable controllers for collections of interconnected subsystems. We derived a semidefinite program for the synthesis of such controllers. These controllers guarantee a fixed performance bound for the interconnected system which is independent of interconnection topology and system scale. This paper exclusively covered the case where all subsystems are identical. Although it is not discussed here, the methods of this paper may readily be extended to the case where subsystems of various types are interconnected.

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