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Nonlinear Phase-Locked Loop Design using Semidefinite Programming

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***Abstract*— The *Phase-Locked Loop*(PLL) is a key component of modern electronic communication and control systems. PLL is designed to extract signals from transmission channels. It plays an important role in systems where it is required to estimate the phase of a received signal, such as carrier tracking from *global positioning system* (GPS) satellites. In order to robustly provide centimeter-level accuracy, it is crucial for the PLL to estimate the instantaneous phase of an incoming signal which is usually buried in random noise or some type of interference. This paper presents an approach that utilizes the recent development in the semi-definite programming and sum-of-squares (SOS) field. A Lyapunov function will be searched as the certificate of the lock-in region of the PLL system. Moreover, the polynomial design technique will be used to further refine the controller parameters for system response away from the equilibrium point. Various simulation results will be provided to show the effectiveness of this approach.**

***Keywords:* Non-linear systems, optimization, computation and communications.**

I. Introduction

For nonlinear control systems, one would often like to know the region of attraction of an equilibrium point. Often, this region is difficult to both find and represent computationally. The usual mathematical tool used for analyzing of the region of attraction is Lyapunov's method. This gives us a sufficient condition for local stability, although it is often difficult to find a Lyapunov function that can be used as a certificate for the whole domain-of-attraction. Several prior approaches have used quadratic functions, for example [1], [2], [3]. In particular, the approach of [3] makes use of semidefinite programming to find a quadratic function whose sublevel-set is a good inner approximation to the region of attraction. For system in which the region is complicated, an ellipsoid may not provide a good approximation, and the above methods leave a large unexplored region within the domain-of-attraction.

With recent developments in algebra and sum-of-squares techniques, it is now possible to solve for a Lyapunov function with a more general polynomial form [4], [5]. Positive definiteness properties are replaced by sum-of-squares constraints which can be efficiently solved using convex optimization. The SOSTOOLS [6] toolbox for MATLAB has been developed as an easy computational tool to solve problems that utilizes the sum-of-squares techniques. This approach has also allowed finding a Lyapunov function within some specified semi-algebraic region [7], [8]. However, while this provides a method to certify a given inner approximation to the region of attraction, it does not immediately provide a way to find it. The level-set method [9] has been developed to find a semi-algebraic representation of the domain-of-attraction. With these polynomial techniques, it is possible to precisely estimate the domain of attraction of a nonlinear polynomial system and to find a suitable Lyapunov function as the stability certificate.

The PLL system is a nonlinear system with limited domain-of-attraction. Due to its importance in communication systems, analyzing and designing a PLL system has attracted many attentions in this field [10], [11], [12], [13], [14], [15], [16]. The current approach for designing a controller for a PLL system is still based on the linear model [16]. Hence, the performance of the resultant system cannot be guaranteed at system states far away from the designed equilibrium point.

In this paper, we utilize the current sum-of-squares techniques to analyze the domain-of-attraction of a PLL system. A local Lyapunov function can then be found as the certificate of the domain-of-attraction using this approach. The Lyapunov function will be further used to improve the stability region and performance of the PLL system. An example of a second order PLL system is used later in this paper to show the benefits of this design approach.

II. Preliminaries

The following are some definitions that will be used frequently in this paper. $\mathbb{R}[x]$ is used to represent the ring of polynomials in x with real coefficients. A polynomial $f \in \mathbb{R}[x]$ is called positive semidefinite (PSD) if $f(x) \geq 0$, for all $x \in \mathbb{R}^n$. A polynomial f is called *sum-of-squares* (SOS) if there exist polynomials $g_1, ..., g_s \in \mathbb{R}[x]$ such that $f = g_1^2 + g_2^2 + \cdots + g_s^2$. Clearly if f is sum-of-squares then f is PSD. It is also well-known that the converse is not true. Σ denotes the set of all sum-of-squares polynomials in $\mathbb{R}[x]$. $\mathbb{R}_+$ is used to represent the set of nonnegative real numbers.

Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is C^1. Define the 0-*sub-level set* of g to be $sub(g) \subset \mathbb{R}^n$ given by $sub(g) = \{ x \in \mathbb{R}^n \mid g(x) \leq 0 \}$. Further define the boundary of $sub(g)$ as $\partial\, sub(g)$.

One feature of the proposed advection algorithm is that the advection problem can be converted into a semidefinite program. The following is a standard form of a semidefinite

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program.

$$\begin{aligned} \min_{X} \quad & \operatorname{trace}(CX) \\ \text{s. t.} \quad & \operatorname{trace}(A_i X) = b_i \quad \text{for } i = 1, \ldots, m \\ & X \succeq 0, \end{aligned}$$

where $X \in \mathbb{R}^{n \times n}$ is symmetric. $X \succeq 0$ means that $z^T X z$ is positive semidefinite for all $z \in \mathbb{R}^n$.

The condition of one semi-algebraic set containing another semi-algebraic set is one of the key constraints used in this paper. The following lemma shows that this kind of relationship can be converted to constraints on the coefficients of the polynomials. The proof can be found in [7] or [4].

Lemma 1: Given $p,\ q \in \mathbb{R}[x]$, suppose there exist $s_0, s_1 \in \Sigma$ such that

$$s_0 - s_1 q + p = 0 \qquad \text{for all } x \in \mathbb{R}^n. \tag{1}$$

Then $sub(q) \subset sub(p)$. Further, given q and the degree bound of p, s_0, and s_1, the set of coefficients of p, s_0 and s_1 satisfying (1) is the feasible set of a semidefinite program.

The following result is similar. Given $q \in \mathbb{R}[x]$, if there exists $s_0,\ s_1 \in \Sigma$ and $\epsilon > 0$ such that

$$s_0 + s_1 q - p + \epsilon = 0,$$

then $sub(p) \subset sub(q)$.

Usually q is a given polynomial and p is the solution to find such that $sub(p)$ and $sub(q)$ approximately represent the same set with some other constraints on p, such as having lower degree or passing through several pre-specified points. The above results are used to construct such constraints.

III. Acquiring the Local Lyapunov Function

Finding a local Lyapunov function is coupled with finding the domain-of-attraction. Without a clear knowledge of the actual shape of the domain-of-attraction, it is hard to find a Lyapunov function that can be used to represent the entire domain-of-attraction [4], [8]. To deal with this difficulty, we utilize the current development in set advection [9].

A. Set advection

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. In this paper, we will consider the following autonomous system

$$\dot{x}(t) = f(x). \tag{2}$$

Given a locally Lipschitz autonomous system (2) and an open subset $U \in \mathbb{R}^n$, the basic local existence and uniqueness theorem [17] states that there exist $c \in \mathbb{R}_+$ such that the autonomous system (2) has a unique solution for any $z \in U$ in the compact time interval $[-c, c]$.

We define the *flow map* $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ to be the local unique solution of

$$\begin{aligned} \frac{\partial \phi_t(z)}{\partial t} &= f(\phi_t(z)) \quad \text{for } t \in [-c, c],\ c(z) \in \mathbb{R}_+,\ z \in \mathbb{R}^n \\ \phi_0(z) &= z. \end{aligned}$$

For any $t \in \mathbb{R}$ such that $\phi_t(x)$ exists, the map $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, invertible and has a continuous inverse; that is it is a topological homeomorphism on $\mathbb{R}^n$ [18].

Given $t \in \mathbb{R}$, we define the time t advection operator $A_t : C(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ by

$$q = A_t p \quad \text{if} \qquad q(x) = p\left(\phi_{-t}(x)\right) \ \text{ for all } x \in \mathbb{R}^n,$$

where $C(X, Y)$ is the set of functions mapping from X to Y. The map A_t is also called the Liouville operator associated with f. A very important property is that it is linear. Figure 1 shows the concept of the advection operator. Given polynomial p, A_t maps the coefficients of p to another polynomial q such that $sub(q) = \phi_t\, sub(p)$. We relate the advection operator to the advection of sets in the following remark.

Remark 1: Suppose $g_1,\ g_2$ are functions mapping $\mathbb{R}^n$ to $\mathbb{R}$. If $g_2 = A_t g_1$ then $sub(g_2) = \phi_t\left(sub(g_1)\right)$.

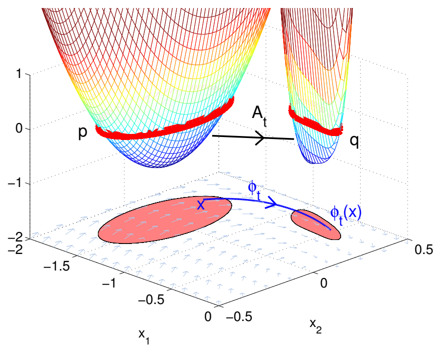

Fig. 1. The advection operator A_t.

B. Time-stepping

Since we are performing advection, we must use an approximation to the flow map ϕ_h with time step h. Many such approximations are possible, and are provided by the theory of numerical integration. The first-order Taylor approximation to $q = A_h p$ is the map $B_h : C(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ given by

$$q = B_h p \qquad \text{if } q(x) = p(x) - hDp(x)f(x),$$

where the derivative $Dp(x)$ is a linear map $Dp(x) : \mathbb{R}^n \to \mathbb{R}^n$ at each point x.

Based on the required accuracy of the advection, we could also choose to use higher order Taylor approximation. However, depending on the system dynamics, this usually will lead to the requirement of using higher degree polynomials in the sum-of-squares constraints. The relationship between the accuracy and the degree of polynomials will be further investigated in future work.

C. Domain-of-attraction estimation

The set advection concept is used to estimate the domain-of-attraction of a system. We use the following definition of the domain-of-attraction in this paper.

Definition 1: Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is analytic with the flow map, ϕ, and the origin is asymptotically stable. Define the *domain-of-attraction* (also called the basin/region of attraction) of f to be $R \subset \mathbb{R}^n$ such that for any $x \in R$, $\phi_t(x)$ is defined for all $t \geq 0$ and $\lim_{t\to\infty} \phi_t(x) = 0$.

The following properties can be easily derived. The detailed proofs can be found in [9].

Lemma 2: Suppose f is analytic and the origin is asymptotically stable and $R \neq \emptyset$. Suppose also $S_1 \subset R$ and $0 \in S_1$, and S_1 is a connected closed positively invariant set. Let $h > 0$ be a positive constant, and define the *backward advection* of S_1 to be S_2, given by

$$S_2 = \phi_{-h} S_1.$$

Then $S_1 \subset S_2 \subset R$, and S_2 is also connected, closed and positively invariant. Furthermore, $\partial S_2 = \phi_{-h} \partial S_1$.

Theorem 1: Suppose f is analytic and the origin is asymptotically stable and $h > 0$. Also suppose $0 \in S_0$ and $S_0 \subset R$ is a closed connected positively invariant set, such that there exists $\epsilon > 0$ such that $\mathcal{B}_\epsilon \subset S_0$.

Define the sequence of sets $S_0, S_1, S_2, \ldots$ by

$$S_{k+1} = \phi_{-h} S_k \qquad \text{for } k = 0, 1, 2, \ldots$$

Then this sequence converges to R in the following sense:

(i) $S_k \subset R$ for all $k \in \mathbb{N}$.
(ii) $S_k \subset S_{k+1}$ for all $k \in \mathbb{N}$.
(iii) For every $x \in R$, there exists n such that $x \in S_n$

D. Star-shaped constraint

For the case of estimating the domain-of-attraction, we introduce the concept of star-shaped sets. The star-shaped sets have many important properties and can be easily implemented as a semidefinite program. We now start with the first property. The detailed information about the star-shaped set can be found in [9].

Definition 2: A set $S \in \mathbb{R}^n$ is called *star-shaped* if for all $x \in S$,

$$\lambda x \in S \qquad \text{for all } \lambda \in [0, 1].$$

The set S is called *strictly star-shaped* if for all $x \in S$,

$$\lambda x \in int(S) \qquad \text{for all } \lambda \in [0, 1).$$

Note that a star shaped set S is connected. We now give a simple sufficient condition that ensures that a sub-level set is star-shaped. We make the following definition.

Definition 3: Suppose $g : \mathbb{R}^n \to \mathbb{R}$. We call g *strictly star-shaped* if g is C^1 and further satisfies $g(0) < 0$ and

$$Dg(x)x > 0 \qquad \text{for all } x \neq 0.$$

The following lemma shows the connection between strictly star-shaped functions and star-shaped sets.

Lemma 3: Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is strictly star-shaped. Then $sub(g)$ is star-shaped.

For the purposes of this paper, we would like to construct a convex set of functions whose sub-level sets are connected. Although the convex set of all convex functions on $\mathbb{R}^n$ will suffice, using it would unnecessarily restrict the class of sets describable to be convex. One cannot simply use the set of all functions whose 0-sub-level set is connected, since this set of functions is not convex. We therefore choose the set of *strictly star-shaped* functions, which is a convex set. We will use strictly star-shaped polynomials to represent sets. This is significantly more general than existing approaches using quadratic functions [2], [1], [3]. Also, it has been shown in Lemma 3 that if g is strictly star-shaped, then $sub(g)$ is strictly star-shaped. By using this property, we can easily pose the star-shaped constraints on g to make $sub(g)$ a connected set.

E. An algorithm for backward advection

Here we will state the result of the backward advection algorithm. The interested readers can find more detailed derivations in [9].

Given a strictly star-shaped polynomial g_{i-1} such that $sub(g_{i-1}) \subset R$, and $sub(g_{i-1})$ is bounded and positively invariant, we compute a polynomial g_i such that $sub(A_h g_i) \approx sub(g_{i-1})$ as follows.

Pick $\alpha > 0$ and positive integer d. Solve, using semidefinite programming, the following feasibility problem for $g_i \in \mathbb{R}[x]$, s_1, s_2, s_3, $s_4 \in \Sigma$.

$$\begin{aligned} g_i(0) &= -1 \\ Dg_i(x)x &> 0 \\ s_3 - s_4 g_{i-1} + B_{(h-\alpha)} g_i &= 0 \\ s_1 + s_2 g_{i-1} - B_h g_i &= 0 \\ \deg(g_i) &\leq d. \end{aligned}$$

Here we introduced an important parameter, α, which we think of as follows. The above algorithm finds a degree d polynomial g_i such that g_i is strictly star shaped, $\phi_h\, sub(g_i) \subset sub(g_{i-1})$, and $\phi_{h-\alpha}\, sub(g_i) \supset sub(g_{i-1})$. Hence the parameter α may be thought of as a tolerance that allows for the constraint that g_i is required to have degree d or less. Then from the result of theorem 1, $\lim_{i\to\infty} sub(g_i)$ converges to the domain-of-attraction. It should be noted that this technique only works in the case that the advected set is positively/negatively invariant.

F. The local Lyapunov Function

We find a local Lyapunov function in order to construct an initial star-shaped positively invariant set. The following result is standard.

Proposition 1: Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is analytic and the origin is a stable equilibrium point. Also suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, $\gamma > 0$, and the set

$$D = \{\, x \in \mathbb{R}^n \mid V(x) \leq \gamma \,\}$$

is compact. Further suppose

$$\begin{aligned} V(x) &> 0 \qquad \text{for all } x \neq 0 \\ V(0) &= 0 \\ DV(x)f(x) &< 0 \qquad \text{for all } x \neq 0, x \in D. \end{aligned}$$

Let $g_0(x) = V(x) - \gamma$. Then $sub(g_0)$ is positively invariant, and $sub(g_0) \subset R$.

One simple approach to finding an initial sub-level set is to find a quadratic Lyapunov function for the linear model of the system, and use a small sub-level set of this quadratic polynomial as the initial set.

An alternative method which often gives a much larger initial set is as follows. Choose a polynomial $p \in \mathbb{R}[x]$ such that $sub(p) \subset R$. We then solve the following convex feasibility problem. Find $V \in \mathbb{R}[x]$ and $s_0, s_1 \in \Sigma$ such that

$$\begin{aligned} DV(x)x &> 0 && \text{for all } x \neq 0 \\ V(x) &> 0 && \text{for all } x \neq 0 \\ V(0) &= 0 && \\ DV(x)f(x) + s_0 - s_1 p &= 0 && \text{for all } x \neq 0. \end{aligned}$$

Similar methods for finding local Lyapunov functions along with details on the construction of the associated semidefinite program may be found in [8], [5]. Here we have added the first constraint to ensure that $V - \gamma$ is strictly star-shaped for $\gamma > 0$. Note that these constraints imply that all sub-level sets of V are compact. Given V, we then solve the convex program

$$\begin{aligned} &\text{maximize} && \gamma \\ &\text{subject to} && V - \gamma - s_0 - s_1 p - \epsilon = 0 \qquad \text{for all } x \\ & && s_0, s_1 \in \Sigma \end{aligned}$$

where $\epsilon > 0$ is small. The optimal γ satisfies $sub(V - \gamma) \subset sub(p)$. Then V and γ satisfy the assumptions of Proposition 1 and so we may use $g_0 = V - \gamma$ as the function defining our initial level-set.

IV. Configuration of a phase-locked loop

Figure 2 shows the basic configuration of a PLL. It has three components; a phase detector, a loop filter, and a *voltage controlled oscillator*(VCO). The VCO generates an output signal whose phase, $\theta_0(t)$, depends on the phase, $\theta_i(t)$, of the input signal. The PLL is phase locked when the phase error $\phi(t) = \theta_i(t) - \theta_0(t)$ is a constant value and the loop is in stable equilibrium state. Usually, it is desired that the phase error, $\phi(t)$, is maintained at zero.

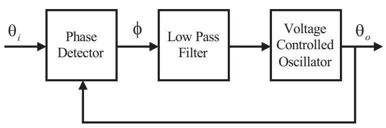

Fig. 2. Basic Configuration of a PLL

Of interest is the behavior of the phase error $\phi(t)$. Because of its sinusoidal nonlinearity in the PLL, the phenomenon of chaos is believed to exist [10], [11] and its inherent chaotic behavior for broadening the lock-in range of PLL has also been realized [12], [13]. A nonlinear controller can drive PLL from chaotic state into periodic state or vice versa [14]. For higher-order PLL, it is not possible to determine whether the loop will or will not slip cycles using the initial frequency alone. In this case, one might define the lock-in range as the separatrix ordinate at $\phi = 0$ [16]. Analyzing the domain-of-attraction of the PLL system provides a better description of the region in which a PLL locks up without slipping. The Lyapunov method has been used for stability analysis in control systems. Here the advection algorithm will be used to find the guaranteed stability boundary of the PLL system and the associated local Lyapunov function is then used to further refine the controller parameters. In [15], a Lyapunov styled analysis for PLL system up to third order is presented. The method shown in this section provides a way to analyze the domain-of-attraction for a more general system. Also, the form of the Lyapunov function used here is much more flexible.

Figure 3 shows the nonlinear model of the PLL. The sine function here represents the phase detector of the system. K in Figure 3 stands for the loop gain of the system. $F(s)$ is equivalent to the low pass filter shown in Figure 2 and it corresponds to the controller of the PLL. Finally, the integrator in Figure 3 is the voltage or numerically controlled oscillator. The key idea of a PLL system is to use the command, y_2, from $F(s)$ to steer the oscillator such that $\theta_0(t)$ tracks $\theta_i(t)$ as closely and quickly as possible.

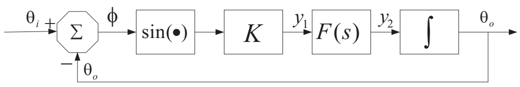

Fig. 3. Model of the Phase-Locked Loop

A. Second order Phase-Locked Loop

To use the nonlinear design approach, start with a reference design. The reference design used in this paper is the linear model of a PLL system. A Proportional-Integrator (PI) controller is chosen to be the filter, $F(s)$, as

$$F(s) = \frac{1 + \tau_2 s}{\tau_1 s}. \tag{3}$$

Using the model shown in Figure 3, it is routine to check that the resulting dynamic equation of the system is

$$\frac{d^2\phi}{dt^2} + K\frac{\tau_2}{\tau_1}\cos(\phi)\frac{d\phi}{dt} + \frac{1}{\tau_1}K\sin(\phi) = \frac{d^2\theta_i}{dt^2}. \tag{4}$$

Assume that the received signal frequency is varying linearly with time and has zero radial acceleration and let $x_1 = \phi$, $x_2 = \dot{\phi}$. The PLL system can be rewritten as the following state space model

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -K\frac{\tau_2}{\tau_1}\cos(x_1)x_2 - \frac{1}{\tau_1}K\sin(x_1) \\ &= k_1\cos(x_1)x_2 + k_2\sin(x_1). \end{aligned} \tag{5}$$

Equation (5) is the final nonlinear model of the second order PLL. A linearized model can then be derived as

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= k_1 x_2 + k_2 x_1. \end{aligned}$$

The filter F can then be designed using existing linear design approach [16]. One typical choice is to let $\omega_n = 15, \zeta = 0.707, K = 1$, where ω_n is the natural frequency, ζ is the damping ratio, and K is the overall gain. The two coefficients of the filter are then given as $\tau_1 = \frac{K}{\omega_n^2}, \tau_2 = \frac{2\zeta}{\omega_n}$

V. Phase-Locked Loop analysis and design

In this section, the second order PLL controller will be used to demonstrate the nonlinear design approach. The same approach can also be applied to the third order controller design.

A. Pull-in range of the traditional PLL system

Now the advection algorithm can be applied to the PLL nonlinear system. To reduce the required number of iterations, a local Lyapunov function is used as the initial set. After a few iterations of the algorithm, it gives us the estimated domain-of-attraction of the system. The result is shown in Figure 4. Note that the estimated region is based on the Taylor series expansion of the sine and cosine functions. In this example, two degree-10 polynomials are used to approximate sine and cosine functions and the estimated region is only valid between $-\pi$ and π. More terms of the Taylor series could also be used to improve the accuracy.

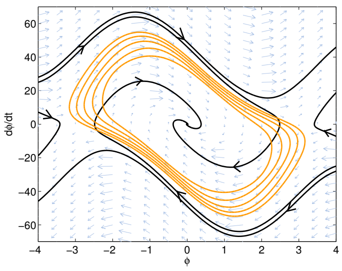

Fig. 4. Result of the original PLL system.

B. PLL system controller design

After getting a good estimated domain-of-attraction, a local Lyapunov function that describes the system behavior can be easily computed using the same sum-of-squares techniques as what we used to get the initial invariant set. This local Lyapunov function can also be used to describe the trajectories of the system in the domain-of-attraction. Figure 5 shows the result gained from the usual sum-of-squares approach.

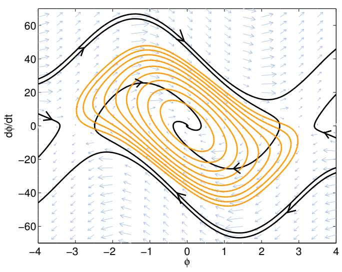

Fig. 5. Local Lyapunov level sets of the original PLL system.

The sum-of-squares techniques can be applied to the design of a controller. For the PLL system, it is desired to design a system which has a larger domain-of-attraction or faster converging speed. Here, the objective is to find possible system parameters such that the same Lyapunov function is still valid and the system converges faster. This can be done by solving a semidefinite program.

Suppose V is the local Lyapunov function for the PLL system. Using the sum-of-squares technique, solve the following optimization problem:

$$\begin{aligned} &\max\ \alpha \\ &\text{s.t.} -(DV)f = s_1 + s_2(a - V) \\ &\quad -(DV)f - \alpha q = s_3 + s_4(b - V) \\ &\quad p(k) \le 0 \end{aligned} \tag{6}$$

where $a,\ b \in \mathbb{R}_+$ specify the domain of the constraints and q is a positive definite performance polynomial specified by the user. $p(k)$ is a linear constraint of controller parameter k. As before, s_1, s_2, s_3, s_4 are sum-of-squares polynomials.

Since V is now a given function, the above constraints are linear in the controller parameters. The first constraint shows that V is a valid Lyapunov function in $sub(V - a)$. This constraint is used to specify the desired domain-of-attraction to maintain. The second constraint along with the objective function will put an upper bound on the derivative of the Lyapunov function in $sub(V - b)$. Faster decreasing rate implies faster converging speed. This specifies the performance requirement of our system. The user could also put different performance requirements in different sub-level sets of V.

Besides dynamic performance constraints, noise bandwidth constraints will be applied as well. The noise bandwidth for this PLL system with PI controller has the following form

$$B_L = \frac{\omega_n}{2}\left(\zeta + \frac{1}{4\zeta}\right).$$

Assume $\zeta \ge 1,\ \omega_n \ge 1$. Then

$$B_L = \frac{\zeta\omega_n}{2} + \frac{\omega_n}{8\zeta} \le -\frac{k_1}{4} - \frac{k_2}{8}.$$

This linear upper bound will be used to find a set of controller parameters that have better dynamic performance while maintaining the same noise bandwidth.

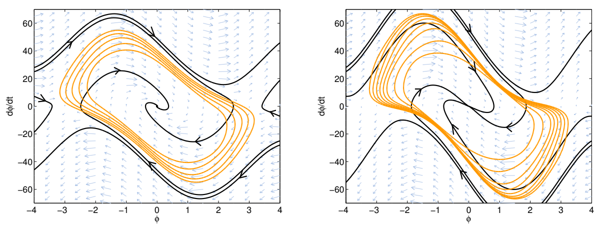

Fig. 6. Comparison of the domain-of-attraction. Left: linear design. Right: nonlinear design.

The system phase portrait as well as the estimated stable region are shown in Figure 6. The nonlinear design has $K = 1$, $\omega_n = 10.813$, and $\zeta = 1.3303$. The noise bandwidth is 8.1082 Hz, which is slightly higher than the noise bandwidth of the linear design, 7.9546 Hz. From the phase portrait, the nonlinear design has approximately 20% larger guaranteed domain-of-attraction. It can also be observed from the phase portrait that the nonlinear design has less overshoot than the linear design. This shows that the proposed method increases the system performance while not sacrificing too much of the noise rejection capability.

A Simulink model is used to compare the nonlinear designed PLL controller with the linear design. In this Simulink model, the sinusoidal input is collapsed by measurement noise and clock noise with zero mean and variances 0.1 and 0.0001, respectively. Figure 7 is the phase error of the two designs. It is clear that the nonlinear designed controller has a much faster convergence rate than the original design.

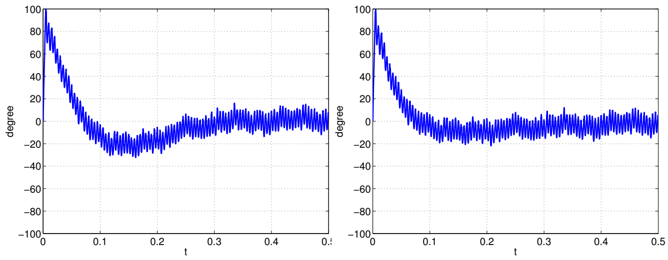

Fig. 7. Phase error of Simulink simulation. Left: linear design. Right: nonlinear design.

Both systems are tested on a NORDNAV R25 software GPS receiver. This receiver collects, down-converts and samples the GPS data by the front end, so that the collected GPS data can be post-processed repeatedly using different tracking-loop filter orders. The results are shown in Figure 8. It can be seen that the the original system has some overshoot and converges around 300ms. The nonlinear design has much less overshoot and converges about five times faster than the original design.

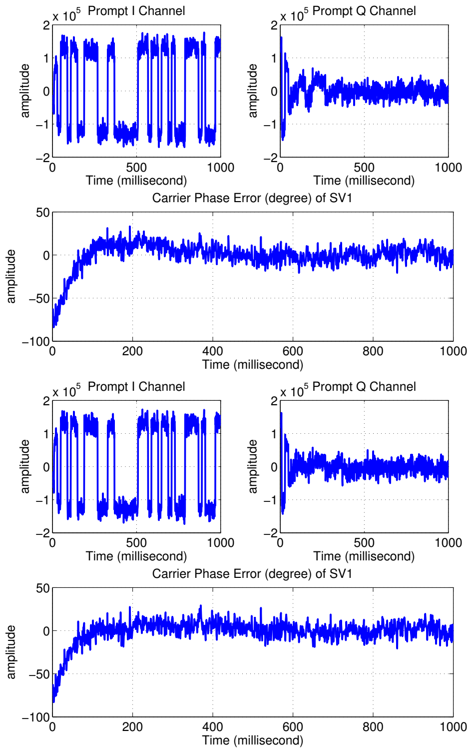

Fig. 8. Real GPS experimental results. Up: linear design. Down: nonlinear design.

VI. Conclusions

In this paper, we presented a method of designing a PI controller of a PLL system. This design approach is based on the polynomial nonlinear model of the PLL system. This approach starts with the linear design of the controller and then estimates the domain-of-attraction of the linear designed system to get the suitable local Lyapunov function for the system. The Lyapunov function is then used as the performance constraints to further refine the performance of the system outside the linear region. The domain-of-attraction of the initial design can also be extended to get a better lock-in region of the PLL system. This approach gives us a way to design a fixed form controller for a nonlinear system.

References

[1] E. J. Davison and E. M. Kurak, "A computational method for determining quadratic Lyapunov functions for non-linear systems," *Automatica*, vol. 7, pp. 627–636, 1971.

[2] A. Levin, "An analytical method of estimating the domain of attraction for polynomial differential equations," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2471–2475, 1994.

[3] G. Chesi, A. Tesi, and A. Vicino, "On optimal quadratic Lyapunov functions for polynomial systems," in *15th Int. Symp. on Mathematical Theory of Networks and Systems*, 2002.

[4] P. A. Parrilo and S. Lall, "Semidefinite programming relaxations and algebraic optimization in control," *European Journal of Control*, vol. 9, no. 2–3, pp. 307–321, Apr. 2003.

[5] A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition," in *Proceedings of the 41st IEEE Conference on Decision and Control*, 2002.

[6] S. Prajna, A. Papachristodoulou, and P. Parrilo, "Introducing sostools: A general purpose sum of squares programming solver," in *Proceedings of the 41st IEEE Conference on Decision and Control*, December 2002.

[7] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, 2000.

[8] Z. W. Jarvis-Wloszek, "Lyapunov based analysis and controllers synthesis for polynomial systems using sum-of-squares optimization," Ph.D. dissertation, University of California, Berkeley, 2003.

[9] T. Wang, S. Lall, and M. West, "Polynomial level-set methods for nonlinear dynamical systems analysis," in *Proceedings of the Allerton Conference on Communication, Control, and Computing*, 2005.

[10] T. Endo and L. O. Chua, "Chaos from phase-locked loops," *IEEE Trans. Circuits Syst.*, vol. 35, no. 8, pp. 987–1003, 1988.

[11] ——, "Chaos from phase-locked loops- part ii: non-hamiltonian case," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 255–633, 1988.

[12] ——, "Synchronization of chaos in phase-locked loops," *IEEE Trans. Circuits Syst.*, vol. 38, no. 12, pp. 1580–1588, 1991.

[13] E. Bradley and D. E. Straub, "Using chaos to broaden the capture range of a phase-locked loop," *IEEE Trans. Circuits Syst.*, vol. 40, no. 11, pp. 808–818, 1993.

[14] Z. F. Liu, K. F. Man, K. S. Tang, and G. Q. Zhong, "Controlling chaos in phase-locked loop," *Proceedings of the 24th Annual Conference of the IEEE*, vol. 3, pp. 1560–1563, 1998.

[15] D. Y. Abramovitch, "Analysis and design of a third order phase-lock loop," *Military Communications Conference of IEEE*, vol. 2, pp. 455–459, 1998.

[16] F. M. Gardner, *Phaselock Techniques*, 3rd ed. Wiley-Interscience, 2005.

[17] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 2002.

[18] H. Khalil, *Nonlinear systems*, 3rd ed. Prentice-Hall Inc., 2000.