Optimal Synthesis and Explicit State-Space Solution for a Decentralized Two-Player Linear-Quadratic Regulator

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Abstract

In this paper, we develop controller synthesis algorithms for decentralized control problems. The system considered here is an information structure, consisting of two interconnected linear subsystems. We construct the optimal controllers via a spectral factorization approach. Explicit state-space formulae are provided, and the order of the optimal controllers are established.

1 Introduction

We are interested in optimal controller synthesis for distributed control problems, consisting of multiple subsystems interacting over a network with limited communication. Many important practical problems fall into this category. Examples include formation flight, teams of vehicles, or large spatially distributed systems such as the internet or the power grid. Unfortunately, decentralized control problems, in general, are currently intractable [2]. Making matters worse, linear control policies may be strictly suboptimal, even when the underlying system dynamics are linear, time-invariant [18].

This paper focuses on a specific information structure, consisting of two interconnected systems with dynamics such that player 1's state affects the state of player 2. Our objective is to find a pair of controllers such that player 1 has access only to the first state, whereas player 2 can measure both states. The controller is chosen to minimize the $\mathcal{H}_2$ norm of the closed-loop transfer function.

This problem has been shown to have a linear optimal controller which may be found via convex optimization [17, 4, 12, 10]. Though convex, most existing formulations of this problem are infinite-dimensional, via a Youla parameterization, and require optimization over this parameter. Since the parameter itself is a linear stable system, a standard approximation would be via a finite basis for the impulse response function. This is in contrast to the centralized case, for which explicit state-space formulae can be constructed.

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This paper follows from our previous work in [16], though our work here is substantially more technical. In that paper, explicit formulae for the optimal controllers were constructed for the finite-horizon, time-varying version of this problem. This paper solves the infinite-horizon $\mathcal{H}_2$ version of the problem. We again show that both controllers separate naturally into a composition of controller and estimator, and each has the same number of states as player 2. Such formulae offer the practical advantages of computational reliability and simplicity, as well as provide understanding and interpretation of the controller structure. Also, it establishes the order of the optimal controller for this system, which is an open problem for general decentralized systems, even in the simplest cases.

Our approach makes use of spectral factorization. The methods used here extend naturally to more general networks, and the results in this paper are a first step towards general state-space solutions.

Previous Work. Since the general decentralized problem is currently intractable, most work has been aimed at classifying those systems that can be reformulated as convex problems [4, 7, 8, 1]. These results were unified and generalized under the concept of quadratic invariance [11]. For systems represented by graph structures and sparsity constraints, conditions for tractability of such systems was provided in [15]. Similar results were obtained in [14] using a poset-based framework.

Many different approaches have been taken to find numerical solutions to some of these problems. Some methods were suggested, though not implemented, in [17]. For the problem considered here, [13] provides a solution based on semidefinite programming. Other SDP approaches have been presented in [9, 19]. For the quadratic case, vectorization [10] provides a finite-dimensional approach, but loses the intrinsic structure and results in high-order controllers.

However, in none of these approaches have explicit state-space formulae been derived. In this paper, we take a spectral factorization approach, similar to [16], to construct explicit state-space formulae for the two-player problem. As a result, we can efficiently and analytically compute the optimal controllers for this distributed problem. Moreover, we gain insight into the form of the solution which previous approaches do not provide.

2 Problem Formulation

We use the following notation in this paper. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The complex unit disc is $\mathbb{D}$, and its boundary, the unit circle, is $\mathbb{T}$. The set $\mathcal{L}_2(\mathbb{T})$ is the Hilbert space of Lebesgue measurable functions on $\mathbb{T}$, which are square integrable, with inner product

$$\langle F, G\rangle = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{trace}(F^*(e^{j\theta})G(e^{j\theta}))\, d\theta$$

As is standard, $\mathcal{H}_2$ denotes the Hardy space

$$\mathcal{H}_2 = \Big\{ f : \{\infty\} \cup \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \mid \exists\, x \in \ell_2(\mathbb{Z}) \text{ s.t. } f(z) = \sum_{k=0}^{\infty} x_k z^{-k} \Big\}$$

of functions analytic outside the closed unit disc, and at infinity, with square-summable power series. The set $\mathcal{H}_2^\perp$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$. The prefix $\mathcal{R}$ indicates the subsets of proper real rational functions. That is, $\mathcal{RL}_2$ is the set of transfer functions with no poles on $\mathbb{T}$, $\mathcal{RH}_2$ is the set of transfer functions with no poles outside $\mathbb{T}$.

Also, we denote the subspace $\mathcal{L}_\infty(\mathbb{T})$ as the set of Lebesgue measurable functions which are bounded on $\mathbb{T}$. Similarly, $\mathcal{H}_\infty$ is the subspace of $\mathcal{L}_\infty$ with functions analytic outside of $\mathbb{T}$, and $\mathcal{H}_\infty^-$ is the subspace of $\mathcal{L}_\infty$ with functions analytic inside $\mathbb{T}$. Consequently, $\mathcal{RH}_\infty$ is the set of transfer functions with no poles outside of $\mathbb{T}$.

Some useful facts about these sets which we will make use of in this paper are [20]:

- if $G(z) \in \mathcal{L}_\infty$, then $G(z)\mathcal{L}_2 \subset \mathcal{L}_2$
- if $G(z) \in \mathcal{H}_\infty$, then $G(z)\mathcal{H}_2 \subset \mathcal{H}_2$
- if $G(z) \in \mathcal{H}_\infty^-$, then $G(z)\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$

For transfer functions $F \in \mathcal{RL}_2$, we use the notation

$$F(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D$$

We are interested in the following state-space system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

This corresponds to a two-player system, in which player 1's state can influence player 2's state. We are interested in finding controllers of the form

$$\begin{aligned} q_1(t+1) &= A_{K1} q_1(t) + B_{K1} x_1(t) \\ u_1(t) &= C_{K1} q_1(t) + D_{K1} x_1(t) \end{aligned}$$

and

$$\begin{aligned} q_2(t+1) &= A_{K2} q_2(t) + B_{K2} x(t) \\ u_2(t) &= C_{K2} q_2(t) + D_{K2} x(t) \end{aligned}$$

That is, player 1 makes decision u_1 based only on the history of his own state x_1, while player 2 makes decision u_2 based on the history of both states x_1 and x_2. This controller can be represented by the transfer functions $\mathcal{K}_{11}, \mathcal{K}_{21}, \mathcal{K}_{22} \in \mathcal{RL}_\infty$, such that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For a set T, we define lower(T) to be the set of 2×2 block lower triangular matrices with elements in T. In other words, $F \in \text{lower}(T)$ if and only if

$$F = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$$

and $F_{ij} \in T$ for all i, j. In particular, our desired controllers are in the set $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$.

Note that the space $\mathcal{S} = \text{lower}(\mathcal{RH}_2) \subset \mathcal{L}_2$ has an orthogonal complement, such that $G \in \mathcal{S}^\perp$ if and only if $G_{11}, G_{21}, G_{22} \in \mathcal{H}_2^\perp$ and $G_{12} \in \mathcal{L}_2$. We will also define $\mathrm{P}_{\mathcal{H}_2} : \mathcal{L}_2 \to \mathcal{H}_2$ as the orthogonal projection onto $\mathcal{H}_2$. Similarly, $\mathrm{P}_{\mathcal{S}} : \mathcal{L}_2 \to \mathcal{S}$ is the orthogonal projection onto $\mathcal{S}$.

Our cost is the vector

$$z(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where, for simplicity, we will assume that $C^T D = 0$ and $D^T D > 0$. Notice that this formulation allows for coupling of the states in the cost. Consequently, our plant can be expressed as the matrix $P \in \mathcal{RL}_\infty$, where

$$\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

and

$$P = \begin{bmatrix} C \\ I \end{bmatrix} (zI - A)^{-1} \begin{bmatrix} H & B \end{bmatrix} + \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \tag{1}$$

where A and B are lower triangular, and H is block diagonal and invertible. Note that H being invertible simply implies that no component of the state is deterministic. This assumption merely simplifies our presentation, while not fundamentally affecting our results.

We define $\mathcal{F}(P, \mathcal{K})$ as the linear fractional transformation

$$\mathcal{F}(P, \mathcal{K}) = P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1} P_{21}$$

Our objective function is the $\mathcal{H}_2$ norm of the closed-loop transfer function from w to z. In other words, we have the following optimization problem.

$$\begin{aligned} &\text{minimize} && \|\mathcal{F}(P, \mathcal{K})\|_2 \\ &\text{subject to} && \mathcal{K} \text{ is stabilizing} \\ & && \mathcal{K} \in \text{lower}(\mathcal{RL}_\infty) \end{aligned} \tag{2}$$

3 Main Results

Having established our notation and problem formulation, we now present the optimal solution for (2). We will develop the proof for this result in the remaining sections.

Theorem 1. *For the system in* (1)*, suppose* $C^T D = 0$ *and* $D^T D > 0$*. Suppose* (A_{11}, B_{11}) *and* (A_{22}, B_{22}) *are stabilizable. Also, suppose there exist stabilizing solutions* X *and* Y *to the algebraic Riccati equations*

$$X = C^T C + A^T X A - A^T X B(D^T D + B^T X B)^{-1} B^T X A \quad (3)$$

$$Y = C_2^T C_2 + A_{22}^T Y A_{22} - A_{22}^T Y B_{22}(D_2^T D_2 + B_{22}^T Y B_{22})^{-1} B_{22}^T Y A_{22} \quad (4)$$

Define

$$K = (D^T D + B^T X B)^{-1} B^T X A \quad (5)$$

$$J = (D_2^T D_2 + B_{22}^T Y B_{22})^{-1} B_{22}^T Y A_{22} \quad (6)$$

and let

$$A^{\mathcal{K}} = A_{22} - B_{21} K_{12} - B_{22} K_{22}$$
$$B^{\mathcal{K}} = A_{21} - B_{21} K_{11} - B_{22} K_{21}$$

Then, there exists a unique optimal $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$ *for* (2) *given by:*

- *Controller* 1 *has realization*

$$q_1(t+1) = A^{\mathcal{K}} q_1(t) + B^{\mathcal{K}} x_1(t)$$
$$u_1(t) = -K_{12} q_1(t) - K_{11} x_1(t)$$

- *Controller* 2 *has realization*

$$q_2(t+1) = A^{\mathcal{K}} q_2(t) + B^{\mathcal{K}} x_1(t)$$
$$u_2(t) = \big(J - K_{22}\big) q_2(t) - K_{21} x_1(t) - J x_2(t)$$

Note that there may not always exist stabilizing solutions to the algebraic Riccati equations (3–4). To simplify our results and avoid confusing the presentation with additional technical assumptions, we will simply assume the existence of stabilizing solutions. For a thorough discussion on algebraic Riccati equations, see [20].

Having established the form of the optimal controller, a number of remarks are in order.

With the inclusion of q_1 and q_2, the optimal controller is not a static gain, despite the fact that we have state feedback in each subsystem and player 2 has complete state information. Contrast this result with the classical LQR controller in which the optimal centralized controller would be the static gain K. In fact, both controllers have dynamics, and each has the same number of states as system 2.

It can be shown that q_1 and q_2 in the optimal controllers are in fact the minimum-mean square error estimate of x_2 given the history of x_1. Letting $\mathrm{E}(\cdot)$ denote expectation, if we define $\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(t), \ldots, x_1(0)\big)$, the optimal control policy can be written as

$$u_1(t) = -K_{11} x_1(t) - K_{12} \eta(t)$$
$$u_2(t) = -K_{21} x_1(t) - K_{22} \eta(t) + J\big(\eta(t) - x_2(t)\big)$$

Thus, the optimal policy is, in fact, attempting to perform the optimal centralized policy, though using η instead of x_2. However, there is an additional term in u_2 which represents the error between x_2 and its estimate η. We also see that in the case where x_2 is deterministic, so that $\eta = x_2$, then the optimal distributed controller reduces to the optimal centralized solution, as it should.

4 Analysis

Before trying to find the optimal controllers, it is important to note when the system can be stabilized. The following lemma provides the necessary and sufficient conditions for the existence of any stabilizing controller.

Lemma 2. *There exists a controller* $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$ *which stabilizes* P *in* (1) *if and only if* (A_{11}, B_{11}) *is stabilizable and* (A_{22}, B_{22}) *is stabilizable.*

Proof. ($\Rightarrow$) If (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable, then there exist matrices F_1 and F_2 such that $A_{11} + B_{11} F_1$ and $A_{22} + B_{22} F_2$ are stable. Consequently, the controller

$$\mathcal{K} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$$

produces the closed loop system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} + B_{11} F_1 & 0 \\ A_{21} + B_{21} F_1 & A_{22} + B_{22} F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which is clearly stable.

($\Leftarrow$) Suppose that (A_{11}, B_{11}) is not stabilizable. Then, there exists a transformation U such that

$$U^{-1} A_{11} U = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \qquad U^{-1} B_{11} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

where a_{22} has at least one unstable eigenvalue λ, with corresponding eigenvector v. Then, it can be readily shown that with the initial condition $x_1(0) = U \begin{bmatrix} 0 \\ v \end{bmatrix}$, the state $x_1(t) \nrightarrow 0$ as $t \to \infty$, for any inputs u. A similar argument holds for the case where (A_{22}, B_{22}) is not stabilizable. ∎

When the system can be stabilized, by choosing stabilizing matrices F_1 and F_2, we can use the standard Youla parametrization to simplify our optimization problem.

Lemma 3. *Let $\mathcal{S} = \text{lower}(\mathcal{RH}_2)$. Suppose (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable, and let F_1 and F_2 be matrices, such that $A_{11}+B_{11}F_1$ and $A_{22}+B_{22}F_2$ have stable eigenvalues. Then, the set of all stabilizing controllers $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$ is parametrized by*

$$\mathcal{K} = Q(I+MQ)^{-1} + F \qquad Q \in \mathcal{S}$$

where $M = (zI - (A+BF))^{-1}B$ and $F = \text{diag}(F_1, F_2)$. Moreover, the set of stable closed-loop transfer functions satisfies

$$\begin{aligned}\{\mathcal{F}(P,\mathcal{K}) \mid \mathcal{K} \in \text{lower}(\mathcal{RL}_\infty),\ \mathcal{K}\ \textit{stabilizing}\} \\ = \{N_{11} + N_{12}QN_{21} \mid Q \in \mathcal{S}\}\end{aligned}$$

where $N_{12} = z^{-1}((C+DF)(zI-(A+BF))^{-1}B + D)$ and

$$\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} = \left[\begin{array}{c|c} A+BF & H \\ \hline C+DF & 0 \\ A+BF & H \end{array}\right]$$

Proof. This result follows from the standard Youla parametrization for the problem. See, for example [3]. ∎

Lemma 4. *For the system in (1), let N be defined as in Lemma 3. Suppose Q is optimal for*

$$\begin{array}{ll} \textit{minimize} & \|N_{11} + N_{12}Q\|_2 \\ \textit{subject to} & Q \in \mathcal{S} \end{array} \tag{7}$$

Then, there exists $\hat{Q} \in \mathcal{S}$, such that $Q = \hat{Q}N_{21}$, and $\hat{Q}$ is optimal for

$$\begin{array}{ll} \textit{minimize} & \|N_{11} + N_{12}\hat{Q}N_{21}\|_2 \\ \textit{subject to} & \hat{Q} \in \mathcal{S} \end{array} \tag{8}$$

Conversely, if $\hat{Q} \in \mathcal{S}$ is optimal for (8), then $Q = \hat{Q}N_{21}$ is optimal for (7).

Proof. This follows from the fact that $N_{21}, N_{21}^{-1} \in \mathcal{S}$, so that $Q \in \mathcal{S}$ if and only if $\hat{Q} \in \mathcal{S}$. ∎

In order to solve the optimization problem in (7), it is convenient to find an equivalent optimality condition, which the following lemma provides.

Lemma 5. *Let $\mathcal{S} = \text{lower}(\mathcal{RH}_2)$. Suppose $F, G \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ minimizes*

$$\begin{array}{ll} \textit{minimize} & \|F + GQ\|_2 \\ \textit{subject to} & Q \in \mathcal{S} \end{array}$$

if and only if

$$G^*F + G^*GQ \in \mathcal{S}^\perp \tag{9}$$

Proof. The proof follows from the classical projection theorem. Since the construction is standard, we omit the proof; for the general idea, see for example [5]. ∎

5 Spectral Factorization

Our goal is now to find a solution $Q \in \mathcal{S}$ which satisfies the optimality condition (9). To this end, we have the following result.

Lemma 6. *Let $\mathcal{S} = \text{lower}(\mathcal{RH}_2)$, and suppose $F, G \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ satisfies*

$$G^*F + G^*GQ \in \mathcal{S}^\perp$$

if and only if the following two conditions both hold:

i) $(G^*F)_{22} + (G^*G)_{22}Q_{22} \in \mathcal{H}_2^\perp$

ii) $\begin{bmatrix} (G^*F)_{11} \\ (G^*F)_{21} \end{bmatrix} + G^*G \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \in \mathcal{H}_2^\perp$

Proof. Let $G^*F + G^*GQ = \Lambda$ where $\Lambda \in \mathcal{S}^\perp$. Note that Λ is partitioned as

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

where $\Lambda_{11}, \Lambda_{21}, \Lambda_{22} \in \mathcal{H}_2^\perp$. Consequently, (i) comes from the fact that $\Lambda_{22} \in \mathcal{H}_2^\perp$, and (ii) since $\begin{bmatrix} \Lambda_{11} \\ \Lambda_{21} \end{bmatrix} \in \mathcal{H}_2^\perp$. ∎

The important aspect of Lemma 6 is that it decomposes our optimality condition (9) over $\mathcal{S}^\perp$ into two separate conditions over $\mathcal{H}_2^\perp$. Each of these conditions can be solved via a spectral factorization approach.

Lemma 7. *Suppose $R_1, R_2 \in \mathcal{RH}_\infty$ have the realizations*

$$\begin{aligned} R_1 &= C(zI - A)^{-1}H \\ R_2 &= z^{-1}(C(zI-A)^{-1}B + D) \end{aligned}$$

Suppose there exists a stabilizing solution X to the algebraic Riccati equation

$$\begin{aligned} X = C^TC + A^TXA - (A^TXB + C^TD) \\ \times (D^TD + B^TXB)^{-1}(B^TXA + D^TC) \end{aligned} \tag{10}$$

Let $W = D^TD + B^TXB$ and $K = W^{-1}(B^TXA + D^TC)$ and $L \in \mathcal{RH}_\infty$ satisfying

$$L = \left[\begin{array}{c|c} A & B \\ \hline W^{\frac{1}{2}}K & W^{\frac{1}{2}} \end{array}\right]$$

Then, $L^{-1} \in \mathcal{RH}_\infty$, $L^{-*} \in \mathcal{RH}_\infty^-$, and $L^*L = R_2^*R_2$. Moreover,

$$\begin{aligned} L^{-*}R_2^*R_1 = W^{-\frac{1}{2}}B^T(z^{-1}I - (A-BK)^T)^{-1}XH \\ + zW^{\frac{1}{2}}K(zI-A)^{-1}H \end{aligned}$$

Proof. This is a standard spectral factorization result. A simple proof follows the approach in [6]. ∎

Lemma 8. *Let $R_1, R_2 \in \mathcal{RH}_\infty$ be defined as in Lemma 7. Suppose there exists a stabilizing solution X to the algebraic Riccati equation* (10)*, and let K and L be defined as in Lemma 7. Then, the unique $Q \in \mathcal{RH}_\infty$ satisfying*

$$R_2^* R_1 + R_2^* R_2 Q \in \mathcal{H}_2^\perp$$

is given by

$$Q = -zK(zI - (A - BK))^{-1} H$$

Proof. From Lemma 7, we know that $R_2^* R_2 = L^* L$. Since $L^{-*} \in \mathcal{RH}_\infty^-$, then $L^{-*}\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$. Hence, the optimality condition is equivalent to

$$L^{-*} R_2^* R_1 + LQ \in \mathcal{H}_2^\perp$$

Since $LQ \in \mathcal{RH}_2$, we can project the optimality condition onto $\mathcal{H}_2$ to obtain

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*} R_2^* R_1) + LQ = 0$$

From Lemma 7, we have

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*} R_2^* R_1) = zW^{\frac{1}{2}} K(zI - A)^{-1} H$$

Consequently, we have

$$\begin{aligned} Q &= -L^{-1}\mathrm{P}_{\mathcal{H}_2}(L^{-*} R_2^* R_1) \\ &= -zK(zI - (A - BK))^{-1} H \end{aligned}$$

■

If we now want to apply this spectral factorization approach to our problem, our Riccati equations would be in terms of the pre-compensator F. However, this difficulty can be avoided with the following result.

Lemma 9. *Suppose $X \in \mathbb{R}^{n\times n}$, and $F \in \mathbb{R}^{m\times n}$. Then,*

$$\begin{aligned} X = C^T C + A^T X A &- (A^T X B + C^T D) \\ &\times (D^T D + B^T X B)^{-1}(B^T X A + D^T C) \end{aligned}$$

and $A - B(D^T D + B^T X B)^{-1}(B^T X A + D^T C)$ is stable, if and only if

$$\begin{aligned} X = C_F^T C_F + A_F^T X A_F &- (A_F^T X B + C_F^T D) \\ &\times (D^T D + B^T X B)^{-1}(B^T X A_F + D^T C_F) \end{aligned}$$

and $A_F - B(D^T D + B^T X B)^{-1}(B^T X A_F + D^T C_F)$ is stable, where $A_F = A + BF$ and $C_F = C + DF$.

Proof. By substitution of A_F and C_F, it can be readily shown that the two Riccati equations are equivalent. ■

In what follows, it will be convenient to define

$$E_1 = \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where the dimensions are defined by the context. We can now solve for the $Q \in \mathcal{S}$ satisfying our optimality condition (9)

Lemma 10. *For the system in* (1)*, suppose $C^T D = 0$ and $D^T D > 0$. Suppose (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable, and let F_1, F_2 be matrices such that $A_{11} + B_{11}F_1$ and $A_{22} + B_{22}F_2$ have stable eigenvalues. Suppose there exist stabilizing solutions X and Y to the algebraic Riccati equations (3–4), and let K, J be given by (5–6). Define*

$$A_F = A + BF, \quad A_K = A - BK, \quad A_J = A_{22} - B_{22}J$$

Finally, let N_{11} and N_{12} be defined as in Lemma 3. Then, the unique optimal $Q \in \mathcal{S}$ for (7) *is given by*

$$Q_{22} = \left[\begin{array}{c|c} A_J & A_J H_2 \\ \hline -J - F_2 & -(J + F_2)H_2 \end{array}\right] \tag{11}$$

$$\begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = \left[\begin{array}{c|c} A_K & A_K E_1 H_1 \\ \hline -K - F & -(K + F)E_1 H_1 \end{array}\right] \tag{12}$$

Proof. From Lemma 5, we know that the optimal $Q \in \mathcal{S}$ for (7) satisfies the optimality condition (9). Using Lemma 6, this can be solved as two separate problems. Condition (i) of the lemma can be solved via Lemma 8, where

$$\begin{aligned} R_1 &= (C_2 + D_2 F_2)(zI - (A_F)_{22})^{-1} H_2 \\ R_2 &= z^{-1}\big((C_2 + D_2 F_2)(zI - (A_F)_{22})^{-1} B_{22} + D_2\big) \end{aligned}$$

to obtain the optimal Q_{22} in (11). Note that (4) and Lemma 9 imply the existence of the required algebraic Riccati equation needed in Lemma 8.

A similar argument is used to solve for Q_{11} and Q_{21} in condition (ii) of Lemma 6, via Lemma 8 where we let $R_1 = N_{11}E_1$ and $R_2 = N_{12}$. ■

Having found the optimal $Q \in \mathcal{S}$, the following result provides the optimal controller for our decentralized problem.

Theorem 11. *For the system in* (1)*, suppose the conditions of Lemma 10 hold, with X, Y, K, J defined by the Riccati equations (3–6). Let $A_K = A - BK$. Then, the unique optimal $\mathcal{K} \in \mathrm{lower}(\mathcal{RL}_\infty)$ for* (2) *is*

$$\mathcal{K} = \begin{bmatrix} -K_{11} - K_{12}\Phi & 0 \\ -K_{21} - (K_{22} - J)\Phi & -J \end{bmatrix} \tag{13}$$

where

$$\Phi = (zI - (A_K)_{22})^{-1}(A_K)_{21}$$

Proof. From Lemma 10, the unique optimal $Q \in \mathcal{S}$ for (7) is given by (11) and (12). Lemma 4 then implies that $\hat{Q} = QN_{21}^{-1}$ is optimal for (8), where N_{21} is defined in Lemma 3. Using Lemma 3, the unique optimal $\mathcal{K}$ for (2) is given by $\mathcal{K} = \hat{Q}(I + M\hat{Q})^{-1} + F$, with M defined in the lemma, and the result follows. ■

Proof of Theorem 1. The result follows directly from Theorem 11, where we let

$$\begin{aligned} A^{\mathcal{K}} &= (A_K)_{22} = A_{22} - B_{21}K_{12} - B_{22}K_{22} \\ B^{\mathcal{K}} &= (A_K)_{21} = A_{21} - B_{21}K_{11} - B_{22}K_{21} \end{aligned}$$

■

Having established our main result, the last step in our analysis of this problem is to discuss the structure of this optimal controller. To this end, we let

$$\eta = \Phi x_1$$

This represents the following state-space system

$$\eta(t+1) = (A_K)_{22}\eta(t) + (A_K)_{21}x_1(t)$$

Combining this with the dynamics in (1), the closed-loop dynamics of the system become

$$\begin{bmatrix} x_1(t+1) \\ \eta(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} (A_K)_{11} & (A_K)_{12} & 0 \\ (A_K)_{21} & (A_K)_{22} & 0 \\ (A_K)_{21} & (A_K)_{22} - A_J & A_J \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} H_1 & 0 \\ 0 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \quad (14)$$

As a result, we obtain a very simple representation for the optimal controller.

Lemma 12. *Suppose x_1, x_2, η are the states of the autonomous system in* (14)*. Then,*

$$\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t)\big)$$

Proof. This result was proved in [16]. ∎

Thus, we see that the dynamics of the optimal controller, which have order equal to the dimension of player 2's state, are actually an estimation process for $\mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t)\big)$.

6 Conclusions

In this paper, we found the optimal controller for a decentralized two-player problem. This was accomplished via a spectral factorization technique. The optimal policy separated into controller and estimator in a non-trivial way, and the order of the optimal control policy was established as the dimension of player 2's state.

This work extended the results for the finite-horizon problem in [16] to the infinite-horizon case. These results are a first step in finding state-space solutions for more general decentralized control problems. Our future work will continue to extend our methodology to more general networks and output feedback problems.

References

[1] B. Bamieh and P. G. Voulgaris. Optimal distributed control with distributed delayed measurements. *Proceedings of the IFAC World Congress*, 2002.

[2] V. D. Blondel and J. N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36(9):1249–1274, 2000.

[3] G. E. Dullerud and F. Paganini. *A Course in Robust Control Theory: A Convex Approach*. Springer, 2000.

[4] Y-C. Ho and K. C. Chu. Team decision theory and information structures in optimal control problems – Part I. *IEEE Transactions on Automatic Control*, 17(1):15–22, 1972.

[5] D. Luenberger. *Optimization by Vector Space Methods*. John Wiley & Sons, Inc., 1969.

[6] D. H. Mee. Factorisation result for optimal discrete-time systems. *Electronics Letters*, 6(8):233–234, apr 1970.

[7] Jr. N. Sandell and M. Athans. Solution of some nonclassical LQG stochastic decision problems. *IEEE Transactions on Automatic Control*, 19(2):108–116, 1974.

[8] X. Qi, M. Salapaka, P. Voulgaris, and M. Khammash. Structured optimal and robust control with multiple criteria: A convex solution. *IEEE Transactions on Automatic Control*, 49(10):1623–1640, 2004.

[9] Anders Rantzer. Linear quadratic team theory revisited. In *Proceedings of American Control Conference*, pages 1637–1641, June 2006.

[10] M. Rotkowitz. Information structures preserved under nonlinear time-varying feedback. In *Proceedings of the American Control Conference*, pages 4207–4212, 2006.

[11] M. Rotkowitz, R. Cogill, and S. Lall. A simple condition for the convexity of optimal control over networks with delays. In *Proceedings of the IEEE Conference on Decision and Control*, pages 6686–6691, 2005.

[12] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2002.

[13] C.W. Scherer. Structured finite-dimensional controller design by convex optimization. *Linear Algebra and its Applications*, 351(352):639–669, 2002.

[14] P. Shah and P. Parrilo. A partial order approach to decentralized control. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4351–4356, 2008.

[15] J. Swigart and S. Lall. A graph-theoretic approach to distributed control over networks. In *Proceedings of the IEEE Conference on Decision and Control*, 2009.

[16] J. Swigart and S. Lall. An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator. In *Proceedings of the American Control Conference*, 2010.

[17] P. Vouglaris. Control of nested systems. In *Proceedings of the American Control Conference*, volume 6, pages 4442–4445, 2000.

[18] H. S. Witsenhausen. A counterexample in stochastic optimum control. *SIAM Journal of Control*, 6(1):131–147, 1968.

[19] D. Zelazo and M. Mesbahi. $\mathcal{H}_2$ analysis and synthesis of networked dynamic systems. In *Proceedings of the American Control Conference*, pages 2966–2971, 2009.

[20] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1995.