A Unifying Condition for Separable Two Player Optimal Control Problems

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Abstract

We derive a simple condition under which a class of two player linear quadratic optimal control problems can be solved by two separate problems. The condition unifies most of the previously known separable two player problems, and further enables us to explicitly solve a new class of problems of which the optimal control was previously unknown. Moreover, natural interpretation of the separated problems provides better understanding of the optimal controllers' dynamics.

1 Introduction

We consider decentralized control problems where a team of multiple players cooperate to achieve a common collective objective, with limited communication and dynamics propagation between each players. A variety of interesting control problems can be described in this framework, which include vehicle formation, salvo attack, structured control, and so on. Systematic and efficient synthesis of decentralized control policies in such architectures has been a fundamental and central issue for networked control problems.

One of the critical factors limiting these technological developments is that the model-based control synthesis procedures which have been so effective at centralized control do not currently have counterparts for decentralized control. Although good heuristics are known in some cases, and certain special cases have been solved exactly, for the general problem there is currently no method that can in general numerically compute, for example, the optimal mean-square performance achievable by decentralized control, even for the highly specialized scenario of low dimensional linear time-invariant state-space systems. It no longer fits within the existing paradigm for optimal centralized control problems such as semidefinite programming, Riccati equations, et cetera. This is the key obstacle to the overall problem, and a tractable algorithm for finding the optimal controller, even the optimal linear controller, does not yet exist [1].

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In this paper, we specifically focus on one of the simplest setup among such problems. We consider two players in the nested structures [10], which implies 1) the first player's state and decision affects the second player, and 2) the first player's measurement is available to the second player. We are interested in finding the structured $\mathcal{H}_2$ optimal control where the first player's decision is computed by its own information, while the second player's decision is made using the information from the both players.

According to the recent work in [5], the two player setup which we focus on is classified under the quadratically invariant category, and in fact the $\mathcal{H}_2$ problems can be solved via convex programming. Numerical solutions based on semidefinite programming or finite basis expansion have been suggested [2, 3, 4, 5, 6]. More recently, an explicit solution for the state feedback problems has been developed using spectral factorization [8], and similar results have been obtained via the poset framework [7]. The result in [8] was extended to partial output feedback cases recently [9].

We present a unified condition which characterizes all the solvable problems described above. Under the condition, the problems are naturally separated into sequential problems whose solutions can be combined to produce the optimal solution of the original problem. The separated problems provide clear interpretation of the optimal controller's dynamics. Furthermore, the problems are separated at the highest level, thus no complicated argument is needed in deriving the results. Finally, the condition enables us to explicitly solve another class of problems, *dynamically decoupled output feedback problems*, of which the optimal solution was not previously known.

2 Problem definition

In this paper, we consider the following two player dynamics. The system evolves with

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}}_{A}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \underbrace{\begin{bmatrix}B_{11} & 0\\ B_{21} & B_{22}\end{bmatrix}}_{B}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + \underbrace{\begin{bmatrix}H_{11} & 0\\ 0 & H_{22}\end{bmatrix}}_{H}\begin{bmatrix}w_1(t)\\ w_2(t)\end{bmatrix}$$

where x_i and u_i denote the state and the control of the i-th player. w_i denotes the independent exogenous Gaussian noise with unit intensity. For simplicity, we let $x_i(0) = 0$, and assume that A is stable. For unstable A, we can work out the prestabilized A with the stabilizability assumption. This would result in more complicated algebra and problem descriptions but does not fundamentally change the derivation and result.

The first player measures some linear function of x_1, while the second player measures another linear function of both x_1 and x_2. They may be corrupted by noise, so

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}}_{M} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix}}_{N} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

where $HN^T = 0$ and $N_{ii}N_{ii}^T > 0$ if $N_{ii} \neq 0$ are assumed.

Let us denote the set of all 2×2 block lower triangular linear operators by $\mathcal{S}$.

$$\mathcal{S} = \left\{ S \mid S \in \begin{bmatrix} \bullet & \circ \\ \bullet & \bullet \end{bmatrix} \right\}$$

The control synthesis problem is finding the optimal lower triangular control $\mathcal{K} \in \mathcal{RL}_\infty \cap \mathcal{S}$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}}_{\mathcal{K}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

minimizing the $\mathcal{H}_2$ norm of the following cost vector with $C^T D = 0$ and $D^T D > 0$.

$$z(t) = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} D_1 & D_2 \end{bmatrix}}_{D} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The problem description in the general framework is

$$\begin{aligned} P : \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} &= \begin{bmatrix} T & U \\ V & G \end{bmatrix} \\ &= \begin{bmatrix} C(sI - A)^{-1}H & C(sI - A)^{-1}B + D \\ M(sI - A)^{-1}H + N & M(sI - A)^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} T_1 & T_2 & U_1 & U_2 \\ V_{11} & 0 & G_{11} & 0 \\ V_{21} & V_{22} & G_{21} & G_{22} \end{bmatrix} \end{aligned}$$

Since $\mathcal{S}$ is quadratically invariant under P_{22}, the controller, $\mathcal{K} \in \mathcal{S}$, and its Youla parameter, $Q = \mathcal{K}(I - P_{22}\mathcal{K})^{-1}$, have the same sparsity pattern. Therefore the two player optimal $\mathcal{H}_2$ control problem can be described as below [5].

Problem 1. *(The two player problem)*

$$\underset{Q \in \mathcal{RH}_\infty \cap \mathcal{S}}{\textit{minimize}} \quad \|T + UQV\|_2^2$$

or in state space representation,

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & H & B \\ \hline C & 0 & D \\ M & N & 0 \end{array} \right] \begin{bmatrix} x \\ w \\ u \end{bmatrix}$$

The optimal controller $\mathcal{K}^* \in \mathcal{RL}_\infty \cap \mathcal{S}$ is given by

$$\mathcal{K}^* = Q^*(I + GQ^*)^{-1}$$

where Q^* is the optimal solution of Problem 1.

3 Separated problems

This section shows that, under some conditions, Problem 1 can be solved by two separated problems of which the optimal solutions are well known explicitly.

Let us call by Π the linear operator to be minimized.

$$\Pi(Q) = T + UQV$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathcal{RH}_\infty$

We can partition $\Pi(Q) = \begin{bmatrix} \Pi_1(Q) & \Pi_2(Q) \end{bmatrix}$ by,

$$\Pi_1(Q) = T_1 + \begin{bmatrix} U_1 & U_2 \end{bmatrix} Q \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}$$

$$\Pi_2(Q) = T_2 + \begin{bmatrix} U_1 & U_2 \end{bmatrix} Q \begin{bmatrix} 0 \\ V_{22} \end{bmatrix}$$

Since we are dealing with the $\mathcal{H}_2$ norm, we have that

$$\|\Pi(Q)\|_2^2 \;=\; \|\Pi_1(Q)\|_2^2 \;+\; \|\Pi_2(Q)\|_2^2$$

and Problem 1 is equivalent to

$$\underset{Q \in \mathcal{RH}_\infty \cap \mathcal{S}}{\text{minimize}} \quad \|\Pi_1(Q)\|_2^2 + \|\Pi_2(Q)\|_2^2$$

But we know that

$$\begin{aligned} &\min_{Q \in \mathcal{RH}_\infty \cap \mathcal{S}} \|\Pi_1(Q)\|_2^2 + \|\Pi_2(Q)\|_2^2 \\ &\quad \geq \min_{Q \in \mathcal{RH}_\infty \cap \mathcal{S}} \|\Pi_1(Q)\|_2^2 + \min_{Q \in \mathcal{RH}_\infty \cap \mathcal{S}} \|\Pi_2(Q)\|_2^2 \\ &\quad \geq \min_{Q \in \mathcal{RH}_\infty} \|\Pi_1(Q)\|_2^2 + \min_{Q \in \mathcal{RH}_\infty \cap \mathcal{S}} \|\Pi_2(Q)\|_2^2 \end{aligned}$$

Furthermore, $\Pi_2(Q)$ is a function of only Q_{22} if $Q \in \mathcal{S}$. So we simplify the notation by

$$\Pi_2(Q) = T_2 + U_2 Q_{22} V_{22} = \Pi_3(Q_{22})$$

and it follows that

$$\begin{aligned} &\min_{Q \in \mathcal{RH}_\infty \cap \mathcal{S}} \|T + UQV\|_2^2 \\ &\quad \geq \min_{Q \in \mathcal{RH}_\infty} \|\Pi_1(Q)\|_2^2 \;+\; \min_{Q_{22} \in \mathcal{RH}_\infty} \|\Pi_3(Q_{22})\|_2^2 \end{aligned}$$

This implies that the right hand side provides a lower bound for the optimal value of Problem 1. We claim that, under a technical condition, this lower bound is actually tight, and the solution of Problem 1 is explicitly given in terms of the solutions of the two problems in the right hand side.

To this end, let us define two minimization problems regarding $\Pi_1(Q)$ and $\Pi_3(Q_{22})$ as follows.

Problem 2. *(Π_1 problem)*

$$\underset{Q \in \mathcal{RH}_\infty}{\text{minimize}} \quad \left\| T_1 + \begin{bmatrix} U_1 & U_2 \end{bmatrix} Q \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \right\|_2^2$$

or in state space representation,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{q} \\ z_{\Pi_1} \\ y_1 \\ r \end{bmatrix} = \left[\begin{array}{cc|ccc} A_{11} & 0 & H_{11} & B_{11} & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} \\ \hline C_1 & C_2 & 0 & D_1 & D_2 \\ M_{11} & 0 & N_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ q \\ w_1 \\ u_1 \\ p \end{bmatrix}$$

Problem 3. *(Π_3 problem)*

$$\underset{Q_{22} \in \mathcal{RH}_\infty}{\text{minimize}} \quad \| T_2 + U_2 Q_{22} V_{22} \|_2^2$$

or in state space representation,

$$\begin{bmatrix} \dot{x}_2 - \dot{q} \\ z_{\Pi_3} \\ y_2 - r \end{bmatrix} = \left[\begin{array}{c|cc} A_{22} & H_{22} & B_{22} \\ \hline C_2 & 0 & D_2 \\ M_{22} & N_{22} & 0 \end{array} \right] \begin{bmatrix} x_2 - q \\ w_2 \\ u_2 - p \end{bmatrix}$$

A set of intermediate variables (p, q, r) were labeled in order to better understand these separated problems.

Note that the Π_3 problem is a typical LQG problem, therefore the optimal solution is explicitly given in terms of two Riccati equations. The Π_1 problem can be more subtle and a little technical complicacy may be involved, but without fundamental increase in computational complexity. We will discuss more on this in the next section.

The following lemma on a simple linear algebraic condition will be helpful in developing our proof.

Lemma 4. *Suppose $Z \in \mathcal{RH}_\infty$ and $R \in \mathcal{RH}_\infty$ given in appropriate sizes such that*

$$\pi(Z) = ZR = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

is defined.

Then there exists $Q \in \mathcal{RH}_\infty \cap \mathcal{S}$ such that

$$\pi(Q) = \pi(Z)$$

if there exists $\Omega \in \mathcal{RH}_\infty$ such that $R_2 = \Omega R_1$.

Proof. Pick any $Q_{22} \in \mathcal{RH}_\infty$ in the same size of Z_{22}, and consider $\begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} - Q_{22} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix} \in \mathcal{RH}_\infty$. Then $Q = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}$ satisfies $\pi(Q) = \pi(Z)$. ■

This lemma gives a sufficient condition under which the range of π can be represented by structured parametrization. Now we are ready to present the main lemma.

Lemma 5. *Suppose that there exist optimal solutions for Problem 2 and Problem 3. Also suppose that there exists $\Omega \in \mathcal{RH}_\infty$ such that*

$$V_{21} = \Omega V_{11}$$

Then the following equality holds.

$$\begin{aligned} &\min_{Q \in \mathcal{RH}_\infty \cap \mathcal{S}} \|T + UQV\|_2^2 \\ &\quad = \min_{Q \in \mathcal{RH}_\infty} \|\Pi_1(Q)\|_2^2 + \min_{Q_{22} \in \mathcal{RH}_\infty} \|\Pi_3(Q_{22})\|_2^2 \end{aligned}$$

Proof. Let $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ and W be optimal for Problem 2 and 3, respectively. Also, let $Q_{22}^* = W$.

Since Ω satisfies $V_{21} = \Omega V_{11}$, Lemma 4 follows that

$$\begin{bmatrix} Q_{11}^* \\ Q_{21}^* \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} - W \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix} \in \mathcal{RH}_\infty$$

satisfies $\Pi_1(Q^*) = \Pi_1(Z)$, where $Q^* = \begin{bmatrix} Q_{11}^* & 0 \\ Q_{21}^* & Q_{22}^* \end{bmatrix}$ achieves the lower bound of the left hand side. ■

Now the optimal solution of Problem 1 is explicitly given as follows.

Corollary 6. *Suppose that the assumptions in Lemma 5 hold. Let $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ and W be the optimal solutions of Problem 2 and Problem 3.*

Then the optimal solution Q^* of Problem 1 is given by

$$Q^* = \begin{bmatrix} Z_{11} + Z_{12}\Omega & 0 \\ Z_{21} + (Z_{22} - W)\,\Omega & W \end{bmatrix}$$

Proof. This follows from Lemma 5. ■

By applying this, we present the explicit solutions to several classes of the two player control problems.

4 Explicit solutions

4.1 State feedback

State feedback problems refer to the instances when each player measures its own state perfectly, that is

$$M = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

in which case, we have

$$\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} (sI - A_{11})^{-1} H_{11} \\ (sI - A_{22})^{-1} A_{21} (sI - A_{11})^{-1} H_{11} \end{bmatrix}$$

therefore

$$\Omega_{\mathrm{S}} = (sI - A_{22})^{-1} A_{21} \in \mathcal{RH}_\infty$$

satisfies $V_{21} = \Omega_{\mathrm{S}} V_{11}$ and the problem can be separated to the following two problems.

Problem 7. Π_1 *problem reduces to*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{q} \\ z_{\Pi_1} \\ y_1 \\ r \end{bmatrix} = \left[\begin{array}{cc|ccc} A_{11} & 0 & H_{11} & B_{11} & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} \\ \hline C_1 & C_2 & 0 & D_1 & D_2 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ q \\ w_1 \\ u_1 \\ p \end{bmatrix}$$

which is a typical LQR problem.

Problem 8. Π_3 *problem reduces to*

$$\begin{bmatrix} \dot{x}_2 - \dot{q} \\ z_{\Pi_3} \\ y_2 - r \end{bmatrix} = \left[\begin{array}{c|cc} A_{22} & H_{22} & B_{22} \\ \hline C_2 & 0 & D_2 \\ I & 0 & 0 \end{array}\right] \begin{bmatrix} x_2 - q \\ w_2 \\ u_2 - p \end{bmatrix}$$

which is another LQR problem.

The solutions to these two problems can be combined to provide the optimal controller for the original two player state feedback problem. The state space solution is presented below.

Theorem 9. *Suppose that there exist stabilizing solutions X and Y for*

$$A^T X + XA - XB(D^T D)^{-1}B^T X + C^T C = 0$$
$$A_{22}^T Y + YA_{22} - YB_{22}(D_2^T D_2)^{-1}B_{22}^T Y + C_2^T C_2 = 0$$

and let

$$K = -(D^T D)^{-1}B^T X$$
$$J = -(D_2^T D_2)^{-1}B_{22}^T Y$$

and $A_K = A + BK$. Then the optimal control for the state feedback problem is given by

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & J \end{bmatrix} \begin{bmatrix} x_1(t) \\ q(t) \\ x_2(t) - q(t) \end{bmatrix}$$

where q is a controller state propagating by

$$q = \left[\begin{array}{c|c} (A_K)_{22} & (A_K)_{21} \\ \hline I & 0 \end{array}\right] x_1$$

Proof. The standard LQR solution together with Corollary 6 show that the optimal solution Q^* is

$$Q^* = \left[\begin{array}{cc|cc} A_K & 0 & \begin{pmatrix} 0 \\ A_{21} \end{pmatrix} + B\begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix} & 0 \\ 0 & A_J & -A_{21} & B_{22}J \\ \hline K & \begin{pmatrix} 0 \\ J \end{pmatrix} & \begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix} & \begin{pmatrix} 0 \\ J \end{pmatrix} \end{array}\right]$$

where $A_J = A_{22} + B_{22}J$.

The optimal controller $\mathcal{K}^* = Q^*(I + GQ^*)^{-1}$ is obtained via a long algebra as follows.

$$\mathcal{K}^* = \left[\begin{array}{c|cc} (A_K)_{22} & (A_K)_{21} & 0 \\ \hline K_{12} & K_{11} & 0 \\ K_{22} - J & K_{21} & J \end{array}\right]$$

This is identical to the optimal control presented in the theorem. ■

This corresponds to the earlier result in [8]. Note that the controller state q represents the first player's prediction of x_2 based on its information.

Before we finish our discussion on the state feedback case, notice that the second measurement, r, in Problem 7 is redundant, meaning that it can be deduced from y_1. Hence the problem can be further reduced to the following, whose solution will be useful for solving the partial output feedback case in the next section.

Problem 10. *Problem 7 is further reduced to*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{q} \\ z_{\Pi_1} \\ y_1 \end{bmatrix} = \left[\begin{array}{cc|ccc} A_{11} & 0 & H_{11} & B_{11} & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} \\ \hline C_1 & C_2 & 0 & D_1 & D_2 \\ I & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ q \\ w_1 \\ u_1 \\ p \end{bmatrix}$$

We show that the above can be solved via Problem 7.

Lemma 11. *Suppose that Q is an optimal Youla parameter for Problem 7. Then,*

$$\tilde{Q} = Q \begin{bmatrix} I \\ \Omega_S \end{bmatrix}$$

is the optimal Youla parameter for Problem 10.

Proof. It follows from $V_{21} = \Omega_{\text{S}} V_{11}$. ■

Corollary 12. *Let X be a stable left inverse of $\begin{bmatrix} I \\ \Omega_S \end{bmatrix}$, and suppose that $\tilde{Q}$ is the optimal Youla parameter for Problem 10. Then,*

$$Q = \tilde{Q}X$$

is an optimal Youla parameter for Problem 7.

Proof. Note that $\begin{bmatrix} I \\ \Omega_{\text{S}} \end{bmatrix}$ is always left invertible in $\mathcal{RH}_\infty$. Then the proof follows from

$$Q \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \tilde{Q}X \begin{bmatrix} I \\ \Omega_{\text{S}} \end{bmatrix} V_{11} = \tilde{Q}V_{11}$$

■

This implies that the optimal solution of Problem 7 is not unique. However the nonuniqueness does not transfer to the original two player problem, since the left inverse expression in Corollary 12 disappears when Corollary 6 generates the optimal solution to the original problem.

4.2 Partial output feedback

Partial output feedback cases arise when the first player perfectly measures its state, while the second player's measurement is a usual linear output measurement with noise. In other words, it corresponds to

$$M = \begin{bmatrix} I & 0 \\ M_{21} & M_{22} \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & N_{22} \end{bmatrix}$$

in which case

$$V_{11} = (sI - A_{11})^{-1}H_{11}$$
$$V_{21} = \left\{M_{21} + M_{22}(sI - A_{22})^{-1}A_{21}\right\}(sI - A_{11})^{-1}H_{11}$$

therefore

$$\Omega_{\text{P}} = M_{21} + M_{22}(sI - A_{22})^{-1}A_{21} \in \mathcal{RH}_\infty$$

satisfies $V_{21} = \Omega_{\text{P}} V_{11}$. The problem can be separated to the following two problems.

Problem 13. Π_1 *problem reduces to*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{q} \\ z_{\Pi_1} \\ y_1 \\ r \end{bmatrix} = \left[\begin{array}{cc|ccc} A_{11} & 0 & H_{11} & B_{11} & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} \\ \hline C_1 & C_2 & 0 & D_1 & D_2 \\ I & 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ q \\ w_1 \\ u_1 \\ p \end{bmatrix}$$

The solution can be obtained by applying Lemma 11 and Corollary 12 to Problem 7.

Lemma 14. *Suppose that Q_S is an optimal Youla parameter for Problem 7, and let X_P be a stable left inverse of $\begin{bmatrix} I \\ \Omega_P \end{bmatrix}$. Then*

$$Q_P = Q_S \begin{bmatrix} I \\ \Omega_S \end{bmatrix} X_P$$

is an optimal Youla parameter of Problem 13.

Proof. Lemma 11 implies $Q_\mathrm{S} \begin{bmatrix} I \\ \Omega_\mathrm{S} \end{bmatrix}$ is optimal for Problem 10. Applying Corollary 12 to Problem 13 follows that $Q_\mathrm{S} \begin{bmatrix} I \\ \Omega_\mathrm{S} \end{bmatrix} X_\mathrm{P}$ is optimal for Problem 13. ■

Problem 15. Π_3 *problem is equal to Problem 3.*

Now we present the optimal solution for the partial output feedback problem using the solutions of Problem 13 and Problem 15.

Theorem 16. *Suppose that there exist stabilizing solutions X, Y, and Ψ for*

$$A^TX + XA - XB(D^TD)^{-1}B^TX + C^TC = 0$$
$$A_{22}^TY + YA_{22} - YB_{22}(D_2^TD_2)^{-1}B_{22}^TY + C_2^TC_2 = 0$$
$$A_{22}\Psi + \Psi A_{22}^T - \Psi M_{22}^T(N_{22}N_{22}^T)^{-1}M_{22}\Psi + H_{22}H_{22}^T = 0$$

and let

$$K = -(D^TD)^{-1}B^TX$$
$$J = -(D_2^TD_2)^{-1}B_{22}^TY \qquad \Gamma = -\Psi M_{22}^T(N_{22}N_{22}^T)^{-1}$$

and $A_K = A + BK$, $A_\Gamma = A_{22} + B_{22}J + \Gamma M_{22}$. Then the optimal control for the partial output feedback problem is given by

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & J \end{bmatrix} \begin{bmatrix} x_1(t) \\ q(t) \\ \xi(t) \end{bmatrix}$$

where q and ξ are controller states propagating by

$$q = \left[\begin{array}{c|c} (A_K)_{22} & (A_K)_{21} \\ \hline I & 0 \end{array}\right] x_1$$

$$\xi = \left[\begin{array}{c|c} A_\Gamma & -\Gamma \\ \hline I & 0 \end{array}\right] (y_2 - M_{21}x_1 - M_{22}q)$$

Proof. Standard LQR/LQG solutions with Corollary 6 show that the optimal solution Q^* is given by

$$Q^* = \left[\begin{array}{ccc|cc} A_K & 0 & 0 & \begin{pmatrix} 0 \\ A_{21} \end{pmatrix} + B\begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix} & 0 \\ 0 & A_\Gamma & -\Gamma M_{22} & \Gamma M_{22} & -\Gamma \\ 0 & B_{22}J & A_{22} & -A_{21} & 0 \\ \hline K & \begin{pmatrix} 0 \\ J \end{pmatrix} & 0 & \begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix} & 0 \end{array}\right]$$

It can be shown via a long algebra that the optimal controller satisfying $\mathcal{K}^* = Q^*(I + P_{22}Q^*)^{-1}$ is

$$\mathcal{K}^* = \left[\begin{array}{cc|cc} (A_K)_{22} & 0 & (A_K)_{21} & 0 \\ \Gamma M_{22} & A_\Gamma & \Gamma M_{21} & -\Gamma \\ \hline K_{12} & 0 & K_{11} & 0 \\ K_{22} & J & K_{21} & 0 \end{array}\right]$$

which is a minimal realization of the optimal control presented in the theorem. ■

Compared to the earlier result in [9], this is a different realization for which the implication of the controller states are more obvious. As in the state feedback case, q is the first player's prediction of x_2 given its own information. Recalling the notation of Problem 3, it is obvious that ξ is the best estimate of $(x_2 - q)$ given the second player's information , *i.e.*, $y_2 - (M_{21}x_1 + M_{22}q)$.

4.3 Dynamically decoupled output feedback

Consider cases when each player evolves and take noisy measurements independently, but the second player has access to the first players measurement information to comply with the two player framework. We call such cases *dynamically decoupled output feedback* problems. Although the dynamics is decoupled in this case, the problem is still coupled through the cost function. The case is characterized by

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$$

For generality B is left block lower triangular. In this case,

$$\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} M_{11}(sI - A_{11})^{-1}H_{11} + N_{11} \\ 0 \end{bmatrix}$$

therefore

$$\Omega_\mathrm{D} = 0$$

satisfies $V_{21} = \Omega_\mathrm{D}V_{11}$. The problem can be separated to the following two problems.

Problem 17. Π_1 *problem reduces to*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{q} \\ z_{\Pi_1} \\ y_1 \end{bmatrix} = \left[\begin{array}{cc|ccc} A_{11} & 0 & H_{11} & B_{11} & 0 \\ 0 & A_{22} & 0 & B_{21} & B_{22} \\ \hline C_1 & C_2 & 0 & D_1 & D_2 \\ M_{11} & 0 & N_{11} & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ q \\ w_1 \\ u_1 \\ p \end{bmatrix}$$

which is an LQG problem with an unobservable mode.

Solving the above involves a reduced size filter Riccati equation, which is computationally cheaper.

Problem 18. Π_3 *problem is equal to Problem 3.*

By solving the above two, we present the optimal control for our last case.

Theorem 19. *Suppose that there exist stabilizing solutions X, Y, Φ, and Ψ for*

$$A^T X + XA - XB(D^T D)^{-1} B^T X + C^T C = 0$$
$$A_{22}^T Y + Y A_{22} - Y B_{22}(D_2^T D_2)^{-1} B_{22}^T Y + C_2^T C_2 = 0$$
$$A_{11}\Phi + \Phi A_{11}^T - \Phi M_{11}^T (N_{11} N_{11}^T)^{-1} M_{11}\Phi + H_{11} H_{11}^T = 0$$
$$A_{22}\Psi + \Psi A_{22}^T - \Psi M_{22}^T (N_{22} N_{22}^T)^{-1} M_{22}\Psi + H_{22} H_{22}^T = 0$$

and let

$$K = -(D^T D)^{-1} B^T X \qquad \Theta = -\Phi M_{11}^T (N_{11} N_{11}^T)^{-1}$$
$$J = -(D_2^T D_2)^{-1} B_{22}^T Y \qquad \Gamma = -\Psi M_{22}^T (N_{22} N_{22}^T)^{-1}$$

and $A_\Theta = A + BK + \begin{pmatrix} \Theta M_{11} & 0 \\ 0 & 0 \end{pmatrix}$, $A_\Gamma = A_{22} + B_{22} J + \Gamma M_{22}$. *Then the optimal control for the dynamically decoupled output feedback problem is given by*

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & J \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ q(t) \\ \xi(t) \end{bmatrix}$$

where $\hat{x}_1$, q and ξ are controller states propagating by

$$\begin{bmatrix} \hat{x}_1 \\ q \end{bmatrix} = \left[\begin{array}{c|c} A_\Theta & -\begin{pmatrix} \Theta \\ 0 \end{pmatrix} \\ \hline I & 0 \end{array}\right] y_1$$

$$\xi = \left[\begin{array}{c|c} A_\Gamma & -\Gamma \\ \hline I & 0 \end{array}\right] (y_2 - M_{22} q)$$

Proof. Standard LQG solutions together with Corollary 6 show that the optimal solution Q^* is given by,

$$Q^* = \left[\begin{array}{cccc|cc} A_\Theta & -\begin{pmatrix} \Theta M_{11} & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & -\begin{pmatrix} \Theta \\ 0 \end{pmatrix} & 0 \\ BK & A & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\Gamma & -\Gamma M_{22} & 0 & -\Gamma \\ 0 & 0 & B_{22} J & A_{22} & 0 & 0 \\ \hline K & 0 & \begin{pmatrix} 0 \\ J \end{pmatrix} & 0 & 0 & 0 \end{array}\right]$$

The optimal controller $\mathcal{K}^* = Q^*(I + P_{22} Q^*)^{-1}$ can be found via a long algebra.

$$\mathcal{K}^* = \left[\begin{array}{cc|cc} A_\Theta & 0 & \begin{pmatrix} -\Theta \\ 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & \Gamma M_{22} \end{pmatrix} & A_\Gamma & 0 & -\Gamma \\ \hline K & \begin{pmatrix} 0 \\ J \end{pmatrix} & 0 & 0 \end{array}\right]$$

This is a minimal realization of the optimal control presented in the theorem. ■

Note that $\hat{x}_1$ and q is the first player's best estimate of x_1 and x_2, respectively, and ξ is the best estimate of $(x_2 - q)$ given the second player's information, , $y_2 - M_{22} q$.

5 Conclusion

In this paper, we characterized a class of explicitly solvable two player optimal control problems. We showed that, under a simple algebraic condition, the two player $\mathcal{H}_2$ optimal control problems can be separated to two conventional optimal control problems.

Previously known problems including state feedback problems and partial output feedback problems fall into this category, therefore we are able to prove the same results using a simple unified technique. In addition, we presented a new class of interesting problems satisfying the condition, and derived an explicit state space solution which was previously unknown.

The problems separate at the very top level, therefore they have natural interpretations which in turn provide clear understanding of the controller dynamics.

Future research directions may include extension to general n-player problems or general output feedback problems.

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