A Sequel to Lighthill's Early Work
—
Aerodynamic Inverse Design and Shape Optimization via Control Theory

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Introduction

Objective of Computational Aerodynamics

1. Capability to predict the flow past an airplane in different flight regimes such as take off, cruise, flutter.
2. Interactive design calculations to allow immediate improvement
3. Automatic design optimization

Early Aerodynamic Design Methods

- 1945 Lighthill (Conformal Mapping, Incompressible Flow)
- 1965 Nieuwland (Hodograph, Power Series)
- 1970 Garabedian - Korn (Hodograph, Complex Characteristics)
- 1974 Boerstoel (Hodograph)
- 1974 Trenen (Potential Flow, Dirichlet Boundary Conditions)
- 1977 Henne (3D Potential Flow, based on FLO22)
- 1985 Volpe-Melnik (2D Potential Flow, Bsed on FLO36)
- 1979 Garabedian-McFadden (Potential Flow, Neumann Boundary Conditions, Iterated Mapping)
- 1976 Sobieczi (Fictitious Gas)
- 1979 Drela-Giles (2D Euler Equations, Streamline Coordinates, Newton Iteration)

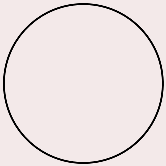
Lighthill's Method

Lighthill's Method

Profile P on z plane

$\Downarrow$

Profile C on σ plane



- Let the profile P be conformally mapped to an unit circle C
- The surface velocity is $q = \frac{1}{h} |\nabla \phi|$ where ϕ is the potential in the circle plane, and h is the mapping modulus $h = \left| \frac{dz}{d\sigma} \right| = \frac{ds}{d\theta}$
- Choose $q = q_T$
- Solve for the maping modulus $h = \frac{1}{q_T} |\nabla \phi|$

Implementation of Lighthill's Method

Design Profile C for Specified Surface speed q_t. Let a profile C be conformally mapped to a circle by

$$log \frac{dz}{d\sigma} = \sum \frac{C_n}{\sigma^n}$$

$$log \frac{ds}{d\theta} + i(\alpha - \theta - \frac{pi}{2}) = \sum (a_n cos(n\theta) + b_n sin(n\theta)) + i \sum (b_n cos(n\theta) - a_n sin(n\theta))$$

where

$$q = \frac{\nabla \Phi}{h}, \; h = \left| \frac{dz}{d\sigma} \right|$$

and

$$\Phi = (r + \frac{1}{r}) cos\theta + \frac{\Gamma}{2\pi}\theta \text{ is known}$$

On C set $q = q_t$

$$\rightarrow \frac{ds}{d\theta} = \frac{\Phi_\theta}{q_t} \rightarrow a_n, \; b_b$$

Constraints with Lighthill's Method

To preserve q_∞

$$c_0 = 0$$

Also, integration around a circuit gives

$$\Delta z = \oint \frac{dz}{d\sigma} d\sigma = 2\pi i c_1$$

Closure $\longrightarrow c_1 = 0$
Thus,

$$\int log(q_t) d\theta = 0$$

$$\int log(q_t) cos(\theta) d\theta = 0$$

$$\int log(q_t) sin(\theta) d\theta = 0$$

Control Theory Approach to Design

Control Theory Approach to Design

A wing is a device to control the flow. Apply the theory of control of partial differential equations (J.L.Lions) in conjunction with CFD.

References

- Pironneau (1964) Optimum shape design for subsonic potential flow
- Jameson (1988) Optimum shape design for transonic and supersonic flow modeled by the transonic potential flow equation and the Euler equations

Control Theory Approach to the Design Method

Define a cost function

$$I = \frac{1}{2}\int_{\mathcal{B}} (p - p_t)^2 d\mathcal{B}$$

or

$$I = \frac{1}{2}\int_{\mathcal{B}} (q - q_t)^2 d\mathcal{B}$$

The surface shape is now treated as the control, which is to be varied to minimize I, subject to the constraint that the flow equations are satisfied in the domain D.

Choice of Domain

ALTERNATIVES

1. Variable computational domain - Free boundary problem
2. Transformation to a fixed computational domain - Control via the transformation function

EXAMPLES

1. 2D via Conformal mapping with potential flow
2. 2D via Conformal mapping with Euler equations
3. 3D Sheared Parabolic Coordinates with Euler equation
4. ...

Formulation of the Control Problem

Suppose that the surface of the body is expressed by an equation

$$f(\underline{x}) = 0$$

Vary f to $f + \delta f$ and find δI.
If we can express

$$\delta I = \int_{\mathcal{B}} g \delta f d\mathcal{B} = (g, \delta f)_{\mathcal{B}}$$

Then we can recognize g as the gradient $\frac{\partial I}{\partial f}$.
Choose a modification

$$\delta f = -\lambda g$$

Then to first order

$$\delta I = -\lambda (g, g)_{\mathcal{B}} \leq 0$$

In the presence of constraints project g into the admissible trial space.

Accelerate by the conjugate gradient method.

Traditional Approach to Design Optimization

Define the geometry through a set of design parameters, for example, to be the weights α_i applied to a set of shape functions $b_i(x)$ so that the shape is represented as

$$f(x) = \sum \alpha_i b_i(x).$$

Then a cost function I is selected , for example, to be the drag coefficient or the lift to drag ratio, and I is regarded as a function of the parameters α_i. The sensitivities $\frac{\partial I}{\partial \alpha_i}$ may be estimated by making a small variation $\delta\alpha_i$ in each design parameter in turn and recalculating the flow to obtain the change in I. Then

$$\frac{\partial I}{\partial \alpha_i} \approx \frac{I(\alpha_i + \delta\alpha_i) - I(\alpha_i)}{\delta\alpha_i}.$$

The gradient vector $\mathcal{G} = \frac{\partial I}{\partial \alpha}$ may now be used to determine a direction of improvement. The simplest procedure is to make a step in the negative gradient direction by setting

$$\alpha^{n+1} = \alpha^n + \delta\alpha,$$

where

$$\delta\alpha = -\lambda\mathcal{G}$$

so that to first order

$$I + \delta I = I - \mathcal{G}^T\delta\alpha = I - \lambda\mathcal{G}^T\mathcal{G} < I$$

Disadvantages

The main disadvantage of this approach is the need for a number of flow calculations proportional to the number of design variables to estimate the gradient. The computational costs can thus become prohibitive as the number of design variables is increased.

Formulation of the Adjoint Approach to Optimal Design

For flow about an airfoil or wing, the aerodynamic properties which define the cost function are functions of the flow-field variables (w) and the physical location of the boundary, which may be represented by the function $\mathcal{F}$, say. Then

$$I = I\left(w, \mathcal{F}\right),$$

and a change in $\mathcal{F}$ results in a change

$$\delta I = \left[\frac{\partial I^T}{\partial w}\right] \delta w + \left[\frac{\partial I^T}{\partial \mathcal{F}}\right] \delta \mathcal{F} \qquad (1)$$

in the cost function. Suppose that the governing equation R which expresses the dependence of w and $\mathcal{F}$ within the flowfield domain D can be written as

$$R\left(w, \mathcal{F}\right) = 0. \qquad (2)$$

Then δw is determined from the equation

$$\delta R = \left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F} = 0. \qquad (3)$$

Since the variation δR is zero, it can be multiplied by a Lagrange Multiplier ψ and subtracted from the variation δI without changing the result.

Formulation of the Adjoint Approach to Optimal Design

$$\begin{aligned}\delta I &= \frac{\partial I^T}{\partial w}\delta w + \frac{\partial I^T}{\partial \mathcal{F}}\delta \mathcal{F} - \psi^T \left(\left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F} \right) \\ &= \left\{ \frac{\partial I^T}{\partial w} - \psi^T \left[\frac{\partial R}{\partial w}\right] \right\} \delta w + \left\{ \frac{\partial I^T}{\partial \mathcal{F}} - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \right\} \delta \mathcal{F}. \end{aligned} \quad (4)$$

Choosing ψ to satisfy the adjoint equation

$$\left[\frac{\partial R}{\partial w}\right]^T \psi = \frac{\partial I}{\partial w} \quad (5)$$

the first term is eliminated, and we find that

$$\delta I = \mathcal{G}^T \delta \mathcal{F}, \quad (6)$$

where

$$\mathcal{G} = \frac{\partial I^T}{\partial \mathcal{F}} - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right].$$

An improvement can be made with a shape change

$$\delta \mathcal{F} = -\lambda \mathcal{G}$$

where λ is positive and small enough that the first variation is an accurate estimate of δI.

Advantages

- The advantage is that (6) is independent of δw, with the result that the gradient of I with respect to an arbitrary number of design variables can be determined without the need for additional flow-field evaluations.
- The cost of solving the adjoint equation is comparable to that of solving the flow equations. Thus the gradient can be determined with roughly the computational costs of two flow solutions, independently of the number of design variables, which may be infinite if the boundary is regarded as a free surface.
- When the number of design variables becomes large, the computational efficiency of the control theory approach over traditional approach, which requires direct evaluation of the gradients by individually varying each design variable and recomputing the flow fields, becomes compelling.

Design using the Transonic Potential Flow Equation

Airfoil Design For Potential Flow using Conformal Mapping

Consider the case of two-dimensional compressible inviscid flow. In the absence of shock waves, an initially irrotational flow will remain irrotational, and we can assume that the velocity vector **q** is the gradient of a potential ϕ. In the presence of weak shock waves this remains a fairly good approximation. Let p, ρ, c, and M be the pressure, density, speed-of-sound, and Mach number q/c. Then the potential flow equation is

$$\nabla \cdot (\rho \nabla \phi) = 0, \tag{7}$$

where the density is given by

$$\rho = \left\{ 1 + \frac{\gamma - 1}{2} M_\infty^2 \left(1 - q^2\right) \right\}^{\frac{1}{(\gamma-1)}}, \tag{8}$$

while

$$p = \frac{\rho^\gamma}{\gamma M_\infty^2}, \quad c^2 = \frac{\gamma p}{\rho}. \tag{9}$$

Here M_∞ is the Mach number in the free stream, and the units have been chosen so that p and q have a value of unity in the far field.

Airfoil Design For Potential Flow using Conformal Mapping

Suppose that the domain D exterior to the profile C in the z-plane is conformally mapped on to the domain exterior to a unit circle in the σ-plane. Let R and θ be polar coordinates in the σ-plane, and let r be the inverted radial coordinate $\frac{1}{R}$. Also let h be the modulus of the derivative of the mapping function

$$h = \left| \frac{dz}{d\sigma} \right| . \tag{10}$$

Now the potential flow equation becomes

$$\frac{\partial}{\partial \theta} \left(\rho \phi_\theta \right) + r \frac{\partial}{\partial r} \left(r \rho \phi_r \right) = 0 \text{ in } D, \tag{11}$$

where the density is given by equation (8), and the circumferential and radial velocity components are

$$u = \frac{r \phi_\theta}{h}, \quad v = \frac{r^2 \phi_r}{h}, \tag{12}$$

while

$$q^2 = u^2 + v^2. \tag{13}$$

Airfoil Design For Potential Flow using Conformal Mapping

The condition of flow tangency leads to the Neumann boundary condition

$$v = \frac{1}{h}\frac{\partial \phi}{\partial r} = 0 \text{ on } C. \tag{14}$$

In the far field, the potential is given by an asymptotic estimate, leading to a Dirichlet boundary condition at $r = 0$.
Suppose that it is desired to achieve a specified velocity distribution q_d on C.
Introduce the cost function

$$I = \frac{1}{2}\int_C (q - q_d)^2 \, d\theta,$$

Design Problem

The design problem is now treated as a control problem where the control function is the mapping modulus h, which is to be chosen to minimize I subject to the constraints defined by the flow equations (7–14).

A modification δh to the mapping modulus will result in variations $\delta\phi$, δu, δv, and $\delta\rho$ to the potential, velocity components, and density. The resulting variation in the cost will be

$$\delta I = \int_C (q - q_d)\, \delta q \, d\theta, \qquad (15)$$

where, on C, $q = u$. Also,

$$\delta u = r\frac{\delta\phi_\theta}{h} - u\frac{\delta h}{h}, \quad \delta v = r^2\frac{\delta\phi_r}{h} - v\frac{\delta h}{h},$$

while according to equation (8)

$$\frac{\partial\rho}{\partial u} = -\frac{\rho u}{c^2}, \quad \frac{\partial\rho}{\partial v} = -\frac{\rho v}{c^2}.$$

Design Problem

It follows that $\delta\phi$ satisfies

$$L\delta\phi = -\frac{\partial}{\partial\theta}\left(\rho M^2 \phi_\theta \frac{\delta h}{h}\right) - r\frac{\partial}{\partial r}\left(\rho M^2 r\phi_r \frac{\delta h}{h}\right)$$

where

$$L \equiv \frac{\partial}{\partial\theta}\left\{\rho\left(1-\frac{u^2}{c^2}\right)\frac{\partial}{\partial\theta} - \frac{\rho uv}{c^2} r\frac{\partial}{\partial r}\right\} + r\frac{\partial}{\partial r}\left\{\rho\left(1-\frac{v^2}{c^2}\right) r\frac{\partial}{\partial r} - \frac{\rho uv}{c^2}\frac{\partial}{\partial\theta}\right\}. \quad (16)$$

Then, if ψ is any periodic differentiable function which vanishes in the far field,

$$\int_D \frac{\psi}{r^2} L\,\delta\phi\, dS = \int_D \rho M^2 \nabla\phi\cdot\nabla\psi \frac{\delta h}{h}\, dS, \quad (17)$$

where dS is the area element $r\,dr\,d\theta$, and the right hand side has been integrated by parts.

Design Problem

Now we can augment equation (15) by subtracting the constraint (17). The auxiliary function ψ then plays the role of a Lagrange multiplier. Thus,

$$\delta I = \int_C (q - q_d)\, q \frac{\delta h}{h}\, d\theta - \int_C \delta\phi \frac{\partial}{\partial\theta}\left(\frac{q - q_d}{h}\right) d\theta - \int_D \frac{\psi}{r^2} L\delta\phi\, dS + \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h}\, dS.$$

Now suppose that ψ satisfies the adjoint equation

$$L\psi = 0 \text{ in } D \tag{18}$$

with the boundary condition

$$\frac{\partial\psi}{\partial r} = \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{q - q_d}{h}\right) \text{ on } C. \tag{19}$$

Then, integrating by parts,

$$\delta I = -\int_C (q - q_d)\, q \frac{\delta h}{h}\, d\theta + \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h}\, dS. \tag{20}$$

Here the first term represents the direct effect of the change in the metric, while the area integral represents a correction for the effect of compressibility. When the second term is deleted the method reduces to a variation of Lighthill's method.

Design Problem

Equation (20) can be further simplified to represent δI purely as a boundary integral because the mapping function is fully determined by the value of its modulus on the boundary. Set

$$\log \frac{dz}{d\sigma} = \mathcal{F} + i\beta,$$

where

$$\mathcal{F} = \log \left| \frac{dz}{d\sigma} \right| = \log h,$$

and

$$\delta\mathcal{F} = \frac{\delta h}{h}.$$

Then $\mathcal{F}$ satisfies Laplace's equation

$$\Delta\mathcal{F} = 0 \text{ in } D,$$

and if there is no stretching in the far field, $\mathcal{F} \to 0$. Introduce another auxiliary function P which satisfies

$$\Delta P = \rho M^2 \nabla\psi \cdot \nabla\psi \text{ in } D, \tag{21}$$

and

$$P = 0 \text{ on } C.$$

Design Problem

Then after integrating by parts we find that

$$\delta I = \int_C \mathcal{G}\, \delta \mathcal{F}_c \, d\theta,$$

where $\mathcal{F}_c$ is the boundary value of $\mathcal{F}$, and

$$\mathcal{G} = \frac{\partial P}{\partial r} - (q - q_d)\, q. \tag{22}$$

Thus we can attain an improvement by a modification

$$\delta \mathcal{F}_c = -\lambda \bar{\mathcal{G}}$$

in the modulus of the mapping function on the boundary, which in turn defines the computed mapping function since F satisfies Laplace's equation. In this way the Lighthill method is extended to transonic flow.

Design using the Euler Equations

Design using the Euler Equations

In a fixed computational domain with coordinates, ξ, the Euler equations are

$$J\frac{\partial w}{\partial t} + R(w) = 0 \qquad (23)$$

where J is the Jacobian (cell volume),

$$R(w) = \frac{\partial}{\partial \xi_i}(S_{ij} f_j) = \frac{\partial F_i}{\partial \xi_i}. \qquad (24)$$

and S_{ij} are the metric coefficients (face normals in a finite volume scheme). We can write the fluxes in terms of the scaled contravariant velocity components

$$U_i = S_{ij} u_j$$

as

$$F_i = S_{ij} f_j = \begin{bmatrix} \rho U_i \\ \rho U_i u_1 + S_{i1} p \\ \rho U_i u_2 + S_{i2} p \\ \rho U_i u_3 + S_{i3} p \\ \rho U_i H \end{bmatrix}.$$

where $p = (\gamma - 1)\rho(E - \frac{1}{2}u_i^2)$ and $\rho H = \rho E + p$.

Design using the Euler Equations

A variation in the geometry now appears as a variation δS_{ij} in the metric coefficients. The variation in the residual is

$$\delta R = \frac{\partial}{\partial \xi_i}\left(\delta S_{ij} f_j\right) + \frac{\partial}{\partial \xi_i}\left(S_{ij}\frac{\partial f_j}{\partial w}\delta w\right) \tag{25}$$

and the variation in the cost δI is augmented as

$$\delta I - \int_D \psi^T \delta R \, d\xi \tag{26}$$

which is integrated by parts to yield

$$\delta I - \int_{\mathcal{B}} \psi^T n_i \delta F_i d\xi_B + \int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi}\left(\delta S_{ij} f_j\right) d\xi + \int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi_i} S_{ij}\frac{\partial f_j}{\partial w}\delta w d\xi$$

Design using the Euler Equations

For simplicity, it will be assumed that the portion of the boundary that undergoes shape modifications is restricted to the coordinate surface $\xi_2 = 0$. Then equations for the variation of the cost function and the adjoint boundary conditions may be simplified by incorporating the conditions

$$n_1 = n_3 = 0, \qquad n_2 = 1, \qquad \mathcal{B}_\xi = d\xi_1 d\xi_3,$$

so that only the variation δF_2 needs to be considered at the wall boundary. The condition that there is no flow through the wall boundary at $\xi_2 = 0$ is equivalent to

$$U_2 = 0, \qquad \text{so that} \qquad \delta U_2 = 0$$

when the boundary shape is modified. Consequently the variation of the inviscid flux at the boundary reduces to

$$\delta F_2 = \delta p \left\{ \begin{array}{c} 0 \\ S_{21} \\ S_{22} \\ S_{23} \\ 0 \end{array} \right\} + p \left\{ \begin{array}{c} 0 \\ \delta S_{21} \\ \delta S_{22} \\ \delta S_{23} \\ 0 \end{array} \right\}. \tag{27}$$

Design using the Euler Equations

In order to design a shape which will lead to a desired pressure distribution, a natural choice is to set

$$I = \frac{1}{2} \int_{\mathcal{B}} (p - p_d)^2 \, dS$$

where p_d is the desired surface pressure, and the integral is evaluated over the actual surface area. In the computational domain this is transformed to

$$I = \frac{1}{2} \iint_{\mathcal{B}_w} (p - p_d)^2 \, |S_2| \, d\xi_1 d\xi_3,$$

where the quantity

$$|S_2| = \sqrt{S_{2j} S_{2j}}$$

denotes the face area corresponding to a unit element of face area in the computational domain.

Design using the Euler Equations

In the computational domain the adjoint equation assumes the form

$$C_i^T \frac{\partial \psi}{\partial \xi_i} = 0 \tag{28}$$

where

$$C_i = S_{ij} \frac{\partial f_j}{\partial w}.$$

To cancel the dependence of the boundary integral on δp, the adjoint boundary condition reduces to

$$\psi_j n_j = p - p_d \tag{29}$$

where n_j are the components of the surface normal

$$n_j = \frac{S_{2j}}{|S_2|}.$$

Design using the Euler Equations

This amounts to a transpiration boundary condition on the co-state variables corresponding to the momentum components. Note that it imposes no restriction on the tangential component of ψ at the boundary.
We find finally that

$$\delta I = -\int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi_i} \delta S_{ij} f_j d\mathcal{D} - \iint_{B_W} \left(\delta S_{21}\psi_2 + \delta S_{22}\psi_3 + \delta S_{23}\psi_4\right) p \, d\xi_1 d\xi_3. \quad (30)$$

Here the expression for the cost variation depends on the mesh variations throughout the domain which appear in the field integral. However, the true gradient for a shape variation should not depend on the way in which the mesh is deformed, but only on the true flow solution. In the next section we show how the field integral can be eliminated to produce a reduced gradient formula which depends only on the boundary movement.

The Reduced Gradient Formulation

Consider the case of a mesh variation with a fixed boundary. Then $\delta I = 0$ but there is a variation in the transformed flux,

$$\delta F_i = C_i \delta w + \delta S_{ij} f_j .$$

Here the true solution is unchanged. Thus, the variation δw is due to the mesh movement δx at each mesh point. Therefore

$$\delta w = \nabla w \cdot \delta x = \frac{\partial w}{\partial x_j} \delta x_j \, (= \delta w^*)$$

and since $\frac{\partial}{\partial \xi_i} \delta F_i = 0$, it follows that

$$\frac{\partial}{\partial \xi_i} \left(\delta S_{ij} f_j \right) = - \frac{\partial}{\partial \xi_i} \left(C_i \delta w^* \right) . \tag{31}$$

It has been verified by Jameson and Kim★ that this relation holds in the general case with boundary movement.

★ "Reduction of the Adjoint Gradient Formula in the Continuous Limit", A.Jameson and S. Kim , 41st AIAA Aerospace Sciences Meeting & Exhibit, AIAA Paper 2003–0040, Reno, NV, January 6–9, 2003.

The Reduced Gradient Formulation

Now

$$\begin{aligned}\int_{\mathcal{D}} \phi^T \delta R \, d\mathcal{D} &= \int_{\mathcal{D}} \phi^T \frac{\partial}{\partial \xi_i} C_i \left(\delta w - \delta w^*\right) d\mathcal{D} \\ &= \int_{\mathcal{B}} \phi^T C_i \left(\delta w - \delta w^*\right) d\mathcal{B} \\ &\quad - \int_{\mathcal{D}} \frac{\partial \phi^T}{\partial \xi_i} C_i \left(\delta w - \delta w^*\right) d\mathcal{D}. \end{aligned} \tag{32}$$

Here on the wall boundary

$$C_2 \delta w = \delta F_2 - \delta S_{2j} f_j. \tag{33}$$

Thus, by choosing ϕ to satisfy the adjoint equation and the adjoint boundary condition, we reduce the cost variation to a boundary integral which depends only on the surface displacement:

$$\begin{aligned}\delta I &= \int_{\mathcal{B}_{\mathcal{W}}} \psi^T \left(\delta S_{2j} f_j + C_2 \delta w^*\right) d\xi_1 d\xi_3 \\ &\quad - \iint_{B_W} \left(\delta S_{21} \psi_2 + \delta S_{22} \psi_3 + \delta S_{23} \psi_4\right) p \, d\xi_1 d\xi_3. \end{aligned} \tag{34}$$

Viscous Adjoint Terms

Derivation of the Viscous Adjoint Terms

The viscous terms will be derived under the assumption that the viscosity and heat conduction coefficients μ and k are essentially independent of the flow, and that their variations may be neglected. This simplification has been successfully used for may aerodynamic problems of interest. In the case of some turbulent flows, there is the possibility that the flow variations could result in significant changes in the turbulent viscosity, and it may then be necessary to account for its variation in the calculation.

Transformation to Primitive Variables

The derivation of the viscous adjoint terms is simplified by transforming to the primitive variables

$$\tilde{w}^T = (\rho, u_1, u_2, u_3, p)^T,$$

because the viscous stresses depend on the velocity derivatives $\frac{\partial U_i}{\partial x_j}$, while the heat flux can be expressed as

$$\kappa \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right).$$

where $\kappa = \frac{k}{R} = \frac{\gamma\mu}{Pr(\gamma-1)}$. The relationship between the conservative and primitive variations is defined by the expressions

$$\delta w = M \delta \tilde{w}, \qquad \delta \tilde{w} = M^{-1} \delta w$$

which make use of the transformation matrices $M = \frac{\partial w}{\partial \tilde{w}}$ and $M^{-1} = \frac{\partial \tilde{w}}{\partial w}$.

Transformation to Primitive Variables

These matrices are provided in transposed form for future convenience

$$
M^T = \begin{bmatrix} 1 & u_1 & u_2 & u_3 & \frac{u_i u_i}{2} \\ 0 & \rho & 0 & 0 & \rho u_1 \\ 0 & 0 & \rho & 0 & \rho u_2 \\ 0 & 0 & 0 & \rho & \rho u_3 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma-1} \end{bmatrix}
$$

$$
M^{-1^T} = \begin{bmatrix} 1 & -\frac{u_1}{\rho} & -\frac{u_2}{\rho} & -\frac{u_3}{\rho} & \frac{(\gamma-1)u_i u_i}{2} \\ 0 & \frac{1}{\rho} & 0 & 0 & -(\gamma-1)u_1 \\ 0 & 0 & \frac{1}{\rho} & 0 & -(\gamma-1)u_2 \\ 0 & 0 & 0 & \frac{1}{\rho} & -(\gamma-1)u_3 \\ 0 & 0 & 0 & 0 & \gamma-1 \end{bmatrix}.
$$

The Viscous Adjoint Field Operator

Collecting together the contributions from the momentum and energy equations, the viscous adjoint operator in primitive variables can be expressed as

$$\begin{aligned}
\left(\tilde{L}\psi\right)_1 &= -\frac{p}{\rho^2}\frac{\partial}{\partial \xi_l}\left(S_{lj}\kappa\frac{\partial \theta}{\partial x_j}\right) \\
\left(\tilde{L}\psi\right)_{i+1} &= \frac{\partial}{\partial \xi_l}\left\{S_{lj}\left[\mu\left(\frac{\partial \phi_i}{\partial x_j}+\frac{\partial \phi_j}{\partial x_i}\right)+\lambda\delta_{ij}\frac{\partial \phi_k}{\partial x_k}\right]\right\} \\
&\quad +\frac{\partial}{\partial \xi_l}\left\{S_{lj}\left[\mu\left(u_i\frac{\partial \theta}{\partial x_j}+u_j\frac{\partial \theta}{\partial x_i}\right)+\lambda\delta_{ij}u_k\frac{\partial \theta}{\partial x_k}\right]\right\} \\
&\quad -\sigma_{ij}\left(S_{lj}\frac{\partial \theta}{\partial x_j}\right) \\
\left(\tilde{L}\psi\right)_5 &= \frac{1}{\rho}\frac{\partial}{\partial \xi_l}\left(S_{lj}\kappa\frac{\partial \theta}{\partial x_j}\right).
\end{aligned}$$

The conservative viscous adjoint operator may now be obtained by the transformation

$$L = {M^{-1}}^T \tilde{L}$$

Viscous Adjoint Boundary Conditions

The boundary conditions satisfied by the flow equations restrict the form of the performance measure that may be chosen for the cost function. There must be a direct correspondence between the flow variables for which variations appear in the variation of the cost function, and those variables for which variations appear in the boundary terms arising during the derivation of the adjoint field equations. Otherwise it would be impossible to eliminate the dependence of δI on δw through proper specification of the adjoint boundary condition. In fact it proves that it is possible to treat any performance measure based on surface pressure and stresses such as the force coefficients, or an inverse problem for a specified target pressure.

Sobolev Inner Product

The Need for a Sobolev Inner Product in the Definition of the Gradient

Another key issue for successful implementation of the continuous adjoint method is the choice of an appropriate inner product for the definition of the gradient. It turns out that there is an enormous benefit from the use of a modified Sobolev gradient, which enables the generation of a sequence of smooth shapes. This can be illustrated by considering the simplest case of a problem in the calculus of variations.
Suppose that we wish to find the path $y(x)$ which minimizes

$$I = \int_a^b F(y, y')dx$$

with fixed end points $y(a)$ and $y(b)$.
Under a variation $\delta y(x)$,

$$\begin{aligned}\delta I &= \int_1^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right) dx \\ &= \int_1^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \delta y dx\end{aligned}$$

The Need for a Sobolev Inner Product in the Definition of the Gradient

Thus defining the gradient as

$$g = \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}$$

and the inner product as

$$(u, v) = \int_a^b uv dx$$

we find that

$$\delta I = (g, \delta y).$$

If we now set

$$\delta y = -\lambda g, \qquad \lambda > 0,$$

we obtain a improvement

$$\delta I = -\lambda(g, g) \leq 0$$

unless $g = 0$, the necessary condition for a minimum.

The Need for a Sobolev Inner Product in the Definition of the Gradient

Note that g is a function of y, y', y'',

$$g = g(y, y', y'')$$

In the well known case of the Brachistrone problem, for example, which calls for the determination of the path of quickest descent between two laterally separated points when a particle falls under gravity,

$$F(y, y') = \sqrt{\frac{1 + y'^2}{y}}$$

and

$$g = -\frac{1 + y'^2 + 2yy''}{2\left[y(1 + y'^2)\right]^{3/2}}$$

It can be seen that each step

$$y^{n+1} = y^n - \lambda^n g^n$$

reduces the smoothness of y by two classes. Thus the computed trajectory becomes less and less smooth, leading to instability.

The Need for a Sobolev Inner Product in the Definition of the Gradient

In order to prevent this we can introduce a weighted Sobolev inner product

$$\langle u, v\rangle = \int (uv + \epsilon u'v')dx$$

where ϵ is a parameter that controls the weight of the derivatives. We now define a gradient $\overline{g}$ such that $\delta I = \langle \overline{g}, \delta y\rangle$. Then we have

$$\begin{aligned}
\delta I &= \int (\overline{g}\delta y + \epsilon \overline{g}'\delta y')dx \\
&= \int \left(\overline{g} - \frac{\partial}{\partial x}\epsilon\frac{\partial \overline{g}}{\partial x}\right)\delta y dx \\
&= (g, \delta y)
\end{aligned}$$

where

$$\overline{g} - \frac{\partial}{\partial x}\epsilon\frac{\partial \overline{g}}{\partial x} = g$$

and $\overline{g} = 0$ at the end points.

The Need for a Sobolev Inner Product in the Definition of the Gradient

Therefore $\overline{g}$ can be obtained from g by a smoothing equation. Now the step

$$y^{n+1} = y^n - \lambda^n \overline{g}^n$$

gives an improvement

$$\delta I = -\lambda^n \langle \overline{g}^n, \overline{g}^n \rangle$$

but y^{n+1} has the same smoothness as y^n, resulting in a stable process.

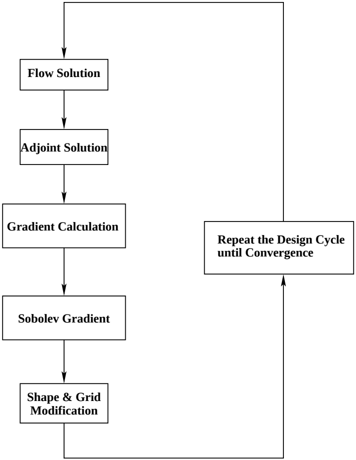
Outline of the Design Process

Outline of the Design Process

The design procedure can finally be summarized as follows:

1. Solve the flow equations for ρ, u_1, u_2, u_3 and p.
2. Solve the adjoint equations for ψ subject to appropriate boundary conditions.
3. Evaluate $\mathcal{G}$ and calculate the corresponding Sobolev gradient $\overline{\mathcal{G}}$.
4. Project $\overline{\mathcal{G}}$ into an allowable subspace that satisfies any geometric constraints.
5. Update the shape based on the direction of steepest descent.
6. Return to 1 until convergence is reached.

Design Cycle

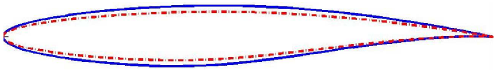


Constraints

- Fixed C_L.
- Fixed span load distribution to present too large C_L on the outboard wing which can lower the buffet margin.
- Fixed wing thickness to prevent an increase in structure weight.
    - Design changes can be can be limited to a specific spanwise range of the wing.
    - Section changes can be limited to a specific chordwise range.

- Smooth curvature variations via the use of Sobolev gradient.

Application of Thickness Constraints



- Prevent shape change penetrating a specified skeleton (colored in red).
- Separate thickness and camber allow free camber variations.
- Minimal user input needed.

Computational Cost★

Cost of Search Algorithm

Steepest Descent	$\mathcal{O}(N^2)$	Steps
Quasi-Newton	$\mathcal{O}(N)$	Steps
Smoothed Gradient	$\mathcal{O}(K)$	Steps

Note: K is independent of N.

★: "Studies of Alternative Numerical Optimization Methods Applied to the Brachistrone Problem", A.Jameson and J. Vassberg, Computational Fluid Dynamics, Journal, Vol. 9, No.3, Oct. 2000, pp. 281-296

Computational Cost★

Total Computational Cost of Design

+	Finite difference gradients Steepest descent	$\mathcal{O}(N^3)$
+	Finite difference gradients Quasi-Newton	$\mathcal{O}(N^2)$
+	Adjoint gradients Quasi-Newton	$\mathcal{O}(N)$
+	Adjoint gradients Smoothed gradient	$\mathcal{O}(K)$

Note: K is independent of N.

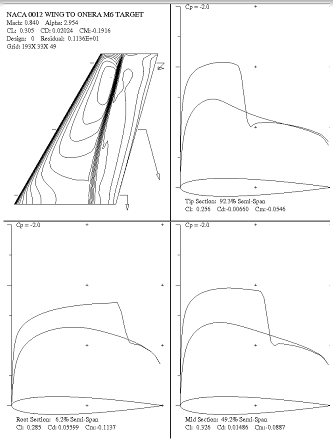
★: "Studies of Alternative Numerical Optimization Methods Applied to the Brachistrone Problem", A.Jameson and J. Vassberg, Computational Fluid Dynamics, Journal, Vol. 9, No.3, Oct. 2000, pp. 281-296

Inverse Design

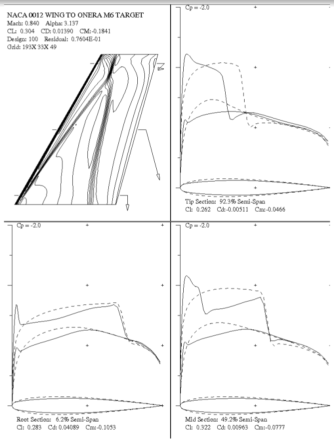
Recovering of ONERA M6 Wing
from its pressure distribution

A. Jameson 2003–2004

NACA 0012 WING TO ONERA M6 TARGET



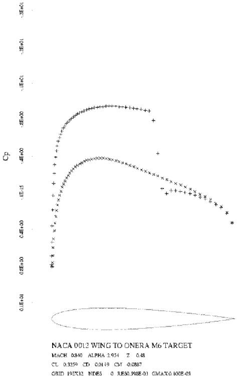
(a) Staring wing: NACA 0012



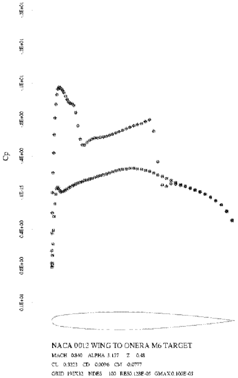
(b) Target wing: ONERA M6

This is a difficult problem because of the presence of the shock wave in the target pressure and because the profile to be recovered is symmetric while the

Pressure Profile at 48% Span



(c) Staring wing: NACA 0012



(d) Target wing: ONERA M6

The pressure distribution of the final design match the specified target, even inside the shock.

Planform and Aero–Structural Optimization

Super B747

Planform and Aero-Structural Optimization

The shape changes in the section needed to improve the transonic wing design are quite small. However, in order to obtain a true optimum design larger scale changes such as changes in the wing planform (sweepback, span, chord, and taper) should be considered. Because these directly affect the structure weight, a meaningful result can only be obtained by considering a cost function that takes account of both the aerodynamic characteristics and the weight.
Consider a cost function is defined as

$$I = \alpha_1 C_D + \alpha_2 \frac{1}{2} \int_{\mathcal{B}} (p - p_d)^2 dS + \alpha_3 C_W$$

Maximizing the range of an aircraft provides a guide to the values for α_1 and α_3.

Choice of Weighting Constants

The simplified Breguet range equation can be expressed as

$$R = \frac{V}{C}\frac{L}{D}\log\frac{W_1}{W_2}$$

where W_2 is the empty weight of the aircraft.
With fixed $V/C, W_1$, and L, the variation of R can be stated as

$$\frac{\delta R}{R} = -\left(\frac{\delta C_D}{C_D} + \frac{1}{\log\frac{W_1}{W_2}}\frac{\delta W_2}{W_2}\right) = -\left(\frac{\delta C_D}{C_D} + \frac{1}{\log\frac{C_{W_1}}{C_{W_2}}}\frac{\delta C_{W_2}}{C_{W_2}}\right).$$

Therefore minimizing

$$I = C_D + \alpha C_W,$$

by choosing

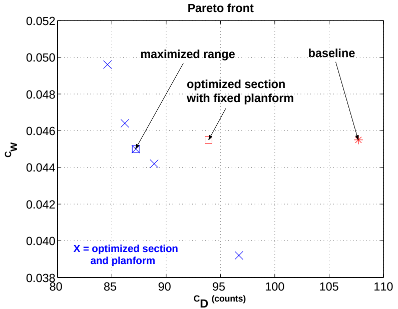
$$\alpha = \frac{C_D}{C_{W_2}\log\frac{C_{W_1}}{C_{W_2}}},$$

corresponds to maximizing the range of the aircraft.

Boeing 747 Euler Planform Results: Pareto Front

Test case: Boeing 747 wing–fuselage and modified geometries at the following flow conditions.

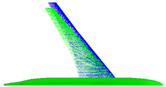
$$M_\infty = 0.87, \qquad C_L = 0.42 \text{ (fixed)}, \qquad \text{multiple } \frac{\alpha_3}{\alpha_1}$$

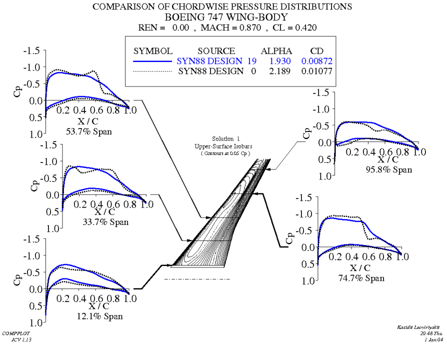


Boeing 747 Euler Planform Results: Sweepback, Span, Chord, and Section Variations to Maximize Range

Geometry	Baseline	Optimized	Variation (%)
Sweep (°)	42.1	38.8	–7.8
Span (ft)	212.4	226.7	+6.7
C_{root}	48.1	48.6	+1.0
C_{mid}	30.6	30.8	+0.7
C_{tip}	10.78	10.75	+0.3
t_{root}	58.2	62.4	+7.2
t_{mid}	23.7	23.8	+0.4
t_{tip}	12.98	12.8	–0.8

Boeing 747 Euler Planform Results: Sweepback, Span, Chord, and Section Variations to Maximize Range





- C_D is reduced from 107.7 drag counts to 87.2 drag counts (19%).
- C_W is reduced from 0.0455 (69,970 lbs) to 0.0450 (69,201 lbs) (1.1%).

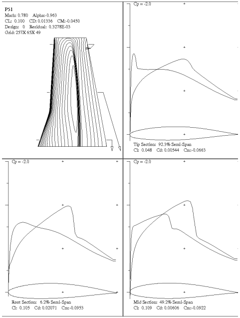
P51 Racer

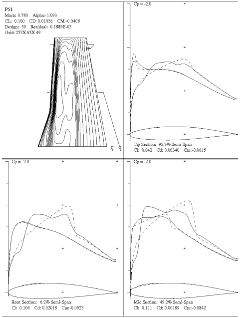
P51 Racer

- Aircraft competing in the Reno Air Races reach speeds above 500 MPH, encounting compressibility drag due to the appearance of shock waves.
- Objective is to delay drag rise without altering the wing structure. Hence try adding a bump on the wing surface.

Partial Redesign

- Allow only outward movement.
- Limited changes to front part of the chordwise range.





Flight at Mach 1

Flight at mach 1

A viable alternative for long range business jets?

Flight at mach 1

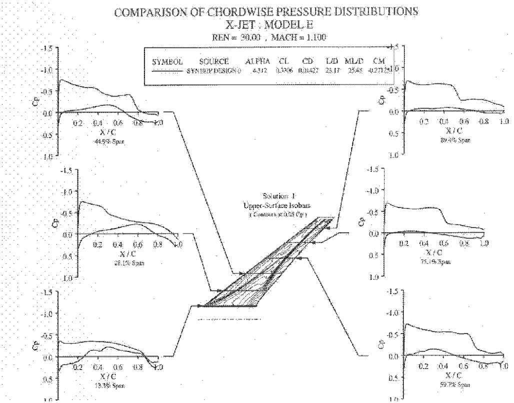
It appears possible to design a wing with very low drag at Mach 1, as indicated in the table below:

C_L	$C_{D_{\text{pres}}}$ (counts)	$C_{D_{\text{friction}}}$ (counts)	$C_{D_{\text{wing}}}$ (counts)
0.300	47.6	41.3	88.9
0.330	65.6	40.8	106.5

- The data is for a wing-fuselage combination, with engines mounted on the rear fuselage simulated by bumps.
- The wing has 50 degrees of sweep at the leading edge, and the thickness to chord ratio varies from 10 percent at the root to 7 percent at the tip.
- To delay drag rise to Mach one requires fuselage shaping in conjunction with wing optimization.

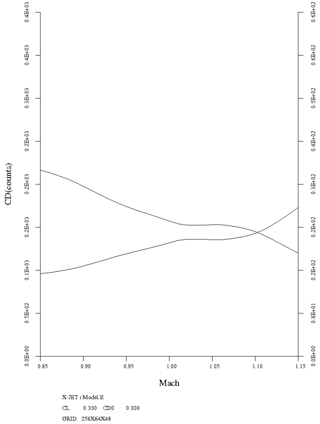
X Jet: Model E

Pressure Distribution on the Wing at Mach 1.1



X Jet: Model E

Drag Rise

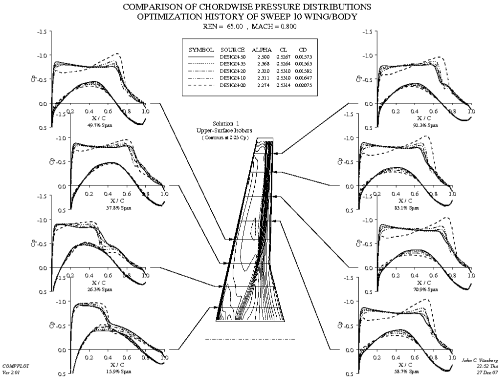


Do We Need Swept Wings on Commercial Jets?

Background for Studies of Reduced Sweep

- Current Transonic Transports
  - Cruise Mach: $0.76 \leq M \leq 0.86$
  - C/4 Sweep: $25^\circ \leq \Lambda \leq 35^\circ$
  - Wing Planform Layout Knowledge Base
    - Heavily Influenced By *Design Charts*
    - Data Developed From *Cut-n-Try* Designs
    - Data Aumented With Parametric Variations
    - Data Collected Over The Years
    - Includes Shifts Due To Technologies
      e.g., Supercritical Airfoils, Composites, etc.

Pure Aerodynamic Optimizations



Evolution of Pressures for $\Lambda = 10^{\circ}$ Wing during Optimization

Pure Aerodynamic Optimizations

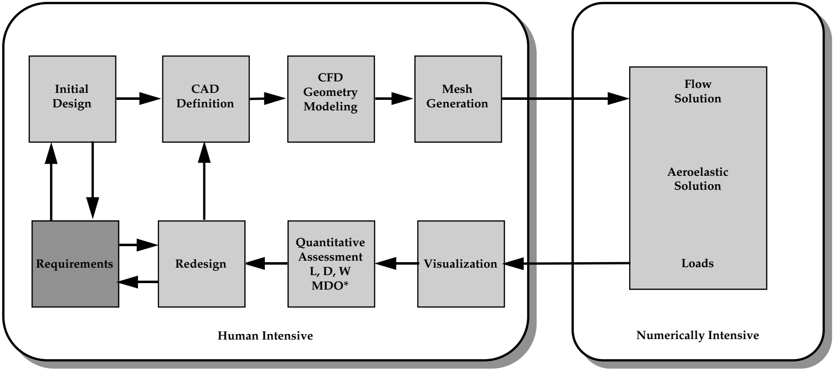
Mach	Sweep	C_L	C_D	$C_{D.tot}$	ML/D	$\sqrt{M}L/D$
0.85	35°	0.500	153.7	293.7	14.47	**15.70**
0.84	30°	0.510	151.2	291.2	14.71	**16.05**
0.83	25°	0.515	151.2	291.2	14.68	**16.11**
0.82	20°	0.520	151.7	291.7	14.62	**16.14**
0.81	15°	0.525	152.4	292.4	14.54	**16.16**
0.80	10°	0.530	152.2	292.2	14.51	**16.22**
0.79	5°	0.535	152.5	292.5	14.45	**16.26**

- C_D in counts
- $C_{D.\text{tot}} = C_D + 140$ counts
- Lowest Sweep Favors $\sqrt{M}L/D \simeq 4.0\%$

Conclusion of Swept Wing Study

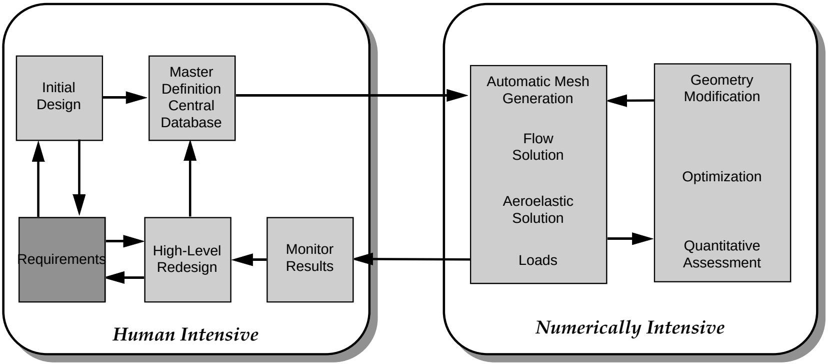
- An unswept wing at Mach 0.80 offers slightly better range efficiency than a swept wing at Mach 0.85.
- It would also improve TO, climb, descent and landing.
- Perhaps B737/A320 replacements should have unswept wings.

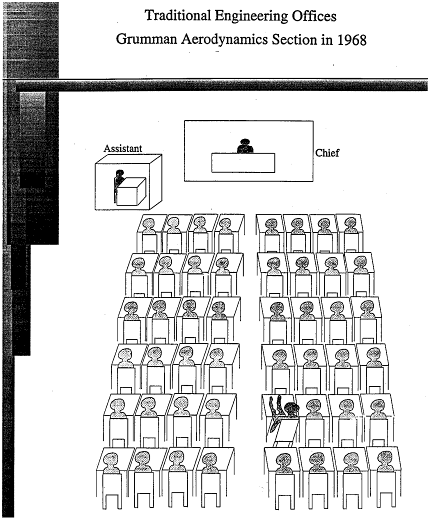
Concept of Numerical Wind Tunnel



*MDO: Multi-Disciplinary Optimization

Advanced Numerical Wind Tunnel




Traditional Engineering Offices
Grumman Aerodynamics Section in 1968
Assistant
Chief