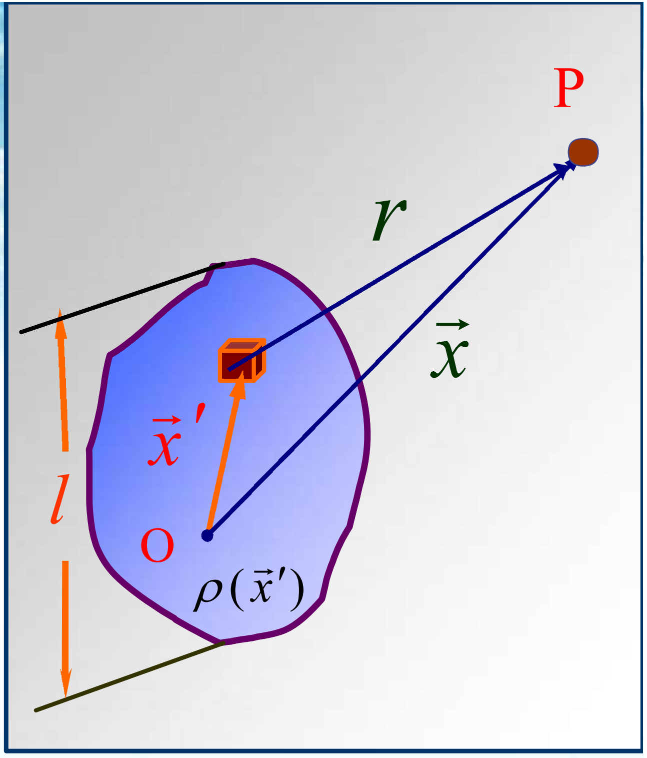
第二章

静电场

2.5 静电场的多极矩

2.5.1静电势多极矩展开

1. 问题分析



理论上: 已知 $\rho(\vec{x}\,')$ ，原则上由

$$\varphi(\vec{x}) = \int_V \frac{\rho(\vec{x}\,')dV'}{4\pi\varepsilon_0 r}$$

求电势。

实际问题: 体电荷分布不均匀或区域不规则，积分困难

特殊情况: 区域的线度远小于该区域到场点的距离，可以近似求解。 条件 $l << r$

$$r \approx |\vec{x}| = R \Rightarrow \varphi(\vec{x}) \approx \frac{Q}{4\pi\varepsilon_0 R}$$

目的:寻求精度更高的级数解

2. $\frac{1}{r}$ 的麦克劳林展开

(1) 一元函数的麦克劳林展开式（在坐标原点展开）

$$f(x) = f(0) + \frac{1}{1!}\frac{df(0)}{dx}x + \frac{1}{2!}\frac{df^{2}(0)}{dx^{2}}x^{2} + \cdots$$

(2) 三元函数的麦克劳林展开

$$\begin{aligned} f(\vec{x}) &= f(x_1, x_2, x_3) \\ &= f(0,0,0) + \frac{1}{1!}\left(x_1\frac{\partial f(0,0,0)}{\partial x_1} + x_2\frac{\partial f(0,0,0)}{\partial x_2} + x_3\frac{\partial f(0,0,0)}{\partial x_3}\right) \\ &+ \frac{1}{2!}\left[x_1^2\frac{\partial^2 f(0,0,0)}{\partial x_1} + x_2^2\frac{\partial^2 f(0,0,0)}{\partial x_2} + x_3^2\frac{\partial^2 f(0,0,0)}{\partial x_3}\right. \\ &\left. + 2x_1x_2\frac{\partial^2 f}{\partial x_1\partial x_2} + 2x_1x_3\frac{\partial^2 f}{\partial x_1\partial x_3} + 2x_2x_3\frac{\partial^2 f}{\partial x_2\partial x_3}\right] + \cdots \end{aligned}$$

$$f(\vec{x}) = f(x_1, x_2, x_3)$$

$$= f(0) + \frac{1}{1!}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) f(0)$$

$$+ \frac{1}{2!}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)^2 f(0) + \cdots$$

$$= f(0) + \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} f(0) + \frac{1}{2}\left[\sum_i x_i \frac{\partial}{\partial x_i} \sum_j x_j \frac{\partial}{\partial x_j}\right] f(0) + \cdots$$

$$= f(0) + \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} f(0) + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} f(0) + \cdots$$

$$= f(0) + (\vec{x}\cdot\nabla) f(0) + \frac{1}{2}(\vec{x}\cdot\nabla)^2 f(0) + \cdots$$

$$f(\vec{x}') = f(x_1', x_2', x_3')$$

$$= f(0) + (\vec{x}'\cdot\nabla') f(0) + \frac{1}{2}(\vec{x}'\cdot\nabla')^2 f(0) + \cdots$$

$$f(\vec{x}') = f(x_1', x_2', x_3')$$

$$= f(0) + (\vec{x}' \cdot \nabla')f(0) + \frac{1}{2}(\vec{x}' \cdot \nabla')^2 f(0) + \cdots$$

(3) 将 $\frac{1}{r}$ 在 $\vec{x}' = 0$ 点展开

$$\frac{1}{r} = \frac{1}{|\vec{x} - \vec{x}'|} = f(\vec{x} - \vec{x}'), \quad \vec{x}' = 0, \quad \frac{1}{r} = \frac{1}{R}$$

对应关系

$$f(\vec{x}') \rightarrow f(\vec{x} - \vec{x}') = \frac{1}{r} = \frac{1}{|\vec{x} - \vec{x}'|}$$

$$f(0) \rightarrow f(\vec{x} - \vec{x}')\Big|_{\vec{x}'=0} = \frac{1}{r}\Big|_{\vec{x}'=0} = f(\vec{x}) = \frac{1}{R}$$

$$f(\vec{x} - \vec{x}') = f(\vec{x}) + (\vec{x}' \cdot \nabla')f(\vec{x}) + \frac{1}{2}(\vec{x}' \cdot \nabla')^2 f(\vec{x}) + \cdots$$

$$f(\vec{x}-\vec{x}') = f(\vec{x}) + (\vec{x}'\cdot\nabla')f(\vec{x}) + \frac{1}{2}(\vec{x}'\cdot\nabla')^2 f(\vec{x}) + \cdots$$

$$\frac{1}{r} = \frac{1}{|\vec{x}-\vec{x}'|} = f(\vec{x}-\vec{x}')$$

$$f(\vec{x}-\vec{x}')\Big|_{\vec{x}'=0} = \frac{1}{r}\Big|_{\vec{x}'=0} = f(\vec{x}) = \frac{1}{R}$$

$$\frac{1}{r} = \frac{1}{R} + (\vec{x}'\cdot\nabla')\frac{1}{r}\Big|_{\vec{x}'=0} + \frac{1}{2}(\vec{x}'\cdot\nabla')^2\frac{1}{r}\Big|_{\vec{x}'=0} + \cdots$$

$$= \frac{1}{R} - (\vec{x}'\cdot\nabla)\frac{1}{R} + \frac{1}{2}(\vec{x}'\cdot\nabla)^2\frac{1}{R} + \cdots$$

$$= \frac{1}{R} - (\vec{x}'\cdot\nabla)\frac{1}{R} + \frac{1}{2}(\vec{x}'\vec{x}' : \nabla\nabla)\frac{1}{R} + \cdots$$

其中 $\left(\nabla'\frac{1}{r}\Big|_{\vec{x}'=0} = -\nabla\frac{1}{r}\Big|_{\vec{x}'=0} = -\nabla\frac{1}{R}\right.$ ， $\left.\vec{a}\vec{a} : \vec{b}\vec{b} = (\vec{a}\cdot\vec{b})^2\right)$

$$\frac{1}{r}=\frac{1}{R}-(\vec{x}'\cdot\nabla)\frac{1}{R}+\frac{1}{2}(\vec{x}'\vec{x}':\nabla\nabla)\frac{1}{R}+\cdots$$

$$\frac{1}{r}=\frac{1}{R}-(\vec{x}'\cdot\nabla)\frac{1}{R}+\frac{1}{2}(\vec{x}'\cdot\nabla)^2\frac{1}{R}+\cdots$$

3. 小区域电荷分布产生的电势

$$\varphi(\vec{x})=\int_\infty\frac{\rho(\vec{x}')dV'}{4\pi\varepsilon_0 r}=\frac{1}{4\pi\varepsilon_0}\int_\infty\rho(\vec{x}')\frac{1}{r}dV'$$

$$\varphi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\int_V\rho(\vec{x}')[\frac{1}{R}-\vec{x}'\cdot\nabla\frac{1}{R}+\frac{1}{2}\vec{x}'\vec{x}':\nabla\nabla\frac{1}{R}+\cdots]dV'$$

$$\varphi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\left[\frac{1}{R}-\vec{x}'\cdot\nabla\frac{1}{R}+\frac{1}{2}\vec{x}'\vec{x}':\nabla\nabla\frac{1}{R}+\cdots\right]dV'$$

$$=\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')dV'\frac{1}{R}$$

$$+\frac{-1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\vec{x}'dV'\cdot\nabla\frac{1}{R}$$

$$+\frac{1}{4\pi\varepsilon_0}\frac{1}{6}\int_V 3\rho(\vec{x}')\vec{x}'\vec{x}'dV':\nabla\nabla\frac{1}{R}$$

$$+\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')[\cdots+\cdots]dV'$$

$$Q=\int_V \rho(\vec{x}')dV'$$ 点电荷零极矩

$$\vec{p}=\int_V \rho(\vec{x}')\vec{x}'dV'$$ 电偶极矩矢量

$$\overleftrightarrow{\mathcal{D}}=\int_V 3\vec{x}'\vec{x}'\rho(\vec{x}')dV'$$

电四极矩张量

分量式: $D_{ij}=\int 3x_i'x_j'\rho(\vec{x}')dV' \qquad i=1\rightarrow 3,\ j=1\rightarrow 3$

$$\varphi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 R} + \frac{-1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla\frac{1}{R} + \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathcal{D}} : \nabla\nabla\frac{1}{R} + \cdots$$

$$\varphi^{(0)} = \frac{Q}{4\pi\varepsilon_0 R}$$

$$Q = \int_V \rho(\vec{x}\,')dV'$$

$$\varphi^{(1)} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla\frac{1}{R}$$

$$\vec{p} = \int_V \rho(\vec{x}\,')\vec{x}\,'dV'$$

$$\varphi^{(2)} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathcal{D}} : \nabla\nabla\frac{1}{R} + \cdots$$

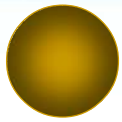
$$\overleftrightarrow{\mathcal{D}} = \int_V 3\vec{x}\,'\vec{x}\,'\rho(\vec{x}\,')dV'$$

$$\varphi(\vec{x}) = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \cdots$$

2.5.2电多极矩

1. 展开式的物理意义

$$\varphi^{(0)} = \frac{Q}{4\pi\varepsilon_0 R}$$

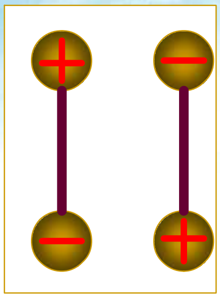


等效于坐标原点点电荷产生的电势。因此小电荷体系在电荷分布区外产生的电势在零级近似下可视为将电荷集中于原点处产生的电势。

$$\varphi^{(1)} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla\frac{1}{R} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot(-\frac{\vec{R}}{R^3}) = \frac{\vec{p}\cdot\vec{R}}{4\pi\varepsilon_0 R^3}$$

等效电偶极矩 $\vec{p}$ 产生的电势。最简单的体系为两个点电荷产生的电势。

$$\varphi^{(2)} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathcal{D}}:\nabla\nabla\frac{1}{R} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\sum_{ij} D_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{R}$$



$$\overleftrightarrow{\mathcal{D}}:\nabla\nabla = \sum_{ij} D_{ij}\vec{e}_i\vec{e}_j : \sum_{kl}\frac{\partial^2}{\partial x_k \partial x_l}\vec{e}_k\vec{e}_l$$

$$= \sum_{ij} D_{ij}\sum_{kl}\frac{\partial^2}{\partial x_k \partial x_l}\delta_{jk}\delta_{il} = \sum_{ij} D_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

$\varphi^{(2)}$等效为体系电四极矩张量产生的电势。最简单的体系为坐标原点附近（+，−，+，−）四个点电荷产生的电势

2. 电四极矩张量 $\overleftrightarrow{\mathscr{D}} = \int_V 3\vec{x}'\vec{x}'\rho(\vec{x}')dV'$

有9个分量 $D_{ij} = \int 3x'_i x'_j \rho(\vec{x}')dV'$

$D_{ij} = D_{ji} \quad (i \neq j)$ 此6个不同分量，只有三个独立

重新定义：$\overleftrightarrow{\mathscr{D}} = \int [3\vec{x}'\vec{x}' - r'^2 \overleftrightarrow{\mathscr{I}}]\rho(\vec{x}')dV'$

$D_{ij} = \int [3x'_i x'_j - \delta_{ij} r'^2]\rho(\vec{x}')dV'$

$\overleftrightarrow{\mathscr{I}} = \sum_{ij} \delta_{ij}\vec{e}_i\vec{e}_j$

可以证明它不改变 $\varphi^{(2)}$

此时有 $D_{11}+D_{22}+D_{33}=0$

所以电四极矩张量 只有5个独立分量

$$\varphi^{(2)} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathscr{D}}:\nabla\nabla\frac{1}{R} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\sum_{ij} D_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{R}$$

3. 证明两种电四极矩表示式对应的电势表示等效

$$D_{ij} = \int 3x_i' x_j' \rho(\vec{x}')dV'$$ 原表示

$$D_{ij} = \int [3x_i' x_j' - \delta_{ij} r'^2]\rho(\vec{x}')dV'$$ 新表示

$$\varphi^{(2)} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\left[\int 3\vec{x}'\vec{x}'\rho(\vec{x}')dV'\right] : \nabla\nabla\frac{1}{R}$$

加入一个0项

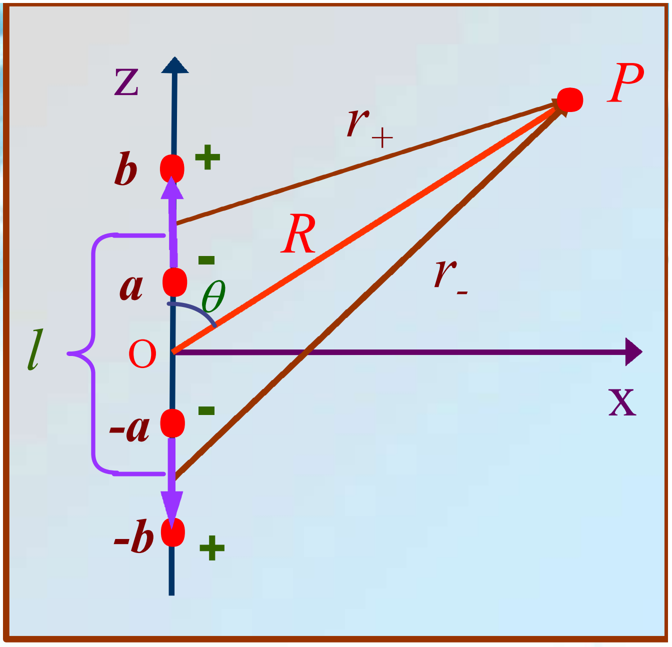
$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\left[\int 3\vec{x}'\vec{x}'\rho(\vec{x}')dV' : \nabla\nabla\frac{1}{R} - \int r'^2 \overleftrightarrow{\mathcal{I}} : \nabla\nabla\frac{1}{R}\rho(\vec{x}')dV'\right]$$

$$= \nabla^2\frac{1}{R} = 0 \quad (R \neq 0)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\left[\int (3\vec{x}'\vec{x}' - r'^2 \overleftrightarrow{\mathcal{I}})\rho(\vec{x}')dV'\right] : \nabla\nabla\frac{1}{R}$$

4.电四极矩最简单体系举例

四个点电荷在一直线上排列，可看作一对正负电偶极子。



（1）运用多极展开式计算电势

$$l = a + b$$

$$\vec{P}_{上} = Q(b-a)\vec{e}_z = \vec{p}$$

$$\vec{P}_{下} = -Q(b-a)\vec{e}_z = -\vec{p}$$

$$Q = \int_V \rho(\vec{x}')dV' = 0$$

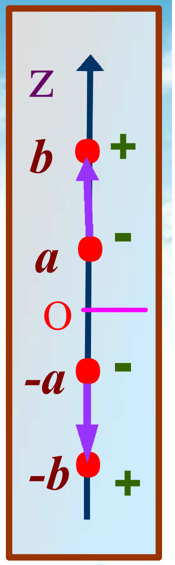
$$\vec{p} = \int_V \rho(\vec{x}')\vec{x}'dV' = 0$$

体系总电荷、总电偶极矩为零

$$\varphi^{(0)} = \frac{Q}{4\pi\varepsilon_0 R} = 0$$

$$\varphi^{(1)} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla\frac{1}{R} = 0$$

$$D_{ij} = \int 3x'_i x'_j \rho(\vec{x}')dV'$$ 依定义 $D_{33} \neq 0$ 其它分量均为零



$$\rho(\vec{x}') = \sum_{i=1}^{4} Q'_i \delta(x')\delta(y')\delta(z' - z_i)$$

$$Q'_1 = Q, \quad z_1 = b, \quad Q'_2 = -Q, \quad z_2 = a,$$

$$Q'_3 = -Q, \quad z_3 = -a, \quad Q'_4 = Q, \quad z_4 = -b$$

$$D_{33} = \int_V 3z'z'\rho(\vec{x}')dV'$$

$$= \iiint_{V'} 3z'z' \sum_{i=1}^{4} Q'_i \delta(x')\delta(y')\delta(z' - z_i)dx'dy'dz'$$

$$= \sum_{i=1}^{4} 3\int_{z'=-\infty}^{z'=\infty} z'z'Q'_i\delta(z' - z_i)dz'$$

$$= 3z_1z_1Q - 3z_2z_2Q - 3z_3z_3Q + 3z_4z_4Q$$

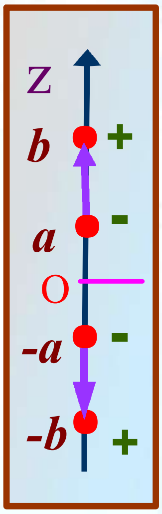
$$= 3(b^2 - a^2 - a^2 + b^2)Q = 6Q(b^2 - a^2)$$

$$= 6Q(b - a)(b + a) = 6pl$$

$$\varphi(\vec{x})=\frac{Q}{4\pi\varepsilon_0 R}+\frac{-1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla\frac{1}{R}+\frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathcal{D}}:\nabla\nabla\frac{1}{R}+\cdots$$

$$=0\quad+\quad0\quad+\quad\frac{1}{4\pi\varepsilon_0}\frac{1}{6}\sum_{ij}D_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{R}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{1}{6}D_{33}\frac{\partial^2}{\partial x_3\partial x_3}\frac{1}{R}$$



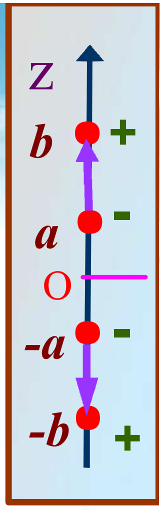
$$\varphi=\varphi^{(2)}=\frac{1}{4\pi\varepsilon_0}pl\frac{\partial^2}{\partial z^2}(\frac{1}{R})$$

忽略更高极势后，系统的势等于第三项的势

（2）运用电势的积分式计算（忽略三级小量保留二级小量）

在近似条件下，可采用两个电偶极子的电势迭加计算。

$$\varphi = \frac{\vec{p}\cdot\vec{r}_+}{4\pi\varepsilon_0 r_+^3} - \frac{\vec{p}\cdot\vec{r}_-}{4\pi\varepsilon_0 r_-^3} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot[\nabla\frac{1}{r_+} - \nabla\frac{1}{r_-}]$$



$$= -\frac{1}{4\pi\varepsilon_0} p \frac{\partial}{\partial z}(\frac{1}{r_+} - \frac{1}{r_-}) \qquad (\vec{p} = p\vec{e}_z)$$

$$r_+ \approx R - \frac{l}{2}\cos\theta \qquad r_- \approx R + \frac{l}{2}\cos\theta \qquad \frac{1}{r_+} - \frac{1}{r_-} = \frac{r_- - r_+}{r_+ r_-} \approx \frac{l\cos\theta}{R^2}$$

$$\frac{\partial}{\partial z}\frac{1}{R} = -\frac{1}{R^2}\frac{\partial R}{\partial z} = -\frac{1}{R^2}\frac{z}{R} = -\frac{\cos\theta}{R^2} \qquad (z = R\cos\theta)$$

$$R = \sqrt{x^2 + y^2 + z^2} \qquad \frac{\partial R}{\partial z} = \frac{z}{R}$$

$$\varphi = \frac{1}{4\pi\varepsilon_0} pl \frac{\partial^2}{\partial z^2}(\frac{1}{R})$$

此为电四极子势

两种方法讲算结果一致，表明 $\varphi^{(2)}$ 为电四极子势

电四极矩其它例子

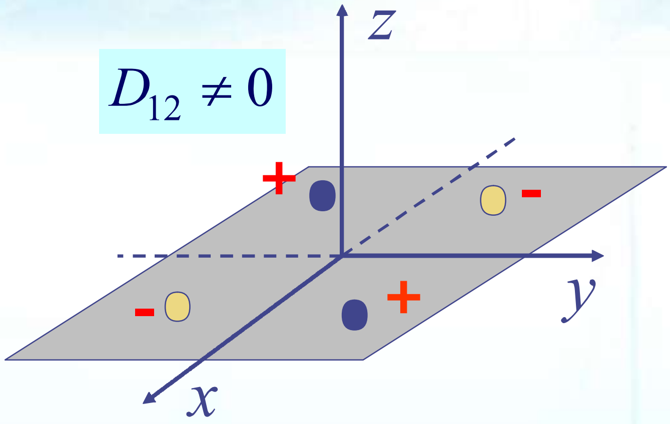
$D_{11} \neq 0$ 四个点电荷在 ***x*** 轴

$D_{22} \neq 0$ 四个点电荷在 ***y*** 轴

$D_{12} = D_{21} \neq 0$ ***x-y*** 平面

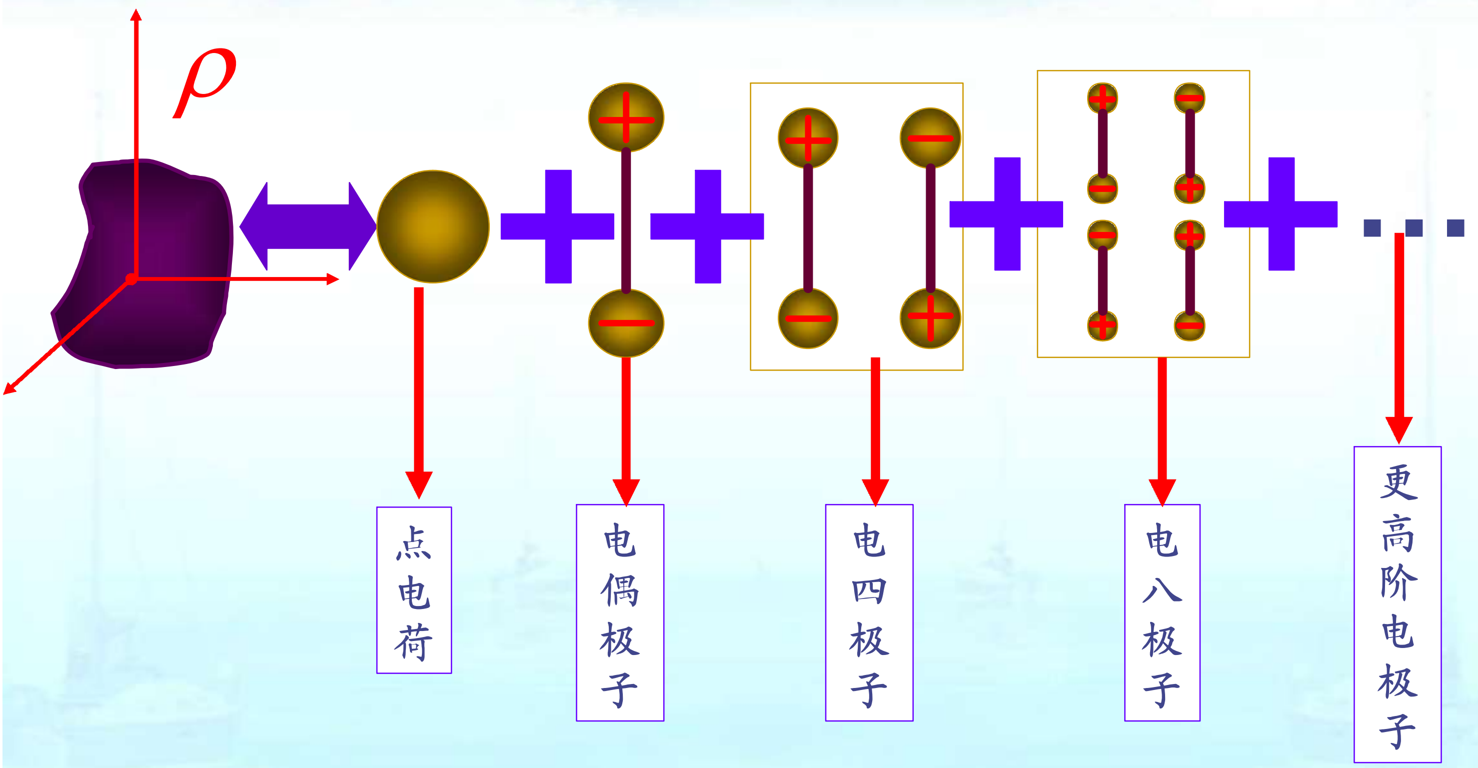
$D_{13} = D_{31} \neq 0$ ***x-z*** 平面

$D_{23} = D_{32} \neq 0$ ***y-z*** 平面



小结：电多极展开要点概述

1.带电体系可等效分解为不同级的理想电多极子迭加



2.带电体系势的级数分解

$$\varphi(\vec{x}) = \int_V \frac{\rho(\vec{x}')dV'}{4\pi\varepsilon_0 r}$$

$$\frac{1}{r} = \frac{1}{R} - (\vec{x}'\cdot\nabla)\frac{1}{R} + \frac{1}{2}(\vec{x}'\cdot\nabla)^2\frac{1}{R} + \cdots$$

$$\varphi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 R} + \frac{-1}{4\pi\varepsilon_0}\vec{p}\cdot\nabla\frac{1}{R} + \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathscr{D}}:\nabla\nabla\frac{1}{R} + \cdots$$

$$\varphi^{(0)} = \frac{Q}{4\pi\varepsilon_0 R}$$

$$Q = \int_V \rho(\vec{x}')dV'$$

点电荷

$$\varphi^{(1)} = -\frac{\vec{p}\cdot\vec{R}}{4\pi\varepsilon_0 R^3}$$

$$\vec{p} = \int_V \rho(\vec{x}')\vec{x}'dV'$$

偶极矩

$$\varphi^{(2)} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\overleftrightarrow{\mathscr{D}}:\nabla\nabla\frac{1}{R}$$

$$\overleftrightarrow{\mathscr{D}} = \int_V 3\vec{x}'\vec{x}'\rho(\vec{x}')dV'$$

四极矩

$$D_{ij} = \int 3x_i'x_j'\rho(\vec{x}')dV'$$

3.电多极展开的特点

①带电体系的势可分解成无穷多项电极子势的迭加，根据问题精度的需要进行取舍，使运算得到简化。

②各项多极势的大小依赖于原点的选取，总势与原点选取无关。

③高极势比低级势效应小

④在实际应用中，往往可以根据势的特征反推体系的电结构特征。