A quantitative approach to Hadwiger's covering conjecture[1]

Chuanming ZONG

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
(Email: cmzong@math.pku.edu.cn)

Dedicated to Professor Yuan Wang on the occasion of his 80th birthday

Abstract: In 1957, Hadwiger made a conjecture that every n-dimensional convex body can be covered by 2^n translates of its interior. Up to now, this conjecture is open for all $n \geq 3$. In this article we encode Hadwiger's conjecture into a series of functions defined on the spaces of convex bodies, propose a four-step program to approach this conjecture, and obtain some partial results.

Keywords: convex body, Hadwiger's conjecture, illumination problem, separation problem, Banach-Mazur metric, β-net

MSC (2000): 52C17; 11H31

1. Introduction

In n-dimensional *Euclidean space* E^n, let K be a *convex body* with boundary $\partial(K)$, interior $int(K)$ and volume $v(K)$, and let $c(K)$ denote the smallest number of translates of $int(K)$ that their union can cover K. In 1955, Levi [16] studied $c(K)$ for the two-dimensional convex domains and proved that

$$c(K) = \begin{cases} 4, & \text{if } K \text{ is a parallelogram,} \\ 3, & \text{otherwise.} \end{cases}$$

Let P denote an n-dimensional *parallelopiped.* Clearly, any translate of $int(P)$ can not cover two vertices of P. Therefore, it can be deduced that

$$c(P) = 2^n.$$

Based on these results and some other observations, in 1957 Hadwiger [11] made the following conjecture: *For every n-dimensional convex body K we have*

$$c(K) \leq 2^n, \tag{1}$$

where the equality holds if and only if K is a parallelopiped.

This conjecture has been studied by many authors. In the course, many partial results have been achieved and several connections with other important

[1] This work is supported by the National Science Foundation of China, a 973 Project and the Chang Jiang Scholars Program.

problems such as the *illumination problem* and the *separation problem* have been discovered (see Bezdek [3], Boltyanski, Martini and Soltan [7], Brass, Moser and Pach [8] and Zong [24] for general references). For example, Lassak [14] proved this conjecture for the three-dimensional centrally symmetric case, Rogers and Zong [19] obtained

$$c(K) \leq \binom{2n}{n}(n\log n + n\log\log n + 5n)$$

for general n-dimensional convex bodies and

$$c(K) \leq 2^n(n\log n + n\log\log n + 5n)$$

for centrally symmetric ones. Nevertheless, we are still far away from the solution of the conjecture, even the three-dimensional case.

Let m be a positive integer and let $\gamma_m(K)$ be the smallest positive number r such that K can be covered by m translates of rK. Clearly, we have

$$\gamma_m(K) = 1$$

for all $m \leq n$, and

$$\gamma_m(K) \geq \gamma_{m+1}(K) \tag{2}$$

for all positive integers m and all convex bodies K.

Let $\mathcal{T}^n$ denote the set of all non-singular linear transformations in E^n and let $\mathcal{K}^n$ denote the space of all n-dimensional convex bodies with the *Banach-Mazur metric* defined by

$$\|K_1, K_2\| = \log\ \min\left\{r :\ K_1 + \mathbf{x} \subseteq T(K_2) \subseteq rK_1 + \mathbf{x};\ \mathbf{x} \in E^n;\ T \in \mathcal{T}^n\right\}.$$

It is well known that $\mathcal{K}^n$ is bounded, connected and compact. On the other hand, for any given positive integer m, it can be shown that $\gamma_m(K)$ as a function of K defined on $\mathcal{K}^n$ is continuous. In addition, we have

$$\gamma_m(K_1) = \gamma_m(K_2)$$

whenever $\|K_1, K_2\| = 0$. Then we define

$$\Gamma(n,m) = \max_{K \in \mathcal{K}^n}\{\gamma_m(K)\}$$

and

$$\gamma(n,m) = \min_{K \in \mathcal{K}^n}\{\gamma_m(K)\}.$$

It is easy to see that (1) holds for all n-dimensional convex bodies K if and only if

$$\gamma_{2^n}(K) < 1$$

holds for all $K \in \mathcal{K}^n$. Therefore, it is equivalent to

$$\Gamma(n, 2^n) < 1.$$

Thus, Hadwiger's conjecture can be encoded in the functions $\gamma_m(K)$ defined on the space $\mathcal{K}^n$.

In this article, we suggest a four-step program (Section 3) to approach Hadwiger's conjecture and study the values of $\gamma_m(K)$ for some particular m and K. Among other things, the following results are proved:

Theorem 1. *Let K be a bounded three-dimensional convex cone (the convex hull of a convex domain and a point which is not in the plane of the domain), then we have*

$$\gamma_8(K) \le \tfrac{2}{3}.$$

Theorem 2. *Let K_p be the unit ball of the three-dimensional ℓ_p norm,*

$$K_p = \{(x,y,z) : \ |x|^p + |y|^p + |z|^p \le 1\}, \quad 1 \le p \le +\infty,$$

then we have

$$\gamma_8(K) \le \sqrt{\tfrac{2}{3}}.$$

2. The two-dimensional case, a brief review

The values of $\gamma(2,m)$ and $\Gamma(2,m)$ have been studied by several authors. Clearly, we have

$$\gamma(2,2) = \Gamma(2,2) = \Gamma(2,3) = 1$$

and, by considering the area measures,

$$\gamma(2,m) \ge \frac{1}{\sqrt{m}}.$$

However, for the nontrivial cases, it is not easy to determine the exact values of $\gamma(2,m)$ and $\Gamma(2,m)$. We list the known results in the following tables.

m	3	4	5
$\gamma(2,m)$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
Authors	J.F. Belousov [1]	S. Krotoszynski [13]	S. Krotoszynski [13]

m	3	4	5	6	7	8
$\Gamma(2,m)$	1	$\frac{\sqrt{2}}{2}$	??	??	$\frac{1}{2}$	$\frac{1}{2}$
Authors		M. Lassak [15]	??	??	F.W. Levi [16]	F.W. Levi [16]

Remark 1. By $\Gamma(2,2^2) = \sqrt{2}/2$ it follows that every two-dimensional convex domain K can be covered by four translates of $\frac{\sqrt{2}}{2}K$. As shown in the second table, the values of $\Gamma(2,5)$ and $\Gamma(2,6)$ have not been determined yet.

3. A four-step program to Hadwiger's conjecture

Let B^n denote the n-dimensional unit ball centered at the origin, and let $\overline{\mathcal{K}^n}$ denote the set of all convex bodies K satisfying

$$B^n \subseteq K \subseteq nB^n. \tag{3}$$

According to John's theorem (see [12]), for each n-dimensional convex body K there is an non-singular linear transformation $T \in \mathcal{T}^n$ such that

$$B^n \subseteq T(K) \subseteq nB^n.$$

Therefore, we have

$$\Gamma(n,m) = \max_{K\in\overline{\mathcal{K}^n}} \gamma_m(K).$$

Definition 1. Let β be a positive number, and let K_1, K_2, $\cdots$, $K_{l(\beta)}$ be $l(\beta)$ convex bodies in $\overline{\mathcal{K}^n}$. If for any $K \in \overline{\mathcal{K}^n}$ there is a corresponding K_i satisfying

$$\|K, K_i\| \le \beta,$$

then we call $\mathcal{N} = \{K_1, K_2, \cdots, K_{l(\beta)}\}$ a β-net in $\overline{\mathcal{K}^n}$.

Remark 2. Defining

$$\mathcal{B}(K_i,\beta) = \{K \in \mathcal{K}^n : \ \|K, K_i\| \le \beta\},$$

it is easy to see that $\mathcal{N} = \{K_1, K_2, \cdots, K_{l(\beta)}\}$ is a β-net in $\mathcal{K}^n$ if and only if

$$\bigcup_{i=1}^{l(\beta)} \mathcal{B}(K_i,\beta) = \mathcal{K}^n.$$

The philosophy of our program. If Hadwiger's conjecture is true in E^n, then there is a positive number $c_n < 1$ such that

$$\gamma_{2^n}(K) \le c_n \tag{4}$$

holds for all $K \in \overline{\mathcal{K}^n}$. On the other hand, since $\gamma_{2^n}(K)$ is continuous on $\overline{\mathcal{K}^n}$, there is a positive number β such that

$$|\gamma_{2^n}(K) - \gamma_{2^n}(K')| \le \tfrac{1}{2}(1-c_n) \tag{5}$$

holds whenever $\|K, K'\| \le \beta$. By (3), we should be able to construct a β-net $\mathcal{N}$ in $\overline{\mathcal{K}^n}$ with the assistance of a computer.

A four-step program for Hadwiger's conjecture.

Step 1. In the considered dimension, for example $n = 3$, study the values of $\gamma_{2^n}(K)$ for some particular convex bodies K and therefore choose a suitable constant c_n for (4).

Step 2. Choose a suitable positive number β to guarantee (5), based on a close study on the function $\gamma_{2^n}(K)$.

Step 3. Based on (3) to construct a suitable β-net $\mathcal{N}$. For example, choose a well-distributed set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p\}$ of points on the surface of nB^n. For each point $\mathbf{x}_i$ we define

$$X_{i,k} = \left\{ \left(\tfrac{1}{n} + \tfrac{j}{k} \left(1 - \tfrac{1}{n} \right) \right) \mathbf{x}_i : \ j = 0, \cdots, k \right\}.$$

Then the set

$$\mathcal{P} = \{conv\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\} : \ \mathbf{y}_i \in X_{i,k}\}$$

of polytopes will be a β-net in $\overline{\mathcal{K}^n}$, provided both p and k are large enough. Here, as usual, $conv\{X\}$ denotes the convex hull of X.

Step 4. By considering several patterns, with the assistance of a computer, verify that

$$\gamma_{2^n}(K_i) \leq c_n$$

holds for all $K_i \in \mathcal{N}$.

Remark 3. In principle, the conjecture can be proved in E^n by our program if it is true in this particular dimension and the computing facility is efficient enough. Clearly the set $\mathcal{P}$ can be much reduced in cardinality.

4. The covering functions on $\mathcal{K}^3$

In this section, among other things, we will prove Theorem 1 and Theorem 2. As a consequence, we give some insight to a reasonable estimate for the constant c_3 defined in the previous section. First, let us introduce two lemmas.

Lemma 1 (Besicovitch [2]). *Each two-dimensional convex domain has an inscribed affine regular hexagon.*

Remark 4. Affine regular hexagons are the imagines of a regular hexagon under non-singular linear transformations.

Lemma 2. *Let K be a two-dimensional convex domain, let λ be a real number satisfying $0 < \lambda < 1$, and let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be an ordered triple on the boundary of K. If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subset \lambda K + \mathbf{y}$, then the whole curve from $\mathbf{x}_1$ to $\mathbf{x}_3$ passing $\mathbf{x}_2$ belongs to $\lambda K + \mathbf{y}$.*

Proof. For convenience, we assume that $\mathbf{o} \in int(K)$ and $\mathbf{y} = (0, a)$. It is well known in convexity (see Eggleston [9]) that the set of *regular convex domains* (each tangent touches K at exactly one point and there is one and only one tangent at each boundary point) is dense in $\mathcal{K}^2$. Therefore, without loss of generality, we assume that K is regular.

Let $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_3 = (x_3, y_3)$ denote the points of $\partial(K) \cap (\lambda K + \mathbf{y})$ with maximal and minimal x-coordinates, respectively. Let $y = f(x)$ denote the curve of $\partial(K)$ from $\mathbf{x}_3$ to $\mathbf{x}_1$, and let $y = g(x)$ denote the above part of

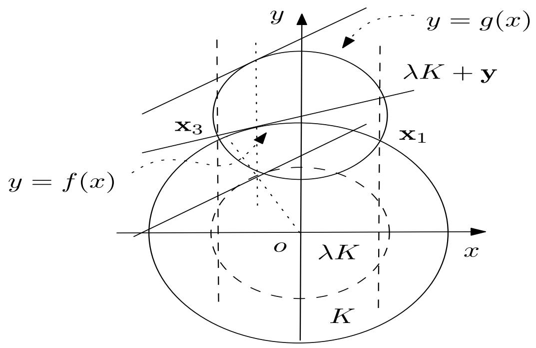

Figure 1

$\partial(\lambda K) + \mathbf{y}$ in the strip of $x_3 \le x \le x_1$. By convexity, as shown in Figure 1, we have

$$g(x_3) \ge f(x_3),$$

$$g(x_1) \ge f(x_1),$$

$$g'(x) = f'(\tfrac{1}{\lambda}x) \ge f'(x)$$

for $x_3 \le x \le 0$, and

$$g'(x) = f'(\tfrac{1}{\lambda}x) \le f'(x)$$

for $0 \le x \le x_1$. Thus, we get

$$g(x) - f(x) = g(x_3) - f(x_3) + \int_{x_3}^{x} (g'(t) - f'(t))dt \ge 0$$

when $x_3 \le x \le 0$, and

$$g(x) - f(x) = g(x_1) - f(x_1) + \int_{0}^{x} (f'(t) - g'(t))dt \ge 0$$

when $0 \le x \le x_1$. Therefore, by convexity, the whole curve $y = f(x)$ belongs to $\lambda K + \mathbf{y}$. The lemma is proved. $\diamondsuit$

Corollary 1. *Let K be an n-dimensional convex body, λ be a real number satisfying $0 < \lambda < 1$, R be a closed region on $\partial(K)$ with boundary Γ and a relatively interior point $\mathbf{p}$. If*

$$\Gamma \cup \{\mathbf{p}\} \subset \lambda K + \mathbf{y}$$

holds for some point $\mathbf{y}$, then we have

$$R \subset \lambda K + \mathbf{y}.$$

Proof of Theorem 1. Let K be a three-dimensional cone over a convex domain D. By Lemma 1, there is an affine regular hexagon H inscribed in D.

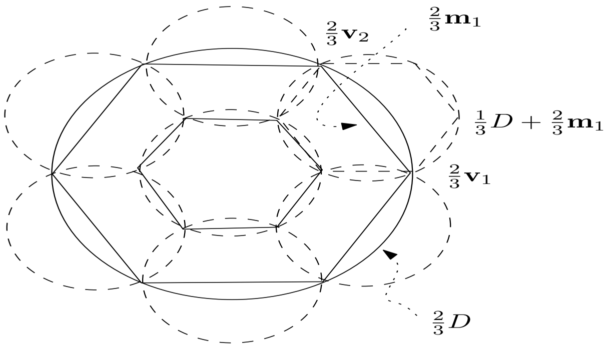

Figure 2

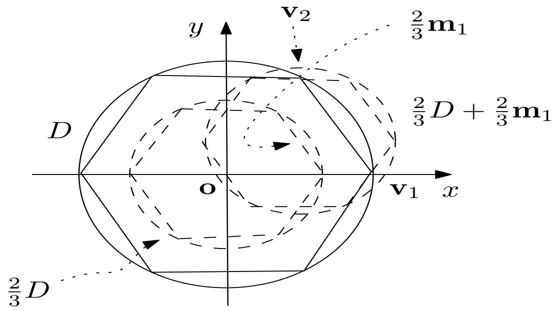

Figure 3

Without loss of generality, we assume that $\mathbf{v} = (0,0,1)$ is the vertex of K, H is perpendicular to $\mathbf{v}$ and centered at the origin $\mathbf{o} = (0,0,0)$.

Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_6$ be the six vertices of H and let $\mathbf{m}_1$, $\mathbf{m}_2$, $\cdots$, $\mathbf{m}_6$ denote the midpoints of $\mathbf{v}_1\mathbf{v}_2$, $\mathbf{v}_2\mathbf{v}_3$, $\cdots$, $\mathbf{v}_6\mathbf{v}_1$, respectively. By elementary argument, as shown in Figure 2, we have

$$\left\{\tfrac{2}{3}\mathbf{v}_1, \tfrac{2}{3}\mathbf{v}_2\right\} \subset \tfrac{1}{3}D + \tfrac{2}{3}\mathbf{m}_1.$$

Thus, by Lemma 2 we get

$$\tfrac{2}{3}D + \tfrac{1}{3}\mathbf{v} \subseteq \bigcup_{i=0}^{6} \left(\tfrac{1}{3}D + \tfrac{2}{3}\mathbf{m}_i + \tfrac{1}{3}\mathbf{v}\right), \tag{6}$$

where $\mathbf{m}_0 = (0,0,0)$. Similarly, as shown in Figure 3, we have

$$\{\mathbf{v}_1, \mathbf{v}_2\} \subset \tfrac{2}{3}D + \tfrac{2}{3}\mathbf{m}_1$$

and therefore

$$D \subseteq \bigcup_{i=0}^{6} \left(\tfrac{2}{3}D + \tfrac{2}{3}\mathbf{m}_i\right). \tag{7}$$

On the other hand, we have

$$\tfrac{1}{3}D + \tfrac{2}{3}\mathbf{m}_i + \tfrac{1}{3}\mathbf{v} \subset \tfrac{2}{3}K + \tfrac{2}{3}\mathbf{m}_i \tag{8}$$

and

$$\tfrac{2}{3}D + \tfrac{2}{3}\mathbf{m}_i \subset \tfrac{2}{3}K + \tfrac{2}{3}\mathbf{m}_i. \tag{9}$$

Therefore, by (6), (7), (8), (9) and convexity we get

$$K \subseteq \bigcup_{i=0}^{7} \left(\tfrac{2}{3}K + \tfrac{2}{3}\mathbf{m}_i\right),$$

where $\mathbf{m}_7 = \frac{1}{2}\mathbf{v}$. Theorem 1 is proved. $\diamondsuit$

Proof of Theorem 2.

Case 1. $1 \le p \le 2$. In this case, we take

$$\Gamma = \left\{\mathbf{x} = (x_1, x_2, x_3) : \ x_1 = \left(\tfrac{1}{3}\right)^{\frac{1}{p}}, \ \mathbf{x} \in \partial(K_p)\right\},$$

$\lambda = \sqrt{\frac{2}{3}}$, $\mathbf{y} = ((\frac{1}{3})^{\frac{1}{p}}, 0, 0)$, and let R denote the part of $\partial(K_p)$ bounded by Γ and containing $(1,0,0)$.

For any point $\mathbf{x} \in \Gamma$, we have

$$\left(\left(\tfrac{1}{3}\right)^{\frac{1}{p}}\right)^p + |x_2|^p + |x_3|^p = 1,$$

$$|x_2|^p + |x_3|^p = \tfrac{2}{3} \le \left(\sqrt{\tfrac{2}{3}}\right)^p$$

and therefore

$$\Gamma \subset \lambda K_p + \mathbf{y}.$$

On the other hand, it can be verified that

$$(1,0,0) \in \lambda K_p + \mathbf{y}.$$

By Corollary 1 we have

$$R \subset \lambda K_p + \mathbf{y}.$$

Therefore, in this case K_p can be covered by the union of $\lambda K_p \pm \left((\frac{1}{3})^{\frac{1}{p}}, 0, 0\right)$, $\lambda K_p \pm \left(0, (\frac{1}{3})^{\frac{1}{p}}, 0\right)$ and $\lambda K_p \pm \left(0, 0, (\frac{1}{3})^{\frac{1}{p}}\right)$ and thus

$$\gamma_8(K_p) \le \gamma_6(K_p) \le \sqrt{\tfrac{2}{3}}.$$

Case 2. $2 \le p \le \infty$. In this case we define

$$\Gamma_i = \{\mathbf{x} = (x_1, x_2, x_3) : \ x_i = 0, \ x_j \ge 0, j \ne i, \ \mathbf{x} \in \partial(K_p)\},$$

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

$\lambda = \sqrt{\frac{2}{3}}$, $\mathbf{y} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and let R denote the part of $\partial(K_p)$ bounded by Γ and containing the point $((\frac{1}{3})^{\frac{1}{p}}, (\frac{1}{3})^{\frac{1}{p}}, (\frac{1}{3})^{\frac{1}{p}})$.

Let J denote the intersection of K_p with the plane $x_1 = 0$, and let J' denote the intersection of $\lambda K_p + \mathbf{y}$ with the plane. It is easy to see that J' is homothetic to J. By a routine computation, for all $2 \le p \le +\infty$, it can be shown that

$$\left(\tfrac{2}{3}\right)^p + 2\left(\tfrac{1}{3}\right)^p \le \left(\tfrac{2}{3}\right)^{\frac{p}{2}}.$$

Thus, both $(0,1,0)$ and $(0,0,1)$ belong to J'. Consequently, we also have

$$\left(0, \left(\tfrac{1}{2}\right)^{\frac{1}{p}}, \left(\tfrac{1}{2}\right)^{\frac{1}{p}}\right) \in J'.$$

By Lemma 2, we get

$$\Gamma_1 \subset J' \subset \lambda K_p + \mathbf{y}$$

and therefore

$$\Gamma \subset \lambda K_p + \mathbf{y}.$$

On the other hand, it can be verified that $\mathbf{o} \in \lambda K_p + \mathbf{y}$,

$$3\left(\left(\tfrac{1}{3}\right)^{\frac{1}{p}} - \tfrac{1}{3}\right)^p \le \left(\tfrac{2}{3}\right)^{\frac{p}{2}}$$

and therefore

$$\left(\left(\tfrac{1}{3}\right)^{\frac{1}{p}}, \left(\tfrac{1}{3}\right)^{\frac{1}{p}}, \left(\tfrac{1}{3}\right)^{\frac{1}{p}}\right) \in \lambda K_p + \mathbf{y}.$$

By Corollary 1 we get

$$R \subset \lambda K_p + \mathbf{y}.$$

Thus, in this case K_p can be covered by the union of the eight translates $\sqrt{\frac{2}{3}}K_p + \left(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$ and hence

$$\gamma_8(K_p) \le \sqrt{\tfrac{2}{3}}.$$

As a conclusion of the two cases, Theorem 2 is proved. ◊

Remark 5. It was shown by Schütte [21] that

$$\gamma_8(K_2) \le \sin 48°9' = 0.744894\cdots < \sqrt{\tfrac{2}{3}}.$$

Thus, following the proof of Theorem 2 we can get that $\sqrt{\frac{2}{3}}$ is not the optimal upper bound for $\gamma_8(K_p)$. However, perhaps one can take $c_3 = \sqrt{\frac{2}{3}}$.

Remark 6. Let T denote a regular tetrahedron. The next table lists the values of $\gamma_m(K)$ for some small m and some particular K, where the values of $\gamma_4(K_2)$ and $\gamma_6(K_2)$ were discovered by L. Fejes Tóth [10] and the values of $\gamma_5(K_2)$ and $\gamma_7(K_2)$ were discovered by Schütte [21].

Remark 7. If Hadwiger's conjecture is true for all dimensions, then we have

$$\Gamma(n, 2^n) < 1$$

for all n. Nevertheless, it seems that

$$\lim_{n\to\infty} \Gamma(n, 2^n) = 1.$$

m	4	5	6	7	8
$\gamma_m(T)$	$\frac{3}{4}$	$\frac{9}{13}$	?	?	?
$\gamma_m(K_1)$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\gamma_m(K_2)$	$0.9428\cdots$	$0.8944\cdots$	$0.8164\cdots$	$0.7775\cdots$	?

References

[1] J.F. Belousov, Theorems on the covering of plane figures, *Ukrain. Geom. Sb.* **20** (1977), 10-17.

[2] A.S. Besicovitch, Measure of asymmetry of convex curves, *J. London Math. Soc.* **23** (1948), 237-240.

[3] K. Bezdek, The illumination conjecture and its extensions, *Period. Math. Hungar.* **53** (2006), 59-69.

[4] K. Bezdek and T. Bisztriczky, A proof of Hadwiger's covering conjecture for dual cyclic polytopes, *Geom. Dedicata* **68** (1997), 29-41.

[5] V. Boltyanski, Solution of the illumination problem for bodies with md$M$ = 2, *Discrete Comput. Geom.* **26** (2001), 527-541.

[6] V. Boltyanski and H. Martini, Illumination of direct vector sums of convex bodies, *Studia Sci. Math. Hungar.* **44** (2007), 367-376.

[7] V.G. Boltyanski, H. Martini and P.S. Soltan, *Excursions into Combinatorial Geometry,* Springer-Verlag, Berlin, 1997.

[8] P. Brass, W. Moser and J. Pach, *Research Problems in Discrete Geometry*, Springer-Verlag, New York, 2005.

[9] H.G. Eggleston, *Convexity*, Cambridge University Press, Cambridge, 1958.

[10] L. Fejes Tóth, Kreisüberdeckungen der hyperbolischen Ebene, *Acta Math. Acad. Sci. Hungar.* **4** (1953), 111-114.

[11] H. Hadwiger, Ungelöste Probleme No. 20, *Elem. Math.* **12** (1957), 121.

[12] F. John, Extremum problems with inequalities as subsidiary conditions, *Courant Anniversary Volume*, Interscience, New York, 1948, 187-204.

[13] S. Krotoszynski, Covering a plane convex body with five smaller homothetical copies, *Beiträge Algebra Geom.* **25** (1987), 171-176.

[14] M. Lassak, Solution of Hadwiger's covering problem for centrally symmetric convex bodies in E^3, *J. London Math. Soc.* **30** (1984), 501-511.

[15] M. Lassak, Covering a plane convex body by four homothetical copies with the smallest positive ratio, *Geom. Dedicata* **21** (1986), 157-167.

[16] F.W. Levi, Ein geometrisches Überdeckungsproblem, *Arch. Math.* **5** (1954), 476-378.

[17] H. Martini and V. Soltan, Combinatorial problems on the illumination of convex bodies, *Aequationes Math.* **57** (1999), 121-152.

[18] I. Papadoperakis, An estimate for the problem of illumination of the boundary of a convex body in E^3, *Geom. Dedicata*, **75** (1999), 275-285.

[19] C.A. Rogers and C. Zong, Covering convex bodies by translates of convex bodies, *Mathematika*, **44** (1997), 215-218.

[20] O. Schramm, Illuminating sets of constant width, *Mathematika* **35** (1988), 180-189.

[21] K. Schütte, Überdeckungen der Kugel mit höchstens acht Kreisen, *Math. Ann.* **129** (1955), 181-186.

[22] I. Talata, Solution of Hadwiger-Levi's covering problem for duals of cyclic $2k$-polytopes, *Geom. Dedicata* **74** (1999), 61-71.

[23] C. Zong, Some remarks concerning kissing numbers, blocking numbers and covering numbers, *Period. Math. Hungar.* **30** (1995), 233-238.

[24] C. Zong, The kissing number, blocking number and covering number of a convex body, *Contemp. Math.* **453** (2008), 529-548.