ENTROPY AND RENORMALIZED SOLUTIONS FOR THE $p(x)$-LAPLACIAN EQUATION WITH MEASURE DATA

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Abstract. In this paper we prove the existence and uniqueness of both entropy solutions and renormalized solutions for the $p(x)$-Laplacian equation with variable exponents and a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. And moreover, we obtain the equivalence of entropy solutions and renormalized solutions.

1. Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. In this paper we study the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \mu & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where the variable exponent $p : \overline{\Omega} \to (1, +\infty)$ is a continuous function, and μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$.

The study of differential equations and variational problems with nonstandard growth conditions arouses much interest with the development of elastic mechanics, electro-rheological fluid dynamics and image processing, etc. We refer the readers to [31, 32, 35, 14] and references therein. $p(x)$-growth conditions can be regarded as a very important class of nonstandard (p, q)-growth conditions. There are already numerous results for such kind of problems (see [1, 2, 3, 19, 22, 18, 5]). The functional spaces to deal with these problems are the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$.

Under our assumptions, it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of renormalized solutions was first introduced by DiPerna and Lions [17] for the study of Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics. We refer to [13, 15, 8, 10, 9, 28] for details. At the same time the notion of entropy solutions has been proposed by Bénilan et al. in [7] for the nonlinear elliptic problems. This framework was extended to related problems with constant p in [12, 11, 4, 30]. The interesting and difficult cases are those of $1 < p \leq N$, since the variational methods of Leray-Lions (see [27]) easily apply for $p > N$, with the solution being bounded and its gradient being in $L^p(\Omega)$.

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Recently, Sanchón and Urbaon in [33] studied a Dirichlet problem of $p(x)$-Laplace equation and obtained the existence and uniqueness of entropy solutions for L^1 data, as well as integrability results for the solution and its gradient. The proofs rely crucially on *a priori* estimates in Marcinkiewicz spaces with variable exponents. Besides, Bendahmane and Wittbold in [6] proved the existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1 data.

The aim of this paper is to extend the results in [7, 12, 33, 15, 6], taking into account a signed measure μ in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. As far as we know, there are few papers concerned with the nonlinear elliptic or parabolic equations involving measure data with variable exponents. So, the study of problem (1.1) is a new and interesting topic. The uniqueness of entropy solutions and renormalized solutions will strongly rely on the structure of the measure μ, that is, μ belongs to $L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$. If not, there is a counterexample in [12] showing that why the definition of entropy solution is not suitable for uniqueness if the measure μ is the Dirac mass.

Inspired by [7], [12], [26] and [33], we develop a refined method. We first construct an approximate solution sequence for problem (1.1) and establish some *a priori* estimates. Next, we draw a subsequence to obtain a limit function, and prove this function is an entropy solution. The advantage of our method is that we can give the concrete expression for the limit function of approximate solutions by means of truncation techniques and prove the strong convergence of the truncations of approximate solutions. Based on this fact, we obtain that the entropy solution of problem (1.1) is also a renormalized solution. It is worth pointing out that we generalize the definition of renormalized solutions from L^1 functions to measurable functions. This allows us to drop some redundant conditions imposed on the exponent $p(x)$ in the previous work. Even for the constant exponent case, to the best of our knowledge, there seems no result about the existence and uniqueness of renormalized solutions to such elliptic equations with a signed measure μ in $L^1(\Omega) + W^{-1,p'}(\Omega)$. By choosing suitable test functions and employing the decomposition for the region of integration according to the different contributions corresponding to different integration sets, we prove the uniqueness of entropy solutions and renormalized solutions, and thus the equivalence of entropy solutions and renormalized solutions.

For the convenience of the readers, we recall some definitions and basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions such that

$$\int_\Omega |u(x)|^{p(x)}\, dx < \infty,$$

endowed with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

which is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. If $p(x)$ is a constant function, then the variable exponent

Lebesgue space coincides with the classical Lebesgue space. The variable exponent Lebesgue spaces is a special case of Orlicz-Musielak spaces treated by Musielak in [29].

For any positive integer k, denote

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

where the norm is defined as

$$\|u\|_{W^{k,p(x)}} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)}.$$

$W^{k,p(x)}(\Omega)$ is also a Banach space. It is called generalized Lebesgue-Sobolev space, which is a special generalized Orlicz-Sobolev space. An interesting feature of a generalized Lebesgue-Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(x)$. This was observed by Zhikov [34] in connection with Lavrentiev phenomenon. However, when the exponent $p(x)$ is **log-Hölder** continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|} \tag{1.2}$$

for every $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(\cdot)}}$ (see [23]).

Throughout this paper we assume that $p(x) \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < N$ satisfies the log-Hölder continuity condition (1.2), and μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, i.e.,

$$\mu = f - \operatorname{div} F, \tag{1.3}$$

where $f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$. Let T_k denote the truncation function at height $k \ge 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \ge k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \le -k. \end{cases}$$

We denote

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is measurable } \mid T_k(u) \in W_0^{1,p(\cdot)}(\Omega), \text{for every } k > 0\}.$$

Next we define the very weak gradient for every measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$. The proof follows from Lemma 2.1 of [7] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega)$.

Proposition 1.1. *For a measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{for a.e. } x \in \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the weak gradient of u.

The notion of the very weak gradient allows us to give the following definitions of entropy solutions and renormalized solutions for problem (1.1).

Definition 1.2. *A measurable function* $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ *is an entropy solution to problem* (1.1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla T_k(u-\phi)\,dx = \int_\Omega fT_k(u-\phi)\,dx + \int_\Omega F\cdot\nabla T_k(u-\phi)\,dx, \tag{1.4}$$

for all $\phi \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

Definition 1.3. *A measurable function* $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ *is a renormalized solution to problem* (1.1) *if the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\int_{\{n\le|u|\le n+1\}} |\nabla u|^{p(x)}\,dx = 0$;

(ii) *For every function* $\varphi \in C_0^\infty(\Omega)$ *and* $S \in W^{1,\infty}(\mathbb{R})$ *satisfying that* S *has a compact support,*

$$\int_\Omega [S(u)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi + S'(u)|\nabla u|^{p(x)}\varphi]\,dx = \int_\Omega [fS(u)\varphi + F\cdot\nabla(S(u)\varphi)]\,dx \tag{1.5}$$

holds.

Now we state our main results as follows.

Theorem 1.4. *Assume that* μ *satisfies* (1.3). *Then there exists a unique entropy solution* $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ *for problem* (1.1).

Theorem 1.5. *Assume that* μ *satisfies* (1.3). *Then the entropy solution* u *in* Theorem 1.4 *is also a renormalized solution for problem* (1.1). *And the renormalized solution is unique.*

Remark 1.6. *The entropy solution for problem* (1.1) *is equivalent to the renormalized solution for problem* (1.1).

Remark 1.7. *When* $p(x)$ *is a constant, we know from* [12] *that* $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ *if and only if* $\mu \in \mathcal{M}_b^p(\Omega)$, *i.e., every signed measure that is zero on the sets of zero* p*-capacity can be decomposed into the sum of a function in* $L^1(\Omega)$ *and an element in* $W^{-1,p'}(\Omega)$, *and conversely, every signed measure in* $L^1(\Omega)+W^{-1,p'}(\Omega)$ *has zero measure for the sets of zero* p*-capacity.*

The rest of this paper is organized as follows. In Section 2, we state some basic results that will be used later. We will prove the main results in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

In this section, we first state some elementary results for the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$. The basic properties of these spaces can be found from [25] by Kováčik and Rákosník, and many of these properties were independently established in [22] by Fan and Zhao.

Lemma 2.1 ([22, 25])**.** (1) *The space* $L^{p(\cdot)}(\Omega)$ *is a separable, uniform convex Banach space, and its conjugate space is* $L^{p'(\cdot)}(\Omega)$ *where* $1/p(x)+1/p'(x)=1$. *For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)};$$

(2) *If* $p_1, p_2 \in C_+(\overline{\Omega}), p_1(x) \le p_2(x)$ *for any* $x \in \Omega$, *then there exists the continuous embedding* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, *whose norm does not exceed* $|\Omega| + 1$.

Lemma 2.2 ([22]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then

$$\min\{|u|_{p(x)}^{p_-}, |u|_{p(x)}^{p_+}\} \le \rho(u) \le \max\{|u|_{p(x)}^{p_-}, |u|_{p(x)}^{p_+}\}.$$

Lemma 2.3 ([22]). $W^{k,p(x)}(\Omega)$ *is a separable and reflexive Banach space.*

Lemma 2.4 ([24, 25]). *Let* $p \in C_+(\overline{\Omega})$ *satisfy the log-Hölder continuity condition* (1.2). *Then, for* $u \in W_0^{1,p(\cdot)}(\Omega)$, *the* $p(\cdot)$*-Poincaré inequality*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}$$

holds, where the positive constant C *depends on* p *and* Ω.

Remark 2.5. *Note that the following inequality*

$$\int_\Omega |u|^{p(x)}\, dx \le C \int_\Omega |\nabla u|^{p(x)}\, dx$$

in general does not hold.

Lemma 2.6 ([16, 21]). *Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded set with Lipschitz boundary and* $p(x) \in C_+(\overline{\Omega})$ *with* $1 < p_- \le p_+ < N$ *satisfy the log-Hölder continuity condition* (1.2). *If* $q \in L^\infty(\Omega)$ *with* $q_- > 1$ *satisfies*

$$q(x) \le p^*(x) := \frac{Np(x)}{N - p(x)}, \quad \forall x \in \Omega,$$

then we have

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

and the imbedding is compact if $\inf_{x \in \Omega} (p^*(x) - q(x)) > 0$.

Now we begin to prove some *a priori* important estimates for entropy solutions of problem (1.1).

Proposition 2.7. *If* u *is an entropy solution of problem* (1.1), *then there exists a positive constant* C *such that for all* $k > 1$,

$$meas\{|u| > k\} \le \frac{C(M+1)^{\frac{p_-^*}{p_-}}}{k^{p_-^*(1-\frac{1}{p_-})}}$$

holds, where

$$M = \frac{p_-}{p_- - 1}\|f\|_{L^1(\Omega)} + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_\Omega |F|^{p'(x)}\, dx.$$

Proof. Choosing $\phi = 0$ in the entropy equality (1.4) and using Young's inequality, we obtain that

$$\int_\Omega |\nabla T_k(u)|^{p(x)}\, dx = \int_{\{|u| \le k\}} |\nabla u|^{p(x)}\, dx$$

$$\le \frac{kp_-}{p_- - 1}\|f\|_{L^1(\Omega)} + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)}\int_\Omega |F|^{p'(x)}\,dx,$$

which implies that for all $k > 1$,

$$\frac{1}{k}\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx \le M_1 + \frac{M_2}{k} \le M := M_1 + M_2, \tag{2.1}$$

where

$$M_1 = \frac{p_-}{p_- - 1}\|f\|_{L^1(\Omega)}, \quad M_2 = \frac{p_-(p_+ - 1)}{p_+(p_- - 1)}\int_\Omega |F|^{p'(x)}\,dx.$$

Recalling Sobolev embedding theorem in Lemma 2.6 and Lemma 2.1, we have the following continuous embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{p^*_-}(\Omega),$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p^*_- = \frac{Np_-}{N-p_-}$. It follows from Lemma 2.2 and (2.1) that for every $k > 1$,

$$\begin{aligned}\|T_k(u)\|_{p^*_-} &= |T_k(u)|_{p^*_-} \le C|\nabla T_k(u)|_{p(x)}\\ &\le C\Big(\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx\Big)^\beta \le C(Mk)^\beta,\end{aligned}$$

where

$$\beta = \begin{cases} \frac{1}{p_-} & \text{if } |\nabla T_k(u)|_{p(\cdot)} \ge 1,\\ \frac{1}{p_+} & \text{if } |\nabla T_k(u)|_{p(\cdot)} \le 1.\end{cases}$$

Noting that $\{|u| \ge k\} = \{|T_k(u)| \ge k\}$, we have

$$\operatorname{meas}\{|u| > k\} \le \Big(\frac{\|T_k(u)\|_{p^*_-}}{k}\Big)^{p^*_-} \le \frac{CM^{\beta p^*_-}}{k^{p^*_-(1-\beta)}} \le \frac{C(M+1)^{\frac{p^*_-}{p_-}}}{k^{p^*_-(1-\frac{1}{p_-})}}.$$

This completes the proof. ☐

Proposition 2.8. *If u is an entropy solution of problem* (1.1), *then for a given $a > 0$ we have*

$$\lim_{k\to+\infty}\int_{\{k\le|u|\le k+a\}} |\nabla u|^{p(x)}\,dx = 0.$$

Proof. For given $a, k > 0$, define the function $T_{k,a}(s) = T_a(s - T_k(s))$ as

$$T_{k,a}(s) = \begin{cases} s - k\operatorname{sign}(s) & \text{if } k \le |s| < k+a,\\ a & \text{if } |s| \ge k+a,\\ 0 & \text{if } |s| \le k.\end{cases}$$

Using $T_{k,a}(u)$ to take the place of $T_k(u-\phi)$ in (1.4), we find

$$\int_{\{k\le|u|\le k+a\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla u\,dx \le \int_\Omega fT_{k,a}(u)\,dx + \int_{\{k\le|u|\le k+a\}} F\cdot\nabla u\,dx.$$

In view of Young's inequality, we have

$$\begin{aligned}&\int_{\{k\le|u|\le k+a\}} |\nabla u|^{p(x)}\,dx\\ &\le a\int_{\{|u|>k\}} |f|\,dx + \int_{\{k\le|u|\le k+a\}} \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{p(x)-1}{p(x)}|F|^{p'(x)}\,dx\end{aligned}$$

$$\leq a\int_{\{|u|>k\}}|f|\,dx+\frac{1}{p_-}\int_{\{k\leq|u|\leq k+a\}}|\nabla u|^{p(x)}\,dx+\frac{p_+-1}{p_+}\int_{\{k\leq|u|\leq k+a\}}|F|^{p'(x)}\,dx,$$

which implies that

$$\begin{aligned}&\int_{\{k\leq|u|\leq k+a\}}|\nabla u|^{p(x)}\,dx\\&\leq\frac{ap_-}{p_--1}\int_{\{|u|>k\}}|f|\,dx+\frac{p_-(p_+-1)}{p_+(p_--1)}\int_{\{k\leq|u|\leq k+a\}}|F|^{p'(x)}\,dx.\end{aligned}$$

Therefore, we obtain that

$$\lim_{k\to+\infty}\int_{\{k\leq|u|\leq k+a\}}|\nabla u|^{p(x)}\,dx=0.$$

Thus we finish the proof of the proposition. □

3. The proofs of main results

Now we are ready to prove the main results. First we prove the existence and uniqueness of entropy solutions for problem (1.1).

Proof of Theorem 1.4.

(1) Existence of entropy solutions.

We first introduce the approximate problems. Find two sequences of $C_0^\infty(\Omega)$ functions $\{f_n\}$ and $\{F_n\}$ strongly converging respectively to f in $L^1(\Omega)$ and to F in $(L^{p'(\cdot)}(\Omega))^N$ such that

$$\|f_n\|_{L^1(\Omega)}\leq\|f\|_{L^1(\Omega)},\int_\Omega|F_n|^{p'(x)}\,dx\leq\int_\Omega|F|^{p'(x)}\,dx. \tag{3.1}$$

Then we consider approximate problem of (1.1)

$$\begin{cases}-\operatorname{div}\left(|\nabla u_n|^{p(x)-2}\nabla u_n\right)=f_n-\operatorname{div}(F_n) & \text{in}\quad\Omega,\\ u_n=0 & \text{on}\quad\partial\Omega.\end{cases} \tag{3.2}$$

By employing the variation methods and arguments in Theorem 4.2 of [20] or [27], we easily find a unique weak solution $u_n\in W_0^{1,p(\cdot)}(\Omega)$ of problem (3.2), which is obviously an entropy solution of problem (3.2). Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution of problem (1.1). We will divide the proof into several steps. Some of the reasoning is based on the ideas developed in [33], [7], [12] and [26].

Choosing $T_k(u_n)$ as a test function in (3.2) and applying Young's inequality in the right-hand side we have

$$\int_\Omega|\nabla T_k(u_n)|^{p(x)}\,dx\leq k\|f_n\|_{L^1(\Omega)}+\int_\Omega\frac{1}{p(x)}|\nabla T_k(u_n)|^{p(x)}\,dx+\int_\Omega\frac{1}{p'(x)}|F_n|^{p'(x)}\,dx,$$

which implies that

$$\begin{aligned}\int_\Omega|\nabla T_k(u_n)|^{p(x)}\,dx&\leq\frac{kp_-}{p_--1}\|f_n\|_{L^1(\Omega)}+\frac{p_-(p_+-1)}{p_+(p_--1)}\int_\Omega|F_n|^{p'(x)}\,dx\\&\leq\frac{kp_-}{p_--1}\|f\|_{L^1(\Omega)}+\frac{p_-(p_+-1)}{p_+(p_--1)}\int_\Omega|F|^{p'(x)}\,dx.\end{aligned} \tag{3.3}$$

Step 1. Construct a subsequence $\{u^n\}$ of $\{u_n\}$ and a limit function u such that, for every positive integer k,

$$T_k u^n \to T_k u \quad \text{strongly in } L^q(\Omega),$$

where $q = \frac{p_- + p_-^*}{2}$.

As $\{T_1 u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ from (3.3), recalling Lemma 2.6, there exist a subsequence $\{u_n^1\}$ of $\{u_n\}$ and a function $v_1 \in L^q(\Omega)$ with $|v_1| \le 1$ such that

$$T_1 u_n^1 \to v_1 \quad \text{strongly in } L^q(\Omega).$$

As $\{T_2 u_n^1\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, there exist a subsequence $\{u_n^2\}$ of $\{u_n^1\}$ and a function $v_2 \in L^q(\Omega)$ with $|v_2| \le 2$ such that

$$T_2 u_n^2 \to v_2 \quad \text{strongly in } L^q(\Omega).$$

By the same procedures, for every positive integer k, we find there exist a subsequence $\{u_n^k\}$ of $\{u_n^{k-1}\}$ and a function $v_k \in L^q(\Omega)$ with $|v_k| \le k$ such that

$$T_k u_n^k \to v_k \quad \text{strongly in } L^q(\Omega).$$

Set $u^n = u_n^n$, $n = 1, 2, \cdots$. Then we have, for every positive integer k,

$$T_k u^n \to v_k \quad \text{strongly in } L^q(\Omega).$$

Let l, k be any two positive integers with $l < k$. Noting the equality

$$T_l(T_k u^n) = T_l u^n,$$

and sending $n \to \infty$, we conclude that

$$T_l v_k = v_l.$$

It follows from the construction of v_k that

$$\begin{aligned}
\Omega &= \{0 \le |v_1| < 1\} \cup \{|v_1| = 1\} \cup \{\text{a negligible set}\} \\
&= \{0 \le |v_1| < 1\} \cup \{1 \le |v_2| < 2\} \cup \{|v_2| = 2\} \cup \{\text{a negligible set}\} \\
&= \cdots\cdots \\
&= \bigcup_{k=1} \{k-1 \le |v_k| < k\} \bigcup \{\text{a negligible set}\}.
\end{aligned}$$

Then we define a measurable function u in Ω by

$$u(x) = \begin{cases} v_k(x) & \text{when } k-1 \le |v_k(x)| < k, \text{ for } k = 1, 2, \cdots, \\ 0 & \text{a negligible set,} \end{cases}$$

which satisfies that

$$T_k u = v_k, \text{ for } k = 1, 2, \cdots.$$

Step 2. Prove the convergence in measure of $\{u^n\}$ and find its subsequence which is almost everywhere convergent in Ω.

For every fixed $\epsilon > 0$, and every positive integer k, we know that

$$\{|u^n - u^m| > \epsilon\} \subset \{|u^n| > k\} \cup \{|u^m| > k\} \cup \{|T_k(u^n) - T_k(u^m)| > \epsilon\}.$$

Recalling the convergence of $\{T_k u^n\}$ in $L^q(\Omega)$ and Proposition 2.7, we conclude that

$$\limsup_{n,m\to\infty} \operatorname{meas}\{|u^n - u^m| > \epsilon\} \le C(p(\cdot), \|f\|_{L^1(\Omega)}, |F|_{p(\cdot)}) k^{-\alpha},$$

where $\alpha = p_-^*(1 - \frac{1}{p_-}) > 0$.

Because of the arbitrariness of k, we prove that

$$\limsup_{n,m\to\infty} \operatorname{meas}\{|u^n - u^m| > \epsilon\} = 0,$$

which implies the convergence in measure of $\{u^n\}$, and then we find an a.e. convergent subsequence (still denoted by $\{u^n\}$) in Ω such that

$$u^n \to u \quad \text{a.e. in } \Omega. \tag{3.4}$$

In the following we denote $\{u^n\}$ by $\{u_n\}$ for simplicity of notation.

Step 3. We prove that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(x)}(\Omega)$, for every $k > 0$.

Since set $\{(k,h)|k,h = 1,2,\cdots\}$ is countable, using the boundedness of $T_k(u_n)$ and $T_{2k}(u_n - T_h(u_n))$ in $W_0^{1,p(\cdot)}(\Omega)$ and the diagonal procedures, we draw a subsequence (still denoted by $\{u_n\}$) from $\{u_n\}$ such that, for every pair of positive integers k, h,

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega), \tag{3.5}$$

$$T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u)) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega) \tag{3.6}$$

and

$$T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rightharpoonup T_{2k}(u - T_h(u)) \text{ weakly in } W_0^{1,p(\cdot)}(\Omega). \tag{3.7}$$

First we prove that this subsequence $\{u_n\}$ satisfies that $T_k u_n$ strongly converges to $T_k u$ in $W_0^{1,p(\cdot)}(\Omega)$ for every positive integer k.

Fix a positive integer k. Let h be a positive integer satisfying $h > k$. We choose

$$w_n = T_{2k}\big(u_n - T_h(u_n) + T_k(u_n) - T_k(u)\big)$$

as a test function in (3.2). If we set $M = 4k + h$, then it is easy to see that $\nabla w_n = 0$ where $|u_n| > M$. Therefore, we may write the weak form of (3.2) as

$$\int_\Omega |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla w_n \, dx = \int_\Omega f_n w_n \, dx + \int_\Omega F_n \cdot \nabla w_n \, dx.$$

Splitting the integral in the left-hand side on the sets where $|u_n| \le k$ and where $|u_n| > k$ and discarding some nonnegative terms, we find

$$\begin{aligned}
&\int_\Omega |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
&\ge \int_\Omega |\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) \cdot \nabla (T_k(u_n) - T_k(u)) \, dx \\
&\quad - \int_{\{|u_n|>k\}} \big||\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)\big| |\nabla T_k(u)| \, dx.
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
&\int_\Omega (|\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u)) \cdot \nabla (T_k(u_n) - T_k(u)) \, dx \\
&\le \int_{\{|u_n|>k\}} \big||\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)\big| |\nabla T_k(u)| \, dx \\
&\quad + \int_\Omega f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
&\quad + \int_\Omega F_n \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \cdot \nabla (T_k(u_n) - T_k(u))\, dx \\
&= I_1 + I_2 + I_3 + I_4. \qquad (3.8)
\end{aligned}
$$

Now we show the limits of I_1, I_2, I_3 and I_4 are zeros when n, and then h tend to infinity respectively.

Limit of I_1. We observe that $|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)$ is bounded in $L^{p'(x)}(\Omega)$, and by the Lebesgue dominated convergence theorem $\chi_{\{|u_n|>k\}}|\nabla T_k(u)|$ converges strongly in $L^{p(x)}(\Omega)$ to $\chi_{\{|u|>k\}}|\nabla T_k(u)|$, which is zero, as n tends to infinity. Thus we obtain

$$
\lim_{n\to+\infty} I_1 = \lim_{n\to+\infty} \int_{\{|u_n|>k\}} \big||\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\big| |\nabla T_k(u)|\, dx = 0. \qquad (3.9)
$$

Limit of I_2. Notice that

$$
\begin{aligned}
I_2 &\le \int_\Omega |f_n - f||T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))|\, dx \\
&\quad + \int_\Omega |f T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))|\, dx \\
&\le 2k \int_\Omega |f_n - f|\, dx + \int_\Omega |f T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))|\, dx.
\end{aligned}
$$

Since f_n is strongly compact in $L^1(\Omega)$, using (3.4) and the Lebesgue dominated convergence theorem, we have

$$
\lim_{h\to+\infty} \lim_{n\to+\infty} |I_2| \le \lim_{h\to+\infty} \int_\Omega |f T_{2k}(u - T_h(u))|\, dx = 0. \qquad (3.10)
$$

Limit of I_3. If we take $T_{2k}(u_n - T_h(u_n))$ as a test function in (3.2), we proceed as in the beginning of the proof to have

$$
\int_\Omega |\nabla T_{2k}(u_n - T_h(u_n))|^{p(x)}\, dx \le C\big(k, p(\cdot), \|f\|_{L^1(\Omega)}, |F|_{p(\cdot)}\big),
$$

where C is a positive constant that does not depend on h. Since

$$
T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u)) \quad \text{weakly in } W_0^{1,p(x)}(\Omega),
$$

we know

$$
\int_\Omega |\nabla T_{2k}(u - T_h(u))|^{p(x)}\, dx \le C\big(k, p(\cdot), \|f\|_{L^1(\Omega)}, |F|_{p(\cdot)}\big).
$$

It follows from the strong convergence of F_n in $(L^{p'(x)}(\Omega))^N$, (3.7), Lemma 2.1 and Lemma 2.2 that

$$
\begin{aligned}
\lim_{n\to+\infty} I_3 &= \int_\Omega F \cdot \nabla T_{2k}(u - T_h(u))\, dx \\
&= \int_{\{h\le|u|\le h+2k\}} F \cdot \nabla T_{2k}(u - T_h(u))\, dx \\
&\le 2|F|_{p'(\cdot),\{h\le|u|\le h+2k\}} \max\Big\{\Big(\int_\Omega |\nabla T_{2k}(u - T_h(u))|^{p(x)}\, dx\Big)^{1/p_-}, \\
&\quad \Big(\int_\Omega |\nabla T_{2k}(u - T_h(u))|^{p(x)}\, dx\Big)^{1/p_+}\Big\} \\
&\le C|F|_{p'(\cdot),\{h\le|u|\le h+2k\}}.
\end{aligned}
$$

In view of the absolute continuity of the integral, we have

$$\lim_{h\to+\infty}\lim_{n\to+\infty} I_3 = \lim_{h\to+\infty}\int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))\,dx = 0. \tag{3.11}$$

Limit of I_4. Recalling (3.5), we have

$$\lim_{n\to+\infty} I_4 = 0. \tag{3.12}$$

Therefore, passing to the limits in (3.8) as n, and then h tend to infinity, by means of (3.9), (3.10), (3.11) and (3.12), we deduce that

$$\lim_{n\to+\infty} E(n) = 0,$$

where

$$E(n) = \int_\Omega \big(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\big)\cdot\nabla(T_k(u_n)-T_k(u))\,dx.$$

We recall the following well-known inequalities: for any two real vectors $a,b\in\mathbb{R}^N$,

$$(a|a|^{p-2} - b|b|^{p-2})(a-b) \ge c(p)|a-b|^p,\ \text{ if } p\ge 2$$

and for every $\varepsilon\in(0,1]$,

$$|a-b|^p \le c(p)\varepsilon^{(p-2)/p}(a|a|^{p-2} - b|b|^{p-2})(a-b) + \varepsilon|b|^p,\ \text{ if } 1<p<2,$$

where $c(p) = \frac{2^{1-p}}{p-1}$ when $p\ge 2$ and $c(p) = \frac{3^{2-p}}{p-1}$ when $1<p<2$.

Therefore, we have

$$\int_{\{x\in\Omega: p(x)\ge 2\}} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)}\,dx \le 2^{p_+-1}(p_+-1)E(n) \tag{3.13}$$

and

$$\begin{aligned}&\int_{\{x\in\Omega: 1<p(x)<2\}} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)}\,dx\\ &\le \frac{3^{2-p_-}}{p_- - 1}\cdot\varepsilon^{(p_- -2)/p_-}E(n) + \varepsilon\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx.\end{aligned} \tag{3.14}$$

Since $E(n)\to 0$ as $n\to+\infty$, then using the arbitrariness of ε and $\nabla T_k(u)$ is bounded in $(L^{p(x)}(\Omega))^N$, we conclude that

$$\lim_{n\to+\infty}\int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)}\,dx = 0, \tag{3.15}$$

which implies that, for every positive integer k,

$$T_k(u_n)\to T_k(u)\quad\text{strongly in } W_0^{1,p(\cdot)}(\Omega).$$

Moreover, for every positive real number r, there exists a positive integer k such that $r\le k$. Recalling the fact that $T_r(T_k(\cdot)) = T_r(\cdot)$ and

$$T_r(T_k(u_n))\to T_r(T_k(u))\quad\text{strongly in } W_0^{1,p(\cdot)}(\Omega),$$

we have, for every $r>0$,

$$T_r(u_n)\to T_r(u)\quad\text{strongly in } W_0^{1,p(\cdot)}(\Omega), \tag{3.16}$$

which also implies that,

$$|\nabla T_r(u_n)|^{p(x)-2}\nabla T_r(u_n)\to |\nabla T_r(u)|^{p(x)-2}\nabla T_r(u)\quad\text{in } (L^{p'(x)}(\Omega))^N. \tag{3.17}$$

Step 4. Show that u is an entropy solution.

Now we choose $v_n = T_k(u_n - \phi)$ as a test function in (3.2) for $k > 0$ and $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. We note that, if $L = k + \|\phi\|_{L^\infty(\Omega)}$ and $n > L$, then

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla T_k(u_n - \phi)\, dx = \int_\Omega |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(u_n - \phi)\, dx$$

and

$$\begin{aligned}&\int_\Omega |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(u_n - \phi)\, dx \\ &\quad = \int_\Omega f_n T_k(u_n - \phi)\, dx + \int_\Omega F_n \cdot \nabla T_k(u_n - \phi)\, dx.\end{aligned}$$

Using (3.5) and (3.17), we pass to the limits as n tends to infinity to conclude

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla T_k(u - \phi)\, dx = \int_\Omega f T_k(u - \phi)\, dx + \int_\Omega F \cdot \nabla T_k(u - \phi)\, dx,$$

for every $k > 0$ and every $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Therefore, we finish the proof of the existence of entropy solutions.

(2) Uniqueness of entropy solutions.

Suppose that u and v are two entropy solutions of problem (1.1). We write the entropy equality (1.4) corresponding to solution u with test function $T_h(v)$ and v with test function $T_h(u)$. Add up both equalities, we find

$$\begin{aligned}&\int_{\{|u - T_h(v)| \le k\}} \left[|\nabla u|^{p(x)-2} \nabla u - F \right] \cdot \nabla T_k(u - T_h(v))\, dx \\ &\quad + \int_{\{|v - T_h(u)| \le k\}} \left[|\nabla v|^{p(x)-2} \nabla v - F \right] \cdot \nabla T_k(v - T_h(u))\, dx \\ &= \int_\Omega f \left[T_k(u - T_h(v)) + T_k(v - T_h(u)) \right] dx. \qquad (3.18)\end{aligned}$$

First we consider the right-hand side of (3.18). Noting that

$$T_k(u - T_h(v)) + T_k(v - T_h(u)) = 0 \quad \text{in} \quad \{|u| \le h, |v| \le h\},$$

we obtain

$$\begin{aligned}&\Big| \int_\Omega f \left[T_k(u - T_h(v)) + T_k(v - T_h(u)) \right] dx \Big| \\ &\qquad \le 2k \Big(\int_{\{|u| > h\}} |f|\, dx + \int_{\{|v| > h\}} |f|\, dx \Big).\end{aligned}$$

Since both meas $\{|u| > h\}$ and meas $\{|v| > h\}$ tend to 0 as h goes to infinity from Proposition 2.7, the right-hand side of (3.18) tends to 0 as h tends to ∞.

We will reach the conclusion $u = v$ after discarding some nonnegative but uninteresting terms and passing to the limits in (3.18). We proceed by splitting the integrals above into the contributions corresponding to different integration sets. For the left-hand side of (3.18), let us denote (we omit the dependence on $x \in \Omega$ for the sake of brevity.)

$$A_0 = \{|u - v| \le k, |u| \le h, |v| \le h\},$$

$$A_1 = \{|u - T_h(v)| \le k, |v| > h\}, \quad A_1' = \{|v - T_h(u)| \le k, |u| > h\},$$

$$A_2 = \{|u - T_h(v)| \le k, |v| \le h, |u| > h\}, \quad A_2' = \{|v - T_h(u)| \le k, |u| \le h, |v| > h\}.$$

Then we have $\{|u-T_h(v)|\le k\}=A_0\cup A_1\cup A_2$ and $\{|v-T_h(u)|\le k\}=A_0\cup A_1'\cup A_2'$. On the set A_0 the left-hand side of (3.18) is equal to

$$\int_{A_0}[|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v]\cdot\nabla(u-v)\,dx.$$

On the set A_1, for the first term of the left-hand side of (3.18) we have

$$\int_{A_1}[|\nabla u|^{p(x)-2}\nabla u-F]\cdot\nabla u\,dx\ge-\int_{A_1}F\cdot\nabla u\,dx.$$

By Lemma 2.1, we get

$$\int_{A_1}F\cdot\nabla u\,dx\le 2|F|_{p'(x),A_1}|\nabla u|_{p(x),A_1}.$$

Since measA_1 tends to zero as h tends to infinity, we have that $|F|_{p'(x),A_1}$ tends to zero. If we prove that $|\nabla u|_{p(x),A_1}$ is bounded with respect to h, then the term with A_1 will converge to zero. We decompose A_1 as

$$A_1=\{v>h,|u-h|\le k\}\cup\{v\le -h,|u+h|\le k\}=A_1^+\cup A_1^-.$$

On A_1^+ (and the same for A_1^-) we have $-k\le u-h\le k$, and so $h-k\le u\le h+k$. Hence $A_1^+\subseteq B_{h-k,2k}$, where $B_{h,k}=\{h\le|u|\le h+k\}$. Choosing $\phi=T_h(u)$ in (1.4) and using Young's inequality, we can get

$$\int_{B_{h,k}}|\nabla u|^{p(x)}\,dx\le C.$$

Thus, from Lemma 2.2 we have

$$\begin{aligned}|\nabla u|_{p(x),A_1}&\le C\max\Big\{\Big(\int_{A_1}|\nabla u|^{p(x)}\,dx\Big)^{1/p_-},\Big(\int_{A_1}|\nabla u|^{p(x)}\,dx\Big)^{1/p_+}\Big\}\\&\le C\max\Big\{\Big(\int_{B_{h-k,2k}}|\nabla u|^{p(x)}\,dx\Big)^{1/p_-},\Big(\int_{B_{h-k,2k}}|\nabla u|^{p(x)}\,dx\Big)^{1/p_+}\Big\}\\&\le C,\end{aligned}$$

that is what we need. In the same way we can estimate the second term of (3.18) on the set A_1'. Therefore, we obtain

$$\limsup_{h\to\infty}\Big(\int_{A_1}\big[|\nabla u|^{p(x)-2}\nabla u-F\big]\cdot\nabla u\,dx+\int_{A_1'}\big[|\nabla v|^{p(x)-2}\nabla v-F\big]\cdot\nabla v\,dx\Big)\ge 0.$$

On the set A_2 (and the same estimates can be done on A_2'), we have

$$\begin{aligned}\int_{A_2}\big[|\nabla u|^{p(x)-2}\nabla u-F\big]\cdot\nabla(u-v)\,dx\ge&-\int_{A_2}\big[|\nabla u|^{p(x)-2}\nabla u-F\big]\cdot\nabla v\,dx\\&-\int_{A_2}F\cdot\nabla u\,dx.\end{aligned}\tag{3.19}$$

By the same reasoning, the second term of the right-hand side of (3.19) tends to zero as h tends to infinity, since measA_2 tends to zero as h tends to infinity. We split A_2 as $A_2^+\cup A_2^-$, where

$$A_2^+=\{|u-v|\le k,|v|\le h,u>h\},\ A_2^-=\{|u-v|\le k,|v|\le h,u<-h\}.$$

On A_2^+ (and A_2^- can be treated in the same way), we have $v \le h$ and $-k \le u - v \le k$, so that $v - k \le u \le v + k \le h + k$; since $v - k > h$, then $A_2^+ \subseteq B_{h,k}$. Using the same argument, we obtain the result. For the first term of the right-hand side of (3.19) we have

$$\int_{A_2} \left[|\nabla u|^{p(x)-2}\nabla u - F\right] \cdot \nabla v \, dx$$
$$\le 2\Big(\left||\nabla u|^{p(x)-1}\right|_{p'(x),\{h\le|u|\le h+k\}} + |F|_{p'(x)}\Big)|\nabla v|_{p(x),\{h-k\le|v|\le h\}},$$

and the right-hand side tends to zero as $h \to \infty$ for every $k > 0$ thanks to Proposition 2.8 and Lemma 2.2. Summing up the results obtained for A_0, A_1, A_1', A_2 and A_2', we have

$$\lim_{h\to\infty} \int_{A_0} \left[|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right] \cdot \nabla(u - v)\, dx = 0,$$

that is

$$\int_{|u-v|\le k} \left[|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right] \cdot \nabla(u - v)\, dx = 0,$$

for every $k > 0$. Thus, we have $\nabla u = \nabla v$ a.e. in Ω.

Finally, from Lemma 2.4, we have

$$|T_k(u - v)|_{p(x)} \le C|\nabla T_k(u - v)|_{p(x)} = 0, \quad \text{for all } k > 0,$$

and hence $u = v$ a.e. in Ω. Therefore we obtain the uniqueness of entropy solutions. This completes the proof of Theorem 1.4. □

Given two bounded measurable functions $p(\cdot), q(\cdot) : \Omega \to \mathbb{R}$, we write

$$q(\cdot) \ll p(\cdot) \quad \text{if} \quad (p - q)_- > 0.$$

Remark 3.1. *If u is an entropy solution of problem* (1.1)*, then following the arguments in* [33] *by some necessary changes and using some a priori estimates in Marcinkiewicz for u and ∇u, we can obtain that $|u|^{q(\cdot)} \in L^1(\Omega)$, for all $0 \ll q(\cdot) \ll q_0(\cdot)$, and $|\nabla u|^{q(\cdot)} \in L^1(\Omega)$, for all $0 \ll q(\cdot) \ll q_1(\cdot)$, where*

$$q_0(\cdot) := \frac{p^*(\cdot)}{p'_+} \quad \textit{and} \quad q_1(\cdot) := \frac{q_0(\cdot)}{q_0(\cdot) + 1} p(\cdot).$$

Next, we prove that the entropy solution u is also a renormalized solution of problem (1.1) and the renormalized solution of problem (1.1) is unique.

Proof of Theorem 1.5.

(1) The entropy solution is a renormalized solution.

First we observe that the entropy solution u in Theorem 1.4 satisfies $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$, for given $k > 0$. Choosing $a = 1$ in Proposition 2.8, we obtain the renormalized condition, i.e.,

$$\lim_{k\to+\infty} \int_{\{k\le|u|\le k+1\}} |\nabla u|^{p(x)}\, dx = 0.$$

Let $\{u_n\}$ be a sequence constructed in (3.2), which satisfies $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(x)}(\Omega)$, for every $k > 0$.

Let $S \in W^{1,\infty}(\mathbb{R})$ be such that supp$S \subset [-M, M]$ for some $M > 0$. For every $\varphi \in C_0^\infty(\Omega)$, $S(u_n)\varphi \in W_0^{1,p(x)}(\Omega)$ is a test function in (3.2). It yields

$$\int_\Omega [S(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi + S'(u_n)|\nabla u_n|^{p(x)}\varphi]\, dx$$

$$= \int_\Omega [f_n S(u_n)\varphi + F_n \cdot \nabla(S(u_n)\varphi)]\, dx. \tag{3.20}$$

For the left-hand side of (3.20), because of supp$S' \subset [-M, M]$ we know

$$S(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n = S(u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)$$

and

$$S'(u_n)|\nabla u_n|^{p(x)} = S'(u_n)|\nabla T_M(u_n)|^{p(x)}.$$

Using (3.4), (3.16) and (3.17), we have

$$S(u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n) \to S(u)|\nabla T_M(u)|^{p(x)-2}\nabla T_M(u) \text{ in } (L^{p'(\cdot)}(\Omega))^N$$

and

$$S'(u_n)|\nabla T_M(u_n)|^{p(x)} \to S'(u)|\nabla T_M(u)|^{p(x)} \quad \text{in } L^1(\Omega).$$

Noting that

$$S(u)|\nabla T_M(u)|^{p(x)-2}\nabla T_M(u) = S(u)|\nabla u|^{p(x)-2}\nabla u,$$
$$S'(u)|\nabla T_M(u)|^{p(x)} = S'(u)|\nabla u|^{p(x)},$$

we deduce

$$S(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \to S(u)|\nabla u|^{p(x)-2}\nabla u \quad \text{in } (L^{p'(\cdot)}(\Omega))^N$$

and

$$S'(u_n)|\nabla u_n|^{p(x)} \to S'(u)|\nabla u|^{p(x)} \quad \text{in } L^1(\Omega).$$

For the right-hand side of (3.20), since $\nabla(S(u_n)\varphi) = \nabla(S(T_M(u_n))\varphi)$, thanks to the strong convergence of $T_M(u_n), f_n$ and F_n, it is easy to pass to the limits in the right-hand side terms. Therefore, we obtain

$$\int_\Omega [S(u)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + S'(u)|\nabla u|^{p(x)}\varphi]\, dx = \int_\Omega [fS(u)\varphi + F \cdot \nabla(S(u)\varphi)]\, dx,$$

which is (ii) in Definition 1.3. This completes the proof of the existence of renormalized solutions.

(2) Uniqueness of renormalized solutions.

Now we prove the uniqueness of renormalized solutions for problem (1.1) by choosing an appropriate test function. Let u and v be two renormalized solutions for problem (1.1). Fix a positive number k. For $\sigma > 0$, let S_σ be the function defined by

$$\begin{cases} S_\sigma(r) = r & \text{if } |r| < \sigma, \\ S_\sigma(r) = (\sigma + \frac{1}{2}) \mp \frac{1}{2}(r \mp (\sigma+1))^2 & \text{if } \sigma \le \pm r \le \sigma + 1, \\ S_\sigma(r) = \pm(\sigma + \frac{1}{2}) & \text{if } \pm r > \sigma + 1. \end{cases} \tag{3.21}$$

It is obvious that

$$\begin{cases} S'_\sigma(r) = 1 & \text{if } |r| < \sigma, \\ S'_\sigma(r) = \sigma + 1 - |r| & \text{if } \sigma \le |r| \le \sigma + 1, \\ S'_\sigma(r) = 0 & \text{if } |r| > \sigma + 1. \end{cases}$$

It is easy to check $S_\sigma \in W^{2,\infty}(\mathbb{R})$ with supp$S'_\sigma \subset [-\sigma - 1, \sigma + 1]$ and supp$S''_\sigma \subset [\sigma, \sigma + 1] \cup [-\sigma - 1, -\sigma]$. Therefore, we may take $S = S'_\sigma$ in (1.5) to have

$$\int_\Omega [S'_\sigma(u)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + S''_\sigma(u)|\nabla u|^{p(x)}\varphi]\, dx = \int_\Omega [fS'_\sigma(u)\varphi + F \cdot \nabla(S'_\sigma(u)\varphi)]\, dx$$

and

$$\int_\Omega [S'_\sigma(v)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla\varphi + S''_\sigma(v)|\nabla v|^{p(x)}\varphi]\,dx = \int_\Omega [fS'_\sigma(v)\varphi + F\cdot\nabla(S'_\sigma(v)\varphi)]\,dx.$$

As $T_k(S_\sigma(u)-S_\sigma(v)) \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we plug $\varphi = T_k(S_\sigma(u)-S_\sigma(v))$ as a test function in the above equalities and subtract them to obtain that

$$J_1 + J_2 = J_3 + J_4 + J_5, \tag{3.22}$$

where

$$\begin{aligned} J_1 &= \int_\Omega \big(S'_\sigma(u)|\nabla u|^{p(x)-2}\nabla u - S'_\sigma(v)|\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla T_k(S_\sigma(u)-S_\sigma(v))\,dx,\\ J_2 &= \int_\Omega [S''_\sigma(u)|\nabla u|^{p(x)} - S''_\sigma(v)|\nabla v|^{p(x)}]T_k(S_\sigma(u)-S_\sigma(v))\,dx,\\ J_3 &= \int_\Omega f(S'_\sigma(u)-S'_\sigma(v))T_k(S_\sigma(u)-S_\sigma(v))\,dx,\\ J_4 &= \int_\Omega T_k(S_\sigma(u)-S_\sigma(v))F\cdot\nabla(S'_\sigma(u)-S'_\sigma(v))\,dx,\\ J_5 &= \int_\Omega F(S'_\sigma(u)-S'_\sigma(v))\cdot\nabla T_k(S_\sigma(u)-S_\sigma(v))\,dx. \end{aligned}$$

We estimate J_1, J_2 and J_3 one by one. Writing

$$\begin{aligned} J_1 =& \int_\Omega \big[|\nabla S_\sigma(u)|^{p(x)-2}\nabla S_\sigma(u) - |\nabla S_\sigma(v)|^{p(x)-2}\nabla S_\sigma(v)\big]\cdot\nabla T_k(S_\sigma(u)-S_\sigma(v))\,dx\\ &+\int_\Omega \big[S'_\sigma(u) - S'_\sigma(u)|S'_\sigma(u)|^{p(x)-2}\big]|\nabla u|^{p(x)-2}\nabla u\cdot\nabla T_k(S_\sigma(u)-S_\sigma(v))\,dx\\ &-\int_\Omega \big[S'_\sigma(v) - S'_\sigma(v)|S'_\sigma(v)|^{p(x)-2}\big]|\nabla v|^{p(x)-2}\nabla v\cdot\nabla T_k(S_\sigma(u)-S_\sigma(v))\,dx\\ :=& J_1^1 + J_1^2 + J_1^3, \end{aligned}$$

and setting $\sigma \geq k$, we have

$$J_1^1 \geq \int_{\{|u-v|\leq k\}\cap\{|u|,|v|\leq k\}} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)\,dx. \tag{3.23}$$

Recalling supp$S'_\sigma \subset [-\sigma-1,\sigma+1]$ and supp$S''_\sigma \subset [\sigma,\sigma+1]\cup[-\sigma-1,-\sigma]$, we obtain

$$\begin{aligned} |J_1^2| \leq& 2\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)}\,dx + \int_{\{\sigma\leq|u|\leq\sigma+1\}\cap\{|v|\leq\sigma+1\}\cap\{|S_\sigma(u)-S_\sigma(v)|\leq k\}} |\nabla u|^{p(x)-1}|\nabla v|\,dx\Big)\\ \leq& 2\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)}\,dx + \int_{\{\sigma\leq|u|\leq\sigma+1\}\cap\{\sigma-k\leq|v|\leq\sigma+1\}} |\nabla u|^{p(x)-1}|\nabla v|\,dx\Big)\\ \leq& C\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)}\,dx + \int_{\{\sigma-k\leq|v|\leq\sigma+1\}} |\nabla v|^{p(x)}\,dx\Big). \end{aligned}$$

And we may get the similar estimate for J_1^3. Furthermore, we have

$$|J_2| \leq C\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)}\,dx + \int_{\{\sigma\leq|v|\leq\sigma+1\}} |\nabla v|^{p(x)}\,dx\Big).$$

From the above estimates and (i) in Definition 1.3, we obtain

$$\lim_{\sigma\to+\infty}(|J_1^2| + |J_1^3| + |J_2|) = 0.$$

Observing

$$f(S'_\sigma(u) - S'_\sigma(v)) \to 0 \quad \text{strongly in } L^1(\Omega)$$

as $\sigma \to +\infty$ and using the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\sigma\to+\infty} |J_3| = 0.$$

From Lemma 2.1 and $S_\sigma \in W^{2,\infty}(\mathbb{R})$, we have

$$\begin{aligned}
|J_4| &\le k \int_\Omega |F|(|S''_\sigma(u)\nabla u| + |S''_\sigma(v)\nabla v|) \\
&\le k \int_{\{\sigma\le|u|\le\sigma+1\}} |F||\nabla u|\,dx + \int_{\{\sigma\le|v|\le\sigma+1\}} |F||\nabla v|\,dx \\
&\le 2k|F|_{p'(\cdot)}\big(|\nabla u|_{p(\cdot),\{\sigma\le|u|\le\sigma+1\}} + |\nabla v|_{p(\cdot),\{\sigma\le|v|\le\sigma+1\}}\big).
\end{aligned}$$

By means of Lemma 2.2 and (i) in Definition 1.3, we obtain that

$$\lim_{\sigma\to+\infty} |J_4| = 0.$$

Next we show

$$\lim_{\sigma\to+\infty} |J_5| = 0.$$

Write

$$\begin{aligned}
|J_5| &\le \int_\Omega |F||S'_\sigma(u) - S'_\sigma(v)|T'_k(S_\sigma(u) - S_\sigma(v))|S'_\sigma(u)\nabla u - S'_\sigma(v)\nabla v|\,dx \\
&= \int_\Omega G(u,v)\,dx. \qquad (3.24)
\end{aligned}$$

We divide the estimate of J_5 into several cases according to the different integration sets. Denote

$$\begin{aligned}
E_1 &= \{|u|\le\sigma, |v|\le\sigma\} \cup \{|u|\ge\sigma+1, |v|\ge\sigma+1\}, \\
E_2 &= \{|u|\le\sigma, |v|\ge\sigma+1\} \cup \{|v|\le\sigma, |u|\ge\sigma+1\} = E_{21}\cup E_{22}, \\
E_3 &= \{|u|\le\sigma, \sigma\le|v|\le\sigma+1\} \cup \{|v|\le\sigma, \sigma\le|u|\le\sigma+1\} = E_{31}\cup E_{32}, \\
E_4 &= \{\sigma\le|u|\le\sigma+1, \sigma\le|v|\le\sigma+1\}, \\
E_5 &= \{\sigma\le|u|\le\sigma+1, |v|\ge\sigma+1\} \cup \{\sigma\le|v|\le\sigma+1, |u|\ge\sigma+1\}.
\end{aligned}$$

It is obvious that $\Omega = E_1\cup E_2\cup E_3\cup E_4\cup E_5$.

From the definition of S_σ and $\operatorname{supp} S'_\sigma \subset [-\sigma-1, \sigma+1]$, we deduce the following estimates.

Estimate on E_1.

$$\int_{E_1} G(u,v)\,dx = 0.$$

Estimate on E_2.

$$\begin{aligned}
\int_{E_{21}} G(u,v)\,dx &\le \int_{\{|S_\sigma(u)-S_\sigma(v)|\le k\}\cap\{|u|\le\sigma\}\cap\{|v|\ge\sigma+1\}} |F||\nabla u|\,dx \\
&\le \int_{\{\sigma-k\le|u|\le\sigma\}} |F||\nabla u|\,dx
\end{aligned}$$

and

$$\int_{E_{22}} G(u,v)\,dx \le \int_{\{\sigma-k\le|v|\le\sigma\}} |F||\nabla v|\,dx.$$

Estimate on E_3.

$$\int_{E_{31}} G(u,v)\,dx \le \int_{\{|S_\sigma(u)-S_\sigma(v)|\le k\}\cap\{|u|\le\sigma\}\cap\{\sigma\le|v|\le\sigma+1\}} G(u,v)\,dx$$
$$\le 2\Big(\int_{\{\sigma-k\le|u|\le\sigma\}} |F||\nabla u|\,dx + \int_{\{\sigma-k\le|v|\le\sigma+1\}} |F||\nabla v|\,dx\Big)$$

and

$$\int_{E_{32}} G(u,v)\,dx \le 2\Big(\int_{\{\sigma-k\le|u|\le\sigma+1\}} |F||\nabla u|\,dx + \int_{\{\sigma-k\le|v|\le\sigma\}} |F||\nabla v|\,dx\Big).$$

Estimate on E_4.

$$\int_{E_4} G(u,v)\,dx \le 2\Big(\int_{\{\sigma\le|u|\le\sigma+1\}} |F||\nabla u|\,dx + \int_{\{\sigma\le|v|\le\sigma+1\}} |F||\nabla v|\,dx\Big).$$

Estimate on E_5.

$$\int_{E_5} G(u,v)\,dx \le \int_{\{\sigma\le|u|\le\sigma+1\}} |F||\nabla u|\,dx + \int_{\{\sigma\le|v|\le\sigma+1\}} |F||\nabla v|\,dx.$$

Summing up the above estimates, we have

$$|J_5| \le \int_\Omega G(u,v)\,dx \le C\Big(\int_{\{\sigma-k\le|u|\le\sigma+1\}} |F||\nabla u|\,dx + \int_{\{\sigma-k\le|v|\le\sigma+1\}} |F||\nabla v|\,dx\Big),$$

which implies from Lemma 2.1, Lemma 2.2 and (i) in Definition 1.3 that

$$\lim_{\sigma\to+\infty} |J_5| = 0.$$

Therefore, sending $\sigma \to +\infty$ in (3.22) and recalling (3.23), we have

$$\int_{\{|u|\le\frac{k}{2},|v|\le\frac{k}{2}\}} (|\nabla u|^{p(x)-2} - |\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)\,dx = 0,$$

which implies $\nabla u = \nabla v$ a.e. on the set $\big\{|u| \le \dfrac{k}{2}, |v| \le \dfrac{k}{2}\big\}$. Since k is arbitrary, we conclude that $\nabla u = \nabla v$ a.e. in Ω.

Moreover, it follows from Lemma 2.4 that

$$|T_k(u-v)|_{p(x)} \le C|\nabla T_k(u-v)|_{p(x)} = 0, \quad \text{for all } k > 0,$$

and hence $u = v$ a.e. in Ω. Therefore we obtain the uniqueness of renormalized solutions. This completes the proof of Theorem 1.5. □

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