10-6 两个自由度体系在简谐荷载下的强迫振动

Force-vibration of Double DOF System Due to Harmonic loads

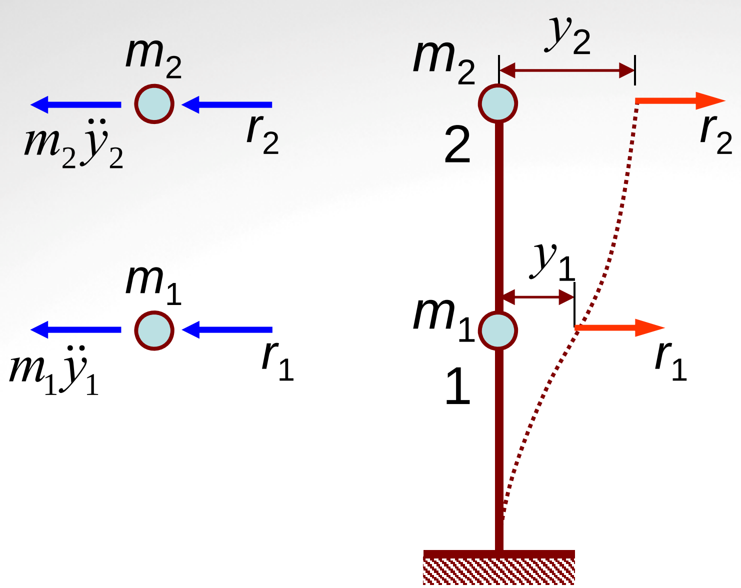
教学目标：

- 理解两个自由度体系在发生强迫振动时微分方程的建立方法。
- 掌握两个自由度体系在简谐荷载作用下的强迫振动的计算。

教学内容：

- 刚度法
- 挠度法

回顾：自由振动、刚度法



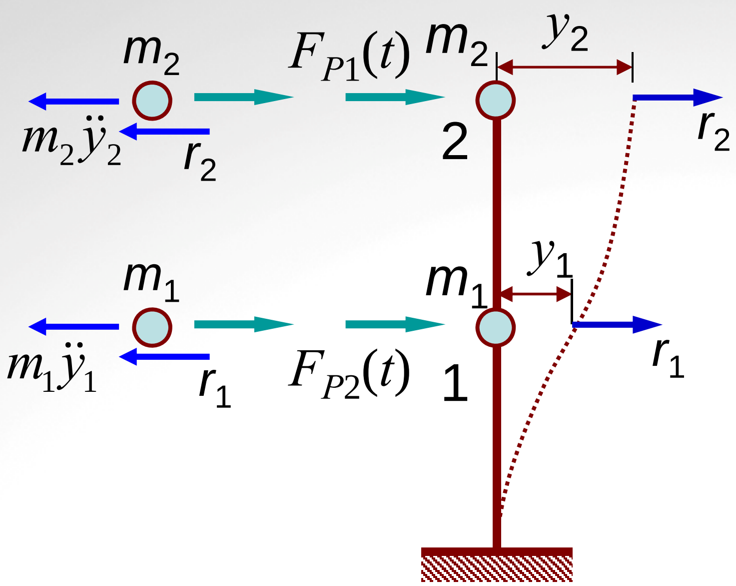
$$\left.\begin{aligned} m_1\ddot{y}_1 + r_1 = 0 \\ m_2\ddot{y}_2 + r_2 = 0 \end{aligned}\right\}$$

$$\left\{\begin{aligned} r_1 = k_{11}y_1 + k_{12}y_2 \\ r_2 = k_{21}y_1 + k_{22}y_2 \end{aligned}\right.$$

$$\left.\begin{aligned} m_1\ddot{y}_1(t) + k_{11}y_1(t) + k_{12}y_2(t) = 0 \\ m_2\ddot{y}_2(t) + k_{21}y_1(t) + k_{22}y_2(t) = 0 \end{aligned}\right\}$$

2自由度体系自由振动微分方程

1. 刚度法



$$m_1\ddot{y}_1 + k_{11}y_1 + k_{12}y_2 = F_{P1}(t)$$
$$m_2\ddot{y}_2 + k_{21}y_1 + k_{22}y_2 = F_{P2}(t)$$

$$\left.\begin{aligned} F_{P1}(t) &= F_{P1}\sin\theta t \\ F_{P2}(t) &= F_{P2}\sin\theta t \end{aligned}\right\}$$

则在平稳振动阶段，各质点也作简谐振动：

$$\left.\begin{aligned} y_1(t) &= Y_1\sin\theta t \\ y_2(t) &= Y_2\sin\theta t \end{aligned}\right\} \Rightarrow \left.\begin{aligned} (k_{11}-\theta^2 m_1)Y_1 + k_{12}Y_2 &= F_{P1} \\ k_{21}Y_1 + (k_{22}-\theta^2 m_2)Y_2 &= F_{P2} \end{aligned}\right\}$$

1. 刚度法

$$\left.\begin{aligned}(k_{11}-\theta^2 m_1)Y_1+k_{12}Y_2&=F_{P1}\\ k_{21}Y_1+(k_{22}-\theta^2 m_2)Y_2&=F_{P2}\end{aligned}\right\}$$

可解得位移的幅值为

$$Y_1=\frac{D_1}{D_0},\quad Y_2=\frac{D_2}{D_0}\qquad D_0=\begin{vmatrix}k_{11}-\theta^2 m_1 & k_{12}\\ k_{21} & k_{22}-\theta^2 m_2\end{vmatrix}$$

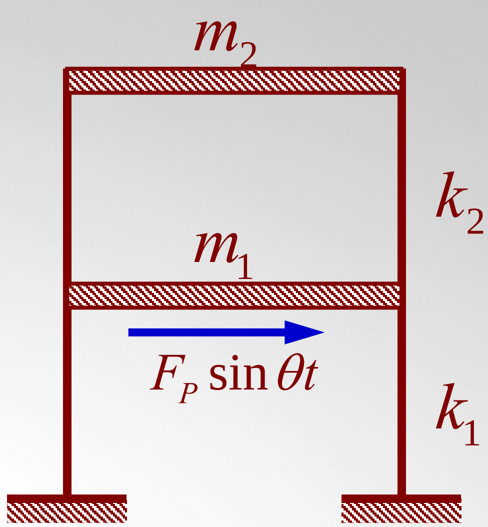
$$D_1=\begin{vmatrix}F_{P1} & k_{12}\\ F_{P2} & k_{22}-\theta^2 m_2\end{vmatrix}\qquad D_2=\begin{vmatrix}k_{11}-\theta^2 m_1 & F_{P1}\\ k_{21} & F_{P2}\end{vmatrix}$$

若θ与ω_1或ω_2重合，则： $D_0=0$，当D_1、D_2不全为零时，

位移幅值即为无限大，出现共振Resonance现象。

1. 刚度法

例10-9：刚架在底层横梁上作用简谐荷载 $F_{P1}(t)=F_P\sin\theta t$，试画出第一、二层横梁的振幅 Y_1、Y_2与荷载频率θ之间的关系曲线。设$m_1=m_2=m$， $k_1=k_2=k$。



解： （1）刚度系数：$k_{11}=k_1+k_2$， $k_{12}=k_{21}=-k_2$， $k_{22}=k_2$

（2）荷载幅值：$F_{P1}=F_P$， $F_{P2}=0$

（3）自振频率： $\omega_1^2=\dfrac{3-\sqrt{5}}{2}\dfrac{k}{m}$， $\omega_2^2=\dfrac{3+\sqrt{5}}{2}\dfrac{k}{m}$

1. 刚度法

（4）位移幅值：

$$\left.\begin{aligned}(k_{11}-\theta^2 m_1)Y_1+k_{12}Y_2&=F_{P1}\\ k_{21}Y_1+(k_{22}-\theta^2 m_2)Y_2&=F_{P2}\end{aligned}\right\}$$

$$\left.\begin{aligned}Y_1&=\frac{(k_2-\theta^2 m_2)F_P}{D_0}\\ Y_2&=\frac{k_2F_P}{D_0}\end{aligned}\right\}$$

$$D_0=\begin{vmatrix}k_{11}-\theta^2 m_1 & k_{12}\\ k_{21} & k_{22}-\theta^2 m_2\end{vmatrix}$$

$m_1=m_2=m$， $k_1=k_2=k$

$$\left.\begin{aligned}Y_1&=\frac{(k-m\theta^2)F_P}{D_0}\\ Y_2&=\frac{kF_P}{D_0}\end{aligned}\right\}$$

$$D_0=\left(2k-\theta^2 m\right)\left(k-\theta^2 m\right)-k^2$$

$\theta=\omega_1$,或 $\theta=\omega_2$

$D_0=0$, 当D_1 D_2不全为0时，发生共振

1. 刚度法

讨论共振现象

$$D_0=\left(2k-\theta^2 m\right)\left(k-\theta^2 m\right)-k^2$$

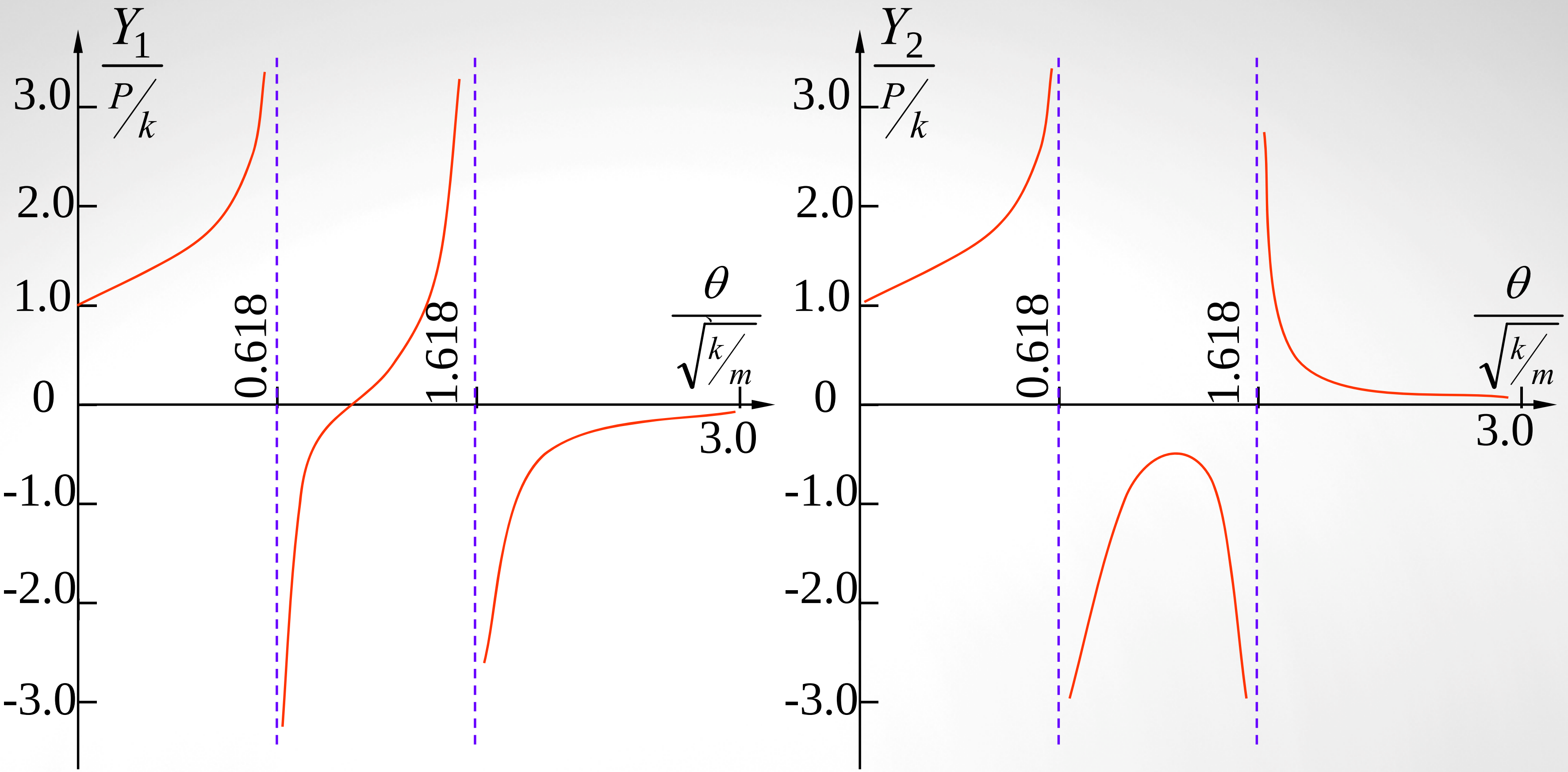
⇓

$$D_0=m^2\theta^4-3km\theta^2+k^2=m^2(\theta^2-\omega_1^2)(\theta^2-\omega_2^2)$$

$$\left.\begin{aligned}Y_1&=\frac{F_P}{k}\frac{1-\frac{m}{k}\theta^2}{\left(1-\frac{\theta^2}{\omega_1^2}\right)\left(1-\frac{\theta^2}{\omega_2^2}\right)}\\Y_2&=\frac{F_P}{k}\frac{1}{\left(1-\frac{\theta^2}{\omega_1^2}\right)\left(1-\frac{\theta^2}{\omega_2^2}\right)}\end{aligned}\right\}$$

1. 刚度法

振幅参数 $Y_1\Big/\frac{F_P}{k}$ $Y_2\Big/\frac{F_P}{k}$ 与荷载频率参数 $\theta\Big/\sqrt{\frac{k}{m}}$ 之间的关系曲线。

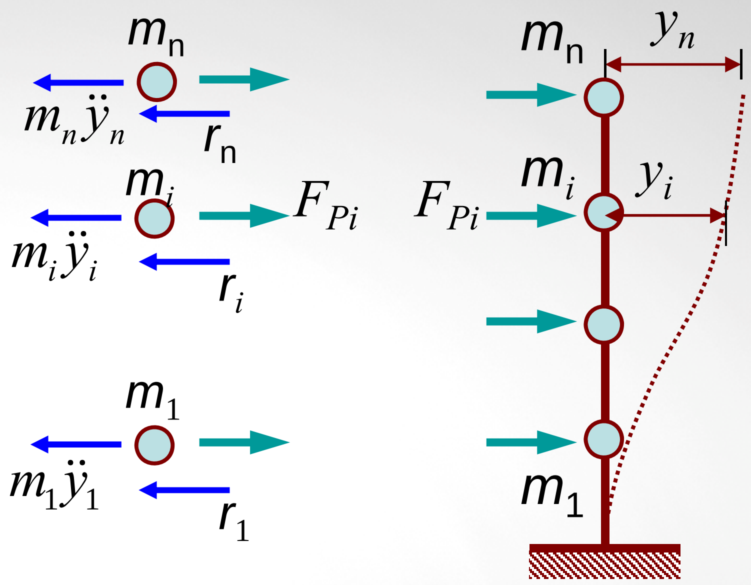


1. 刚度法

n个自由度体系

回顾：2自由度：

$$m_1\ddot{y}_1 + k_{11}y_1 + k_{12}y_2 = F_{P1}(t)$$
$$m_2\ddot{y}_2 + k_{21}y_1 + k_{22}y_2 = F_{P2}(t)$$



n自由度：

$$\left.\begin{array}{l}
m_1\ddot{y}_1 + k_{11}y_1 + k_{12}y_2 + \cdots + k_{1n}y_n = F_{P1}(t) \\
m_2\ddot{y}_2 + k_{21}y_1 + k_{22}y_2 + \cdots + k_{2n}y_n = F_{P2}(t) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
m_n\ddot{y}_n + k_{n1}y_1 + k_{n2}y_2 + \cdots + k_{nn}y_n = F_{Pn}(t)
\end{array}\right\} \quad \mathbf{M}\ddot{y} + \mathbf{K}y = \mathbf{F}_P(t)$$

1. 刚度法

$$(\mathbf{K}-\theta^2\mathbf{M})\mathbf{Y}=\mathbf{F}_P$$

⇓

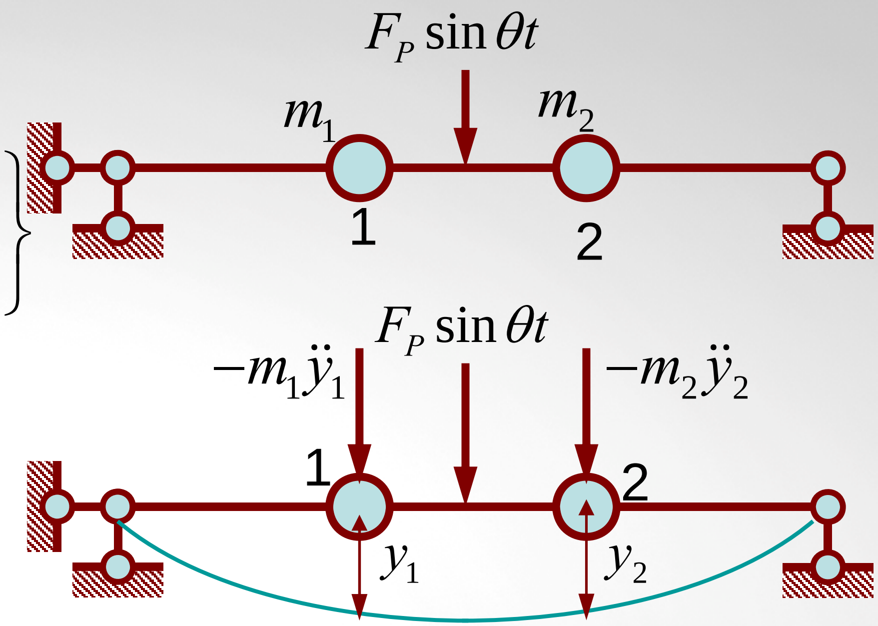
$$D_0=\left|\mathbf{K}-\theta^2\mathbf{M}\right|$$

- 如果$D_0 \neq 0$，可求得任意时刻t各质点的位移。
- 如果$D_0=0$的情形。当荷载频率θ与体系的自振频率中的任一个ω_i相等时，就可能出现共振现象。

对于具有n个自由度的体系来说，在 n 种情况下 $(\theta=\omega_i,\ i=1,2,\ ...,n)$都有可能出现共振现象。

2. 挠度法

$$\left.\begin{aligned} y_1 &= (-m_1\ddot{y}_1)\delta_{11} + (-m_2\ddot{y}_2)\delta_{12} + \Delta_{1P}\sin\theta t \\ y_2 &= (-m_1\ddot{y}_1)\delta_{21} + (-m_2\ddot{y}_2)\delta_{22} + \Delta_{2P}\sin\theta t \end{aligned}\right\}$$



Δ_{1P} Δ_{2P} 为荷载幅值在质点1、2产生的静力位移。

$$\left.\begin{aligned} m_1\ddot{y}_1\delta_{11} + m_2\ddot{y}_2\delta_{12} + y_1 &= \Delta_{1P}\sin\theta t \\ m_1\ddot{y}_1\delta_{21} + m_2\ddot{y}_2\delta_{22} + y_2 &= \Delta_{2P}\sin\theta t \end{aligned}\right\}$$

平稳振动阶段的解为：

$$\left.\begin{aligned} y_1(t) &= Y_1\sin\theta t \\ y_2(t) &= Y_2\sin\theta t \end{aligned}\right\} \rightarrow \left.\begin{aligned} (m_1\theta^2\delta_{11}-1)Y_1 + m_2\theta^2\delta_{12}Y_2 + \Delta_{1P} &= 0 \\ m_1\theta^2\delta_{21}Y_1 + (m_2\theta^2\delta_{22}-1)Y_2 + \Delta_{2P} &= 0 \end{aligned}\right\}$$

2. 挠度法

$$\left.\begin{aligned}(m_1\theta^2\delta_{11}-1)Y_1+m_2\theta^2\delta_{12}Y_2+\Delta_{1P}=0\\ m_1\theta^2\delta_{21}Y_1+(m_2\theta^2\delta_{22}-1)Y_2+\Delta_{2P}=0\end{aligned}\right\}$$

平稳振动阶段的解为：

$$Y_1=\frac{D_1}{D_0},\quad Y_2=\frac{D_2}{D_0}$$

$$\left.\begin{aligned}D_0&=\begin{vmatrix}(m_1\theta^2\delta_{11}-1) & m_2\theta^2\delta_{12}\\ m_1\theta^2\delta_{21} & (m_2\theta^2\delta_{22}-1)\end{vmatrix}\\ D_1&=\begin{vmatrix}-\Delta_{1P} & m_2\theta^2\delta_{12}\\ -\Delta_{2P} & (m_2\theta^2\delta_{22}-1)\end{vmatrix}\\ D_2&=\begin{vmatrix}(m_1\theta^2\delta_{11}-1) & (-\Delta_{1P})\\ m_2\theta^2\delta_{21} & (-\Delta_{2P})\end{vmatrix}\end{aligned}\right\}$$

2. 挠度法

位移：

$$\left.\begin{aligned} y_1(t) &= Y_1 \sin\theta t \\ y_2(t) &= Y_2 \sin\theta t \end{aligned}\right\}$$

惯性力：

$$\left.\begin{aligned} -m_1\ddot{y}_1(t) &= m_1\theta^2 Y_1 \sin\theta t \\ -m_2\ddot{y}_2(t) &= m_2\theta^2 Y_2 \sin\theta t \end{aligned}\right\}$$

因为位移、惯性力和动荷同时到达幅值，动内力也在振幅位置到达幅值。如任一截面的弯矩幅值，可由下式求出：

$$M_{i\max} = \bar{M}_{i1} I_1 + \bar{M}_{i2} I_2 + M_{iP}$$

I_1、I_2分别为质点1、2的惯性力幅值

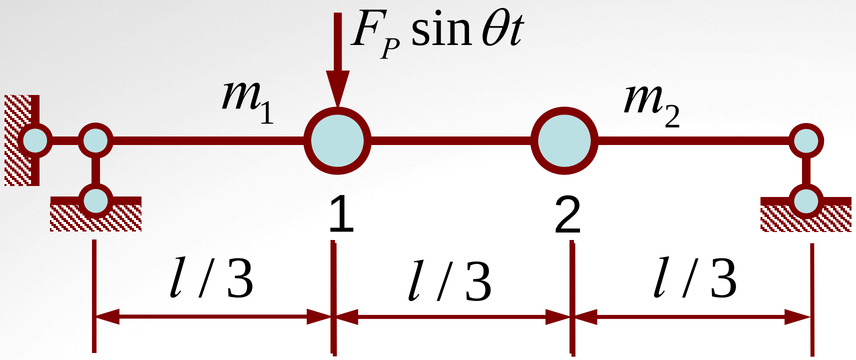
$\bar{M}_1$ $\bar{M}_2$ 分别为单位惯性力I_1=1、I_2=1作用时，同一截面的弯矩值

M_P为动荷载幅值作为静力作用下同一截面的弯矩值

2. 挠度法

例10-10：求图示体系的动位移和动弯矩的幅值图。

已知：$m_1=m_2=m$，EI=常数，$\theta=0.6\omega_1$。



$$\left.\begin{aligned}(m_1\theta^2\delta_{11}-1)Y_1+m_2\theta^2\delta_{12}Y_2+\Delta_{1P}=0\\ m_1\theta^2\delta_{21}Y_1+(m_2\theta^2\delta_{22}-1)Y_2+\Delta_{2P}=0\end{aligned}\right\}$$

解：（1）自振频率

$$\delta_{11}=\delta_{22}=\frac{4l^3}{243EI}\qquad \delta_{12}=\delta_{21}=\frac{7l^3}{486EI}$$

$$\omega_1=5.692\sqrt{\frac{EI}{ml^3}}\qquad \theta=0.6\omega_1=3.415\sqrt{\frac{EI}{ml^3}}$$

（2）求 Δ_{1P}　Δ_{2P}　　$\Delta_{1P}=\dfrac{4F_pl^3}{243EI}$，　$\Delta_{2P}=\dfrac{7F_pl^3}{486EI}$

2. 挠度法

$$\left.\begin{aligned}(m_1\theta^2\delta_{11}-1)Y_1+m_2\theta^2\delta_{12}Y_2+\Delta_{1P}=0\\ m_1\theta^2\delta_{21}Y_1+(m_2\theta^2\delta_{22}-1)Y_2+\Delta_{2P}=0\end{aligned}\right\}$$

（3）计算D$_0$、D$_1$和D$_2$

$$D_0=\begin{vmatrix}(m_1\theta^2\delta_{11}-1) & m_2\theta^2\delta_{12}\\ m_1\theta^2\delta_{21} & (m_2\theta^2\delta_{22}-1)\end{vmatrix}=0.642\,7$$

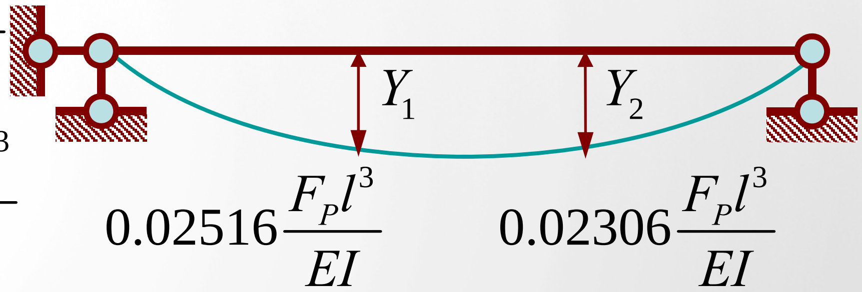
$$D_1=\begin{vmatrix}-\Delta_{1P} & m_2\theta^2\delta_{12}\\ -\Delta_{2P} & (m_2\theta^2\delta_{22}-1)\end{vmatrix}=0.015\,72\frac{F_Pl^3}{EI}$$

$$D_2=\begin{vmatrix}(m_1\theta^2\delta_{11}-1) & (-\Delta_{1P})\\ m_2\theta^2\delta_{21} & (-\Delta_{2P})\end{vmatrix}=0.014\,40\frac{F_Pl^3}{EI}$$

（4）计算位移幅值

$$Y_1=\frac{D_1}{D_0}=\frac{0.015\,72F_pl^3}{0.624\,7EI}=0.025\,16\frac{F_pl^3}{EI}$$

$$Y_2=\frac{D_2}{D_0}=\frac{0.014\,40F_pl^3}{0.624\,7EI}=0.023\,06\frac{F_pl^3}{EI}$$



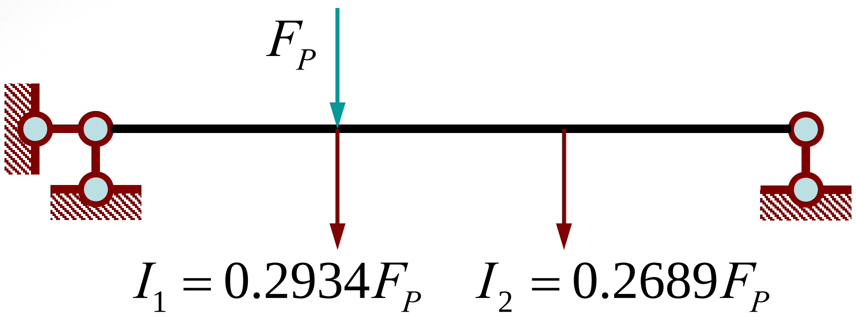
2. 挠度法

（5）计算惯性力幅值

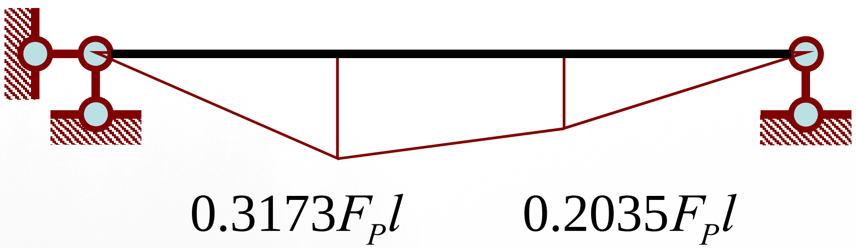
$$I_1 = m_1\theta^2 Y_1 = 11.66\frac{EI}{l^3}\times 0.025\,16\frac{F_P l^3}{EI} = 0.293\,4F_P$$

$$I_2 = m_2\theta^2 Y_2 = 11.66\frac{EI}{l^3}\times 0.023\,06\frac{F_P l^3}{EI} = 0.268\,9F_P$$

（6）计算质点1、2的动弯矩幅值



$$M(t)_{\max} = \bar{M}_1 I_1 + \bar{M}_2 I_2 + M_P$$



$$M_{1\max} = \bar{M}_{11} I_1 + \bar{M}_{12} I_2 + M_{1P}$$

$$M_{2\max} = \bar{M}_{21} I_1 + \bar{M}_{22} I_2 + M_{2P}$$

2. 挠度法

（7）计算质点1的位移、弯矩动力系数

位移

$$y_{1st}=\Delta_{1P}=\frac{4F_P l^3}{243EI}=0.016\,46\frac{F_P l^3}{EI}$$

$$\beta_{y1}=\frac{Y_1}{y_{1st}}=\frac{\dfrac{0.025\,16F_P l^3}{EI}}{\dfrac{0.016\,46F_P l^3}{EI}}=1.529$$

弯矩

$$M_{1st}=\frac{2F_P l}{9}=0.222\,2F_P l$$

$$\beta_{M1}=\frac{M_{1\max}}{M_{1st}}=\frac{0.317\,3F_P l}{0.222\,2F_P l}=1.428$$

没有统一的动力系数

小结

- 刚度法

$$\left.\begin{aligned}(k_{11}-\theta^2 m_1)Y_1+k_{12}Y_2&=F_{P1}\\ k_{21}Y_1+(k_{22}-\theta^2 m_2)Y_2&=F_{P2}\end{aligned}\right\}$$

- 挠度法

$$\left.\begin{aligned}(m_1\theta^2\delta_{11}-1)Y_1+m_2\theta^2\delta_{12}Y_2+\Delta_{1P}&=0\\ m_1\theta^2\delta_{21}Y_1+(m_2\theta^2\delta_{22}-1)Y_2+\Delta_{2P}&=0\end{aligned}\right\}$$

作业

习题　P484：10-23