Testing for structural change in the predictability of asset returns.

Luis M. Viceira[1]

This draft: April 1997

[1]Harvard Business School, Boston MA 02163 and NBER. Email lviceira@hbs.edu. The author is grateful for invaluable comments, suggestions and discussions with John Y. Campbell, James H. Stock and Jonathan H. Wright, and for comments from Guido Imbens, Ken Snowden and participants in the Harvard Finance and Econometrics Workshops. The author gratefully acknowledges the financial support of the Bank of Spain and also La Caixa Fellowship, under whose support an earlier version of this paper was written.

Abstract

There is growing empirical evidence that expected returns on financial assets are time-varying. Moreover, the evidence calls for highly persistent expected returns. Testing for time-varying, persistent expected returns requires the computation of returns over long-horizons and so the use of long time series to have as many independent, non-overlapping observations as possible. When testing hypotheses using long time series, the researcher always faces the risk of structural changes that invalidate any inference based on full sample estimators. This paper tests for structural change in the model generating expected stock returns. To do so we develop some asymptotic results on the asymptotic distribution of recursive, reverse-recursive and sequential least squares estimators under general conditions of heteroscedasticity and autocorrelation. We apply these results to fluctuation tests based on the most frequently statistics used in the literature to measure persistence in expected stock returns. Finally, we pursue an empirical exercise with data on the US stock market, and we find that the higher equity premium in the post-World War II period is also associated with a structural break in the stochastic structure generating expected returns, towards more univariate persistence in expected stock returns. However, we do not find evidence of a structural break in the forecasting relation between log dividend yields and log returns at a one-month horizon. Moreover, conditional on structural stability, the full-sample statistical evidence on this forecasting relation dos not reject the null hypothesis that the log dividend yield does not forecast one-month log returns, once we take into account the highly persistent nature of the process for the log dividend yield.

JEL classification: C12, C22, G12,.

Keywords: Time-varying expected returns, structural change, recursive estimators, variance-ratio, long-horizon regressions.

1 Introduction

During the last decade there has been abundant empirical research on the predictability, or "mean-reversion", properties of stock returns. This research has focused either on the serial correlation of stock returns (Poterba and Summers 1988, Fama and French 1988a) or on the predictability of returns from variables other than past returns, particularly financial ratios such as the the dividend-price ratio (Campbell and Shiller 1988a-b, Fama and French 1988b, 1989) or the book-to-market ratio (Kothari and Shanken 1995), and stochastically detrended short-term interest rates (Campbell, Lo and Mackinlay 1997). A common feature of this research is the use of overlapping multiperiod (or long-horizon) returns to analyze time-variation in expected returns, which requires the use of long-time time series of returns to have as many independent observations as possible.

Overall this literature concludes there is weak evidence about univariate predictability in stock returns (Fama and French, 1988a, Richardson and Stock 1989), but the evidence about multivariate predictability is stronger (Campbell, Lo and MacKinlay, 1997). However, these conclusions are far from going unchallenged, because they depend on the sample period researchers use to test for predictability. For the U.S. experience, the evidence on predictability varies depending on whether the pre-WWII years are included in the sample or not. While the tests for univariate mean-reversion indicate that the effect seems to be concentrated around the Great-Depression and is almost non-existent after WWII (Fama and French 1988, Kim, Nelson and Startz, 1991, Poterba and Summers 1988), the evidence about multivariate predictability is the opposite, i.e., the statistical evidence is stronger after WWII (Campbell, Lo and MacKinlay 1997,). Campbell (1997) shows that predictability is also present in a sample of international stock markets in the last 25 years.

Thus, the available empirical evidence about the existence of predictability in stock returns suggests that an stability analysis may help clarify the issues at hand. As Fama and French (1988a, p. 266) have noted, "(a)utocorrelation may reflect inefficiency or time-varying equilibrium expected returns generated by rational investor behavior. Neither view suggest, however, that patterns of autocorrelation should be stable for a sample period as long as 60 years (...). Stationary price components may be less important after 1940, or perhaps prices no longer have such temporary components. Resolution of this issue will require more powerful statistical techniques". The main aim of this paper is to develop and apply an asymptotic theory that helps us to discern whether temporal instability characterizes the behavior of long-horizon stock returns or, by contrast, the differences we observe in the data across periods are merely due to sampling variability rather than time-variation of the true process generating predictability (or absence of) in stock returns. Moreover, we want to test whether the parameters of a particular model that generates expected returns are constant over time, without conditioning on a particular null hypothesis about the true value of those parameters. That is, we want to separate the issue of stability from the issue of predictability. The advantage of disentangling both hypotheses is obvious: If we do not reject the null hypothesis that the model generating expected returns is constant over time, we can then use the full sample of stock returns to test any null hypothesis about the parameters of the model—for example, if the parameters are such that expected returns are constant—, hence increasing the power of the tests for this null.

To do that we use recursive computations of the statistics most widely used in the literature on predictability in stock returns. By defining our statistics in terms of deviations of the recursive estimates with respect to full sample estimates—"fluctuation tests" à la Ploberger-Krämer-Kontrus (1989)—we are able to separate the issue of testing for stability from that of testing for the existence of predictability in returns. However, the compounding nature of multiperiod returns induces an spurious autocorrelation in long-horizon returns that invalidate the use of standard results on the distribution of the recursive estimators of univariate mean-reversion statistics. Hence we develop some theoretical results on the asymptotic distribution of recursive and reverse-recursive estimators of these statistics, under general structures of autocorrelation and heteroskedasticity, using the approach to inference with multiperiod returns in Richardson and Stock (1989). We find these estimators are continuous functionals of standard Brownian processes in the unit interval.

Testing for stability and predictability in multivariate models present some additional problems. There is considerable evidence that shocks to the log dividend yield, the stock return predictor most often used in the literature, are highly persistent and negatively correlated with shocks to stock returns (see, for example, Campbell and Viceira 1996). Elliott and Stock (1994) have shown that standard inference procedures based on the assumption that regressors are stationary may lead to over-rejections of the null hypothesis in significance tests when the explanatory variables in the regression equation contain local-to-unity roots. They develop an alternative asymptotic theory for this case based on the theory of nearly integrated processes of Chan and Wei (1987), Phillips (1987) and Stock (1991). We apply their theory to test for predictability in stock returns from the log dividend yield, and extend it to find the limiting representation of the OLS recursive and reverse-recursive estimators of the slope in this predictive relationship when the regressors are endogenous and have roots local-to-unity[2]. We find that these estimators are continuous functionals of Ornstein-Uhlenbeck processes and standard Brownian Motion processes in the unit interval. We also use these results to find the limiting distribution of the recursive and reverse-recursive fluctuation statistics and the sequential QLR statistic (Quandt 1960, Stock 1994) we use in our empirical analysis. For the QLR statistic, we also find evidence

[2] Wright (1996) has derived the asymptotic distribution of some leading stability tests when the regressors are nearly integrated but strictly exogenous.

that its asymptotic distribution is robust to regressor endogeneity.

The main empirical results of the paper are the following. When we apply our stability tests to monthly log real returns, log excess returns and dividends of the CRSP portfolio including all traded securities in the NYSE-AMEX-NASDAQ markets from 1926 through 1995, we find some statistical evidence against the null hypothesis of structural stability in the univariate process for stock returns but no evidence against this null in the regression equation that relates dividend yields to expected future stock returns. The evidence calls for a structural break in the mid 70's towards univariate predictability in returns. There is also some weak evidence of a possible structural break around the 50's.

Conditional on the null of structural stability, we also find some evidence of non-zero autocorrelation in stock returns, corroborating earlier evidence by Richardson and Stock (1989) for the period ending in 1985. This evidence is stronger in returns on the equally-weighted portfolio than in returns on the value-weighted portfolio, and stronger in log excess returns than in log real returns. However, while under conventional asymptotics there is some evidence of predictability in one-month log returns (specially in equally-weighted returns) from the log dividend yield, there is none under local-to-unity asymptotics.

This paper is organized as follows. Section 2 outlines the statistics on which our empirical analysis is based, and develops their limiting distributions. In section 3 we discuss Monte Carlo evidence on the behavior of these distribution. Section 4 reports and discusses the empirical results of this paper. Finally, section 5 concludes.

2 Assessing the Stability of Predictability in Asset Returns

2.1 The statistics of interest

Our stability tests are based on three of the most widely used statistics in the recent literature on the predictability of asset returns: The variance ratio statistic of Poterba and Summers (1988) and Lo and MacKinlay (1988), the univariate Fama-French (1988a) regression statistic, and the multivariate regression statistic of Campbell and Shiller (1988) and Fama and French (1988b, 1989). Both the variance ratio and the univariate Fama-French regression statistic test for univariate predictability in stock returns, focusing on the ability of past returns to predict future returns.

The variance ratio statistic $vr(k)$ is the sample counterpart of the ratio of the variance of k-period log returns per unit of time to the variance of one-period returns:

$$VR\,(k) = \frac{\mathrm{Var}(r_t^k)}{k\ \mathrm{Var}(r_t)}\,, \tag{1}$$

where r_t^k denotes the log asset return over the last k periods, which obtains by accumulating the last k one-period log returns, i.e., $r_t^k = \sum_{i=0}^{k-1} r_{t-i}$, and r_{t+1} is the one-period log return, $r_{t+1} = \log(P_{t+1}+D_{t+1})-\log(P_t)$. The univariate mean-reversion regression statistic proposed by Fama and French (1988) is the OLS estimate $b(k)$ of the slope in the regression equation

$$r_{t+k}^k = \alpha\,(k) + \beta\,(k)\, r_t^k + u_{t+k}^k\ . \tag{2}$$

If one-period returns are uncorrelated, we have $VR(k) = 1$ and $\beta(k) = 0$.

When the horizon k is fixed relative to T, both $vr(k)$ and $b(k)$ converge in probability to $VR(k)$ and $\beta(k)$ and have asymptotic gaussian distributions under the null hypothesis of uncorrelated returns. However, Lo and MacKinlay (1988, 1989) have shown that these asymptotic distributions perform poorly in small samples. By contrast, Richardson and Stock (1989) have shown that large-sample representations of full sample estimators of long-horizon return statistics based on varying horizons k such that $k/T \to \delta$, where δ is fixed and non-zero asymptotically, provide better approximations to sampling distributions than those based on fixed k, though they are not gaussian, but functionals of Brownian Motion processes defined in the unit interval. This approach is also particularly useful to find the asymptotic distribution of the recursive estimators of these statistics.

Campbell (1991) has shown that it is possible for expected returns to be time-varying and realized returns to be serially uncorrelated. The multivariate tests that look for predictability in returns consider variables other than the own lagged returns. Hence they test the statistical significance of the vector of slopes in the regression equation of r_{t+k}^k onto x_t, a set of observable variables that may predict returns. We will focus on this regression equation when $k = 1$,

$$r_{t+1} = \alpha_0 + {\alpha_1}' x_t + \varepsilon_{t+1}. \tag{3}$$

Testing for stability and predictability of expected returns in the multivariate predictive model (3) presents some econometric problems. There is considerable evidence that shocks to the log dividend yield, the stock return predictor most often used in the literature, are highly persistent and negatively correlated with shocks to stock returns—see, for example, Campbell and Viceira (1996). Elliott and Stock (1994) have shown that standard inference procedures based on the assumption that regressors are stationary may lead to over-rejections of the null hypothesis in significance tests when the explanatory variables in the regression equation contain local-to-unity roots. They develop an alternative asymptotic theory for this case based on the theory of nearly integrated processes of Chan and Wei

(1987), Phillips (1987) and Stock (1991). We also compute the asymptotic distributions of our stability tests assuming up front that x_t is a nearly-integrated process whose innovations are possibly contemporaneously correlated with the innovations in stock returns.

2.2 Stability tests

Our stability tests are based on both recursive and reverse-recursive estimators of the statistics of interest. When we deal with non-recursive estimation, we perform a unique significance test for the parameters of the model, based on the entire sample estimation of the model. However, in the case of recursive estimation, we want to perform a simultaneous test over the whole sequence of recursive estimates of the parameters of the model, for which we need to find the distribution of the "plot" of recursive estimates. A recursive estimate of a certain parameter is given by the sequence of estimates of that parameter computed over the sequence of increasing subsamples $t = \{1, 2, ..., [T\lambda] : 0 < \lambda_0 \leq \lambda \leq 1\}$, where T is the sample size, $[T\lambda]$ denotes the largest integer that is less than or equal to $T \cdot \lambda$ —the greatest lesser function—and λ_0 is a trimming parameter. A reverse-recursive estimator is computed in analogous manner, with the sequence of subsamples given by $t = \{[T\lambda], ..., T : 0 < \lambda \leq \lambda_0 \leq 1\}$.

We are going to focus on two types of stability tests. The first type of tests are fluctuation tests à la Ploberger-Krämer-Kontrus (1989) in which the stability tests is performed on a properly scaled version of the absolute differences between the recursive—or reverse-recursive— estimates and the full sample estimate of the parameters of interest . The scaling factor is chosen as to have asymptotic distributions that do not depend on unknown parameters. The main advantage of this procedure is that we do not need to impose a priori any null hypothesis about the value of the parameters of interest, and we only test whether they are constant or not.

We consider fluctuation tests based on both recursive and reverse-recursive estimators. The reason is that recursive estimation requires to choose a trimming parameter λ_0, i.e. a number of initial observations for the first recursion large enough to have a meaningful first estimation. The cost of this choice is that we lose any opportunity to detect breaks early in the sample. In our case, this cost is particularly high, since even using overlapping observations, regressions with long-horizon returns require a relatively large number of initial observations. But in our empirical application these observations correspond to those of the Great Depression period which, a priori, we expect it to play an important role in any stability analysis of long-horizon stock returns in the US. To solve this trade-off, Banerjee, Lumsdaine and Stock (1992) propose to use reverse recursions, i.e. to estimate recursively the statistics of interest by going backwards instead of forward.

The second type of test we consider is Quandt's (1960) Likelihood Ratio (QLR) test, which selects the maximal Chow-Wald test for structural break over a sequence of dates. Stock (1994) derives the asymptotic distribution of the QLR test when regressors are stationary and strictly exogenous, and Wright (1996) does it when regressors are nearly integrated processes and strictly exogenous. We derive its distribution under the assumption that regressors are nearly integrated processes and endogenous.

Most of the standard asymptotic results for recursive estimators are built under the joint null hypothesis that the parameters in the regression model are stable over time, the regressors are stationary and exogenous, and the disturbances are either martingale difference sequences or i.i.d. sequences. However, in the univariate regression test, the disturbance term is known to be autocorrelated up to order k and possibly heteroskedastic even under the null of no predictability, due to compounding in multi-period returns. Moreover, the regressor is endogenous. In the multivariate regression test, regressors are also endogenous and highly persistent. These problems prevent us from using standard results on the distribution of recursive estimators in the previous literature, and force us to look for the asymptotic distribution of the stability tests based on these statistics. The following two section present those distributions.

2.3 Asymptotic representation of the recursive and reverse-recursive variance ratio and Fama-French univariate regression statistics

The sequence of subsamples for which we compute recursive estimators is $t = \{(1, 2, ..., [T\lambda]) : 0 < \lambda_0 \leq \lambda \leq 1\}$, and the sequence of subsamples for which we compute reverse-recursive estimators is $t = \{([T\lambda], ..., T) : 0 < \lambda \leq \lambda_0 \leq 1\}$, where T is the sample size, $[T\lambda]$ denotes the largest integer that is less than or equal to $T \cdot \lambda$ and λ_0 is a trimming parameter.

We assume throughout this section that:

(A1) r_t admits the following representation:

$$r_t = \mu + \Psi(L)\,\varepsilon_t,$$

where μ is constant, $\Psi(L) = \sum_{s=0}^{\infty} \psi_s \varepsilon_{t-s}$, $\sum_{s=0}^{\infty} s|\psi_s| < \infty$, $\Psi(1) \neq 0$ and $\{\varepsilon_t\}$ is a martingale difference sequence (m.d.s.) with $E[\varepsilon_t^2|\varepsilon_{t-1}, ...] = \sigma_t^2 > 0$, $\lim_{T\to\infty} \mathrm{E}[T^{-1}\sum_{t=1}^{\infty} \sigma_t^2] = E[\varepsilon_t^2] = \sigma_\varepsilon^2 < \infty$, $E[\varepsilon_t^4|\varepsilon_{t-1}, ...] = \mu_4 < \infty$ and $E[r_0^2] < \infty$.

(A2) The return horizon k is such that $\lim_{T\to\infty} k/T = \delta$.

(A3) The recursive estimator of the sample mean and variance of one-period log returns converges in probability uniformly in λ to the unconditional mean and variance:

$$\mu(1, T\lambda) = \frac{1}{[T\lambda]} \sum_{t=1}^{[T\lambda]} r_t \to \mu,$$

$$\gamma_0(1, T\lambda) = \frac{1}{[T\lambda]} \sum_{t=1}^{[T\lambda]} (r_t - \mu(1, T\lambda))^2 \to \gamma_0,$$

where $\to$ denotes convergence in probability and $\gamma_0 = \mathrm{Var}(r_t)$ denotes the unconditional variance of the one-period log return.

(A3') The reverse-recursive estimator of the sample mean and variance of one-period log returns converges uniformly in $(1-\lambda)$ to the unconditional mean and variance:

$$\mu^r(1, T\lambda) = \frac{1}{[T(1-\lambda)]+1} \sum_{t=[T\lambda]}^{T} r_t \to \mu.$$

$$\gamma_0^r(1, T\lambda) = \frac{1}{[T(1-\lambda)]+1} \sum_{t=[T\lambda]+1}^{T} (r_t - \mu^r(1, T\lambda))^2 \to \gamma_0.$$

Assumption (A1), which follows Stock (1994), states that one-period log returns admit an invertible moving average representation whose innovations ε_t follow a martingale difference sequence process. That is, ε_t is a mean-zero uncorrelated process. It allows ε_t to be conditionally heteroskedastic as long as it is unconditionally homoskedastic and its fourth moment exists and it is finite. These are the weakest possible restrictions on r_t so that $(r_t - \mu)$ —or r_t minus any consistent estimator for μ—obeys the Functional Central Limit Theorem (FCLT) for general $I(0)$ processes (Stock, 1994). For ease of reference, we state this result as a Lemma:

Lemma 1 (FCLT for I(0) processes) *Under assumption (A1), $T^{-1/2}$ times the partial sums of demeaned one-period returns obey the Functional Central Limit Theorem (FCLT) for general $I(0)$ processes. That is, they converge in distribution to a univariate standard Brownian Motion process in the unit interval:*

$$\sqrt{\frac{1}{T}} \sum_{t=1}^{[Tv]} (r_t - \mu) \Rightarrow V^{1/2} W_r(v),$$

where $\Rightarrow$ *denotes weak convergence or convergence in distribution and* $W_r(\bullet)$ *denotes a univariate standard Brownian Motion in the unit interval and* $V = \lim_{T\to\infty} Var\left(\sum_{t=1}^{T} r_t\right) = \Psi(1)^2\sigma_\epsilon^2$ *—or* 2π *times the spectral density of* r_t *at frequency zero.*

Proof. See Stock (1994). ■

Assumption (A2) emphasizes that k, the horizon return, is not fixed in this approach, but it increases with T at a rate such that k/T remains constant. This approach has the following implication for the limiting distribution of the k-horizon log return:

Lemma 2 *From Lemma 1, assumption (A2) and the Continuous Mapping Theorem (CMT),*

$$\sqrt{\frac{1}{T}}\left(r_{[Tv]}^{([T\delta])} - [T\delta]\,\mu\right) \Rightarrow V^{1/2}\left[W(v) - W(v-\delta)\right],$$

where $r_{[Tv]}^{([T\delta])} = \sum_{i=0}^{[T\delta]-1} r_{[Tv]-i}$.

Proof. See Appendix A. ■

We can now state the main results in this section.

Definition 1 *Define the recursive estimator of the variance ratio for* $[T\delta]$*-horizon log returns as*

$$vr\left(T\delta, T\lambda\right) = \frac{\sum_{t=[T\delta]}^{[T\lambda]}\left[r_t^{([T\delta])} - [T\delta]\,\mu\left(1, T\lambda\right)\right]^2}{[T\delta]\sum_{t=1}^{[T\lambda]}\left[r_t - \mu\left(1, T\lambda\right)\right]^2},$$

where $\delta \le \lambda_0 \le \lambda \le 1$ *and* $\mu(1, T\lambda) = (\sum_{t=1}^{[T\lambda]} r_t)/[T\lambda]$.

The following proposition gives us the limiting distribution of this recursive statistic.

Proposition 1 *Under assumptions (A1)–(A3),* $vr(T\delta, T\lambda)$ *has the following limiting representation:*

$$\begin{aligned} vr\left(T\delta, T\lambda\right) &\Rightarrow \frac{V}{\gamma_0} VR\left(\delta, \lambda\right) \\ &= \frac{V}{\gamma_0}\frac{1}{\delta\lambda}\int_\delta^\lambda \left[W_r(v) - W_r(v-\delta) - \frac{\delta}{\lambda}W_r(\lambda)\right]^2 dv, \end{aligned} \tag{4}$$

where $\delta \leq \lambda_0 \leq \lambda \leq 1$, γ_0 is the unconditional variance of r_t and V is 2π times the spectral density of r_t at frequency zero. When $\lambda = 1$ and r_t is uncorrelated, so that $V = \gamma_0$, the limiting distribution for the full sample estimator of the variance ratio in Richardson and Stock (1989) obtains.

Proof. See Appendix A. ■

Corollary 1 *Proposition 1 and the Continuous Mapping Theorem (CMT) imply the following limiting distribution for the maximal absolute fluctuation of the recursive variance ratio statistic relative to its full-sample estimate:*

$$\max_{\lambda_0 \leq \lambda \leq 1} \left| \frac{vr\left(T\delta, T\lambda\right) - vr\left(T\delta, T\right)}{vr\left(T\delta, T\right)} \right| \Rightarrow \sup_{\lambda_0 \leq \lambda \leq 1} \left| \frac{VR\left(\delta, \lambda\right) - VR\left(\delta, 1\right)}{VR\left(\delta, 1\right)} \right|. \tag{5}$$

Proof. In Proposition 1 we have shown that $vr(T\delta, T\cdot)$ has a limiting representation as a random function in $C[0,1]$. Since both $\max\{\cdot\}$ and $\div$ define continuous functionals, the result in the corollary follows immediately from the CMT (see proof of Lemma 2). ■

Proposition 1 shows that the variance-ratio statistic has an asymptotic representation as a functional of a Brownian Motion process in the unit interval. However, this distribution depends on the autocorrelation properties of one-period returns, as summarized by V. By contrast, Corollary 1 shows that percentage deviations of the recursive estimator of the variance ratio from the full sample estimator have an asymptotic distribution that do not depend on any unknown parameter and it is independent of the autocorrelation properties of stock returns. This allows us to test for the null hypothesis of stability without imposing any null hypothesis on the mean-reversion properties of returns. $[vr(T\delta, T\lambda) - vr(T\delta, T)]/vr(T\delta, T)$ is analogous to the fluctuation test proposed by Ploberger, Krämer and Kontrus (1989), where the scaling factor is $vr(T\delta, T)$. Hence the name "fluctuation statistic" we apply to the argument of the maximal operator in (5). Our empirical results will be based on this statistic.

We now define the reverse-recursive estimator of the variance ratio statistic and characterize its asymptotic distribution.

Definition 2 *Define the reverse-recursive estimator of the variance ratio for $[T\delta]$-horizon log returns as*

$$vr^r\left(T\delta, T\lambda\right) = \frac{\sum_{t=[T\lambda]}^{T} \left[r_t^{([T\delta])} - [T\delta]\,\mu^r\left(1, T\left(\lambda - \delta\right)\right)\right]^2}{[T\delta] \sum_{t=[T(\lambda-\delta)]+1}^{T} \left[r_t - \mu^r\left(1, T\left(\lambda - \delta\right)\right)\right]^2},$$

where $\delta \leq \lambda \leq \lambda_0 \leq 1$ *and* $\mu^r(1, T(\lambda-\delta)) = (\sum_{t=[T(\lambda-\delta)]+1}^{T} r_t)/[T(1-\lambda+\delta)]$.

Proposition 2 *Under assumptions (A1)–(A4'),* $vr^r(T\delta, T\lambda)$ *has the following limiting representation:*

$$\begin{aligned} vr^r\left(T\delta, T\lambda\right) &\Rightarrow \frac{V}{\gamma_0} VR^r\left(\delta, \lambda\right) \\ &= \frac{V}{\gamma_0} \frac{1}{\delta\left(1-\lambda+\delta\right)} \int_{\lambda}^{1} \left\{ W_r\left(v\right) - W_r\left(v-\delta\right) - \frac{\delta}{1-\lambda+\delta}\left[W_r\left(1\right) - W_r\left(\lambda-\delta\right)\right] \right\}^2 dv, \end{aligned} \tag{6}$$

where $\delta \leq \lambda \leq \lambda_0 \leq 1$*,* γ_0 *is the unconditional variance of* r_t *and* V *is* 2π *times the spectral density of* r_t *at frequency zero.*

Proof. See Appendix A. ■

Corollary 2 *Proposition 2 and the CMT imply the following limiting distribution for the maximal absolute fluctuation of the reverse-recursive variance ratio statistic relative to its full-sample estimate:*

$$\max_{\delta \leq \lambda \leq \lambda_0} \left| \frac{vr^r\left(T\delta, T\lambda\right) - vr(T\delta, T)}{vr(T\delta, T)} \right| \Rightarrow \sup_{\delta \leq \lambda \leq \lambda_0} \left| \frac{VR^r\left(\delta, \lambda\right) - VR\left(\delta, 1\right)}{VR\left(\delta, 1\right)} \right|. \tag{7}$$

Proof. The proof follows the same lines as the proof of Corollary 1. ■

The results for the reverse-recursive estimator for the variance ratio statistic are analogous to those for the recursive variance ratio statistic. Nevertheless, it is important to emphasize that a reverse-recursive statistic is not redundant with respect to a recursive statistic but a useful complement, since it helps to detect structural breaks in the early part of the sample which a recursive estimator misses because the first recursive estimate uses the first $[T\lambda_0]$ observations.

We now turn to the limiting distributions of the recursive and reverse-recursive estimates of the Fama-French regression statistic for univariate predictability in asset returns.

Definition 3 *Let* $b(T\delta, T\lambda)$ *denote the OLS recursive estimator of* $\beta(\delta)$ *in*

$$r_t^{(T\delta)} = \alpha(\delta) + \beta(\delta) r_{t-[T\delta]}^{[T\delta]} + u_t^{[T\delta]}. \tag{8}$$

We define the recursive h.a.c. fluctuation t-statistic as

$$t\,(T\delta,T\lambda)=\frac{\sqrt{T}\,[b\,(T\delta,T\lambda)-b\,(T\delta,T)]}{\sqrt{Q\,(T\delta,T\lambda)\,\Omega\,(T\delta,T)\,Q\,(T\delta,T\lambda)}},$$

where $2\delta\leq\lambda_0\leq\lambda\leq 1$, $Q(T\delta,T\lambda)^{-1}=([T\lambda]-2[T\delta]+1)^{-1}\sum_{t=2[T\delta]}^{[T\lambda]}\left[r_{t-[T\delta]}^{[T\delta]}-\mu_{-[T\delta]}(T\delta,T\lambda)\right]^2$, $\mu_{-[T\delta]}(T\delta,T\lambda)$ *is the recursive arithmetic mean of* $r_{t-[T\delta]}^{[T\delta]}$ *(see Appendix A), and* $\Omega(T\delta,T)$ *is a heteroskedasticity and autocorrelation consistent (h.a.c.) full-sample estimator of* $\Omega=\lim_{T\rightarrow\infty}Var\left(\sum_{t=1}^{T}z_t\right)$, *where* $z_t=r_{t-[T\delta]}^{[T\delta]}\widehat{u}_t^{[T\delta]}$ *and* $\widehat{u}_t^{[T\delta]}$ *are the full-sample OLS residuals from (8).*

Proposition 3 *Under assumptions (A1)-(A3),* $b(T\delta,T\lambda)$ *has the following asymptotic distribution,*

$$\begin{aligned} b(T\delta,T\lambda) &\Rightarrow \beta\,(\delta,\lambda) \qquad (9)\\ &= \frac{\int_{2\delta}^{\lambda}\left[W_r\,(v-\delta)-W_r\,(v-2\delta)-\mu_{-\delta}\,(\delta,\lambda)\right]\left[W_r\,(v)-W_r\,(v-\delta)-\mu\,(\delta,\lambda)\right]dv}{\int_{2\delta}^{\lambda}\left[W_r\,(v-\delta)-W_r\,(v-2\delta)-\mu_{-\delta}\,(\delta,\lambda)\right]^2dv}, \end{aligned}$$

and $T^{-1}t(T\delta,T\lambda)$,

$$\frac{1}{T}t(T\delta,T\lambda)\Rightarrow t(\delta,\lambda)=\frac{1}{\lambda-2\delta}\frac{\beta\,(\delta,\lambda)-\beta\,(\delta,1)}{\sqrt{Q\,(\delta,\lambda)\,\Omega\,(\delta,1)\,Q\,(\delta,\lambda)}}, \qquad (10)$$

where $2\delta\leq\lambda_0\leq\lambda\leq 1$, *and* $\mu(\delta,\lambda)$, $\mu_{-\delta}(\delta,\lambda)$, $Q(\delta,\lambda)$ *and* $\Omega(\delta,1)$, *given Appendix A, are functionals of standard Brownian Motion processes and they do not depend on any unknown parameter. The limiting distribution of the full-sample estimator* $b(T\delta,T)$ *in Richardson and Stock (1989) obtains from (9) for* $\lambda=1$.

Proof. See Appendix A. ■

Proposition 3 shows that, under the $k/T\rightarrow\delta$ approach, the limiting distribution of the recursive OLS estimator of $\beta(\delta)$ in the Fama-French univariate regression model (8) does not depend on any unknown parameters. Moreover, it does not depend on the autocorrelation properties of r_t, since (9) is not a function of V. This result is convenient to test for structural stability, because it allows us to formulate the stability test in terms of $b(T\delta,T\lambda)-b(T\delta,T)$, with no need of a scaling factor. However, since both under the null hypothesis of no autocorrelation in r_t and the alternative of non-zero autocorrelation, the full sample estimator $b(T\delta,T)$ has the same limiting distribution, one might be tempted to conclude that the $k/T \rightarrow \delta$ approach of Richardson and Stock (1989) is not adequate to test for univariate predictability in returns..

To gain intuition on why the distribution of the estimator does not depend on the autocorrelation properties of one-period returns, it is useful to write the alternative representation of the slope in (8) in terms of one-period log return autocorrelations under the fixed-horizon approach (Kim, Nelson, Startz, 1991):

$$\beta\left(k\right) = \frac{\sum_{j=1}^{k} j\rho_j + \sum_{j=1}^{k-1}\left(k-j\right)\rho_{k+j}}{k + 2\sum_{j=1}^{k}\rho_j}, \tag{11}$$

where ρ_j is the jth.-order autocorrelation coefficient of r_t. Under the fixed-horizon approach, $\beta(k) = 0$ when r_t is uncorrelated and, if r_t is autocorrelated and covariance stationary, $\beta(k) \neq 0$ for low order horizons, but it approaches zero as the horizon becomes very large, since the numerator of (11) approaches a fixed limit but the denominator grows without bound. Hence, under both the null of no mean-reversion and the alternative of mean-reversion, $\beta(k)$ has the same limiting representation when $k \rightarrow \infty$. This is what happens under the $k/T \rightarrow \delta$ approach, because we allow the horizon $k = [T\delta] \rightarrow \infty$ as $T \rightarrow \infty$. However, the point that Richardson and Stock (1989) made convincingly is that the $k/T \rightarrow \delta$ approach behaves much better in small samples than the fixed-horizon approach does. They showed that, under the $k/T \rightarrow \delta$ approach, convergence is achieved even for very low values of T, which translates into large gains in terms of size of the tests in finite samples.

It is interesting to note that the OLS recursive t-statistic itself diverge. It is $O_p(T)$, while the OLS recursive estimator of the slope is $O_p(1)$. The distribution of T^{-1} times the recursive t-statistic does not depend on any unknown parameter either. However, it is computationally much more involved than the distribution of the slope. From this point of view, $b(T\delta, T\lambda)$ is more suitable than $t(T\delta, T\lambda)$ to test for stability. We will use the maximum absolute deviations of $b(T\delta, T\lambda)$ from $b(T\delta, T)$ as the basis for our empirical analysis. The distribution of this fluctuation statistic is given in the following corollary.

Corollary 3 *Proposition 3 and the CMT imply the following limiting distribution for the maximal absolute fluctuation of the recursive Fama-French regression statistic relative to its full-sample estimate:*

$$\max_{\lambda_0 \leq \lambda \leq 1} \left|b(T\delta, T\lambda) - b(T\delta, T)\right| \Rightarrow \sup_{\lambda_0 \leq \lambda \leq 1} \left|\beta(\delta, \lambda) - \beta(\delta, 1)\right|. \tag{12}$$

Proof. The proof follows the same lines as the proof of Corollary 1. ■

Definition 4 *Let* $b^r(T\delta, T\lambda)$ *denote the OLS reverse-recursive estimator of* $\beta(\delta)$ *in (8). We define the reverse-recursive h.a.c. fluctuation t-statistic as*

$$t^r\left(T\delta, T\lambda\right) = \frac{\sqrt{T}\left[b^r\left(T\delta, T\lambda\right) - b\left(T\delta, T\right)\right]}{\sqrt{Q^r\left(T\delta, T\lambda\right)\Omega\left(T\delta, T\right)Q^r\left(T\delta, T\lambda\right)}},$$

where $2\delta \leq \lambda \leq \lambda_0 \leq 1$, $Q^r(T\delta, T\lambda)^{-1} = ([T(1-\lambda)]+1)^{-1}\sum_{t=[T\lambda]}^{T}\left[r_{t-[T\delta]}^{[T\delta]} - \mu_{-[T\delta]}^r(T\delta, T\lambda)\right]^2$, $\mu_{-[T\delta]}^r(T\delta, T\lambda)$ *is the reverse-recursive arithmetic mean of* $r_{t-[T\delta]}^{[T\delta]}$ *(see Appendix A), and* $\Omega(T\delta, T)$ *is a h.a.c. full-sample estimator of* Ω*—which is defined in Proposition 3.*

Finally, we characterize the asymptotic behavior of the reverse-recursive estimator of $\beta(\delta)$.

Proposition 4 *Under assumptions (A1)-(A3),* $b^r(T\delta, T\lambda)$ *has the following stochastic limit,*

$$b^r(T\delta, T\lambda) \Rightarrow \beta^r\left(\delta, \lambda\right) \tag{13}$$

$$= \frac{\int_\lambda^1\left[W_r\left(v-\delta\right) - W_r\left(v-2\delta\right) - \mu_{-\delta}^r\left(\delta, \lambda\right)\right]\left[W_r\left(v\right) - W_r\left(v-\delta\right) - \mu^r\left(\delta, \lambda\right)\right]dv}{\int_\lambda^1\left[W_r\left(v-\delta\right) - W_r\left(v-2\delta\right) - \mu_{-\delta}^r\left(\delta, \lambda\right)\right]^2 dv},$$

and $T^{-1}t^r(T\delta, T\lambda)$,

$$\frac{1}{T}t^r\left(T\delta, T\lambda\right) \Rightarrow t^r(\delta, \lambda) = \frac{1}{1-\lambda}\frac{\beta^r\left(\delta, \lambda\right) - \beta\left(\delta, 1\right)}{\sqrt{Q^r\left(\delta, \lambda\right)\Omega\left(\delta, 1\right)Q^r\left(\delta, \lambda\right)}}, \tag{14}$$

where $\mu^r(\delta, \lambda)$, $\mu_{-\delta}^r(\delta, \lambda)$, $Q^r(\delta, \lambda)$ *and* $\Omega(\delta, 1)$*, given Appendix A, are functionals of Brownian Motion processes and they do not depend on any unknown parameter.*

Proof. See Appendix A. ■

Corollary 4 *Proposition 4 and the CMT imply the following limiting distribution for the maximal absolute fluctuation of the recursive Fama-French regression statistic relative to its full-sample estimate:*

$$\max_{2\delta \leq \lambda \leq \lambda_0}\left|b^r(T\delta, T\lambda) - b(T\delta, T)\right| \Rightarrow \sup_{2\delta \leq \lambda \leq \lambda_0}\left|\beta^r(\delta, \lambda) - \beta(\delta, 1)\right|. \tag{15}$$

Proof. The proof follows the same lines as the proof of Corollary 1. ■

2.4 Asymptotic representation of recursive and reverse-recursive statistics in the dividend-price ratio regression

We are interested in the limiting distribution of the OLS recursive, reverse-recursive and full-sample estimators of the slope in the regression model,

$$r_t = \alpha_0 + \alpha_1 x_{t-1} + \varepsilon_t, \tag{16}$$

where r_t is the one-period log return and x_t is a predictor variable (such as the log dividend-price ratio or the book-to-market ratio). We are going to assume throughout this section that:

(B1) $\{\varepsilon_t\}$ is a martingale difference sequence (m.d.s.) with $E[\varepsilon_t^2|\varepsilon_{t-1},...] = \sigma_{\varepsilon,t}^2 > 0$, $\lim_{T\to\infty}\mathrm{E}[T^{-1}\sum_{t=1}^{\infty}\sigma_{\varepsilon,t}^2] = E[\varepsilon_t^2] = \sigma_\varepsilon^2 < \infty$, $E[\varepsilon_t^4|\varepsilon_{t-1},...] < \infty$ and $E[r_0^2] < \infty$.

(B2) x_t admits the following representation as a local-to-unity process:

$$x_t = \phi x_{t-1} + \eta_t, \tag{17}$$

where $\phi = 1 + c/T$, $c < 0$, and $\{\eta_t\}$ is a m.d.s. with $E[\eta_t^2|\eta_{t-1},...] = \sigma_{\eta,t}^2 > 0$, $\lim_{T\to\infty}\mathrm{E}[T^{-1}\sum_{t=1}^{\infty}\sigma_{\eta,t}^2] = E[\eta_t^2] = \sigma_\eta^2 < \infty$, $E[\eta_t^4|\eta_{t-1},...] < \infty$ and $E[x_0^2] < \infty$.

(B3) Innovations in r_t and x_t are contemporaneously correlated: $\mathrm{E}\left[\varepsilon_t\eta_t\right] = \sigma_{\varepsilon\eta} \neq 0$.

Assumption (B1) imposes the weakest possible restrictions on the innovations in (16) that allow us to obtain the asymptotic distributions of the recursive estimators that follow. In particular, it ensures that ε_t verifies the FCLT (see Lemma 1), so $\sum_{t=1}^{[Tv]}\varepsilon_t \Rightarrow \sigma_e W_r(v)$. Assumption (B2) allows for high persistence in the predictor variable x_t by assuming that ϕ, the autorregressive root in (17), is local to unity, while assumption (B3) allows x_t *not* to be strictly exogenous, but with innovations correlated with innovations in log returns[3]. Both persistence and non-zero correlation with log returns are observed in typical regressions of log returns onto the log dividend price ratio—see, for example, Campbell and Viceira (1996).

Under assumption (B2) we have that $T^{-1/2}x_t$ itself, rather than the accumulated sum of x_t, converges to σ_η times a diffusion process (Phillips, 1987). We state this result as a Lemma for ease of reference.

[3] Wright (1996) has considered the asymptotic distribution of some leading structural stability tests when the regressors have roots near to unity, but they are strictly exogeneous.

Lemma 3 (FCLT for near integrated processes) *Under assumption (B2), $T^{-1/2}$ times x_t obeys the FCLT for near-integrated processes. That is, it converges in distribution to a diffusion process:*

$$\sqrt{\frac{1}{T}} x_{[Tv]} \Rightarrow \sigma_\eta J_c(v),$$

where $J_c(\bullet)$ denotes a 1-dimensional Ornstein-Uhlenbeck process, i.e., $J_c(v)$ is the solution to the stochastic differential equation $dJ_c(v) = cJ_c(v)dv + dW_\eta(v)$, where $W_\eta(\bullet)$ is a standard Brownian Motion process in the unit interval.

Proof. See Phillips (1987, Lemma 1) and Stock (1994, Example 4). ■

Lemma 3 is adapted from results in Chan and Wei (1987), Phillips (1988), Hansen (1992) and Hamilton (1994). It is key to find the limiting distribution of the estimators of the slope in (16) that we present below:

Proposition 5 *Let $a_1(T\lambda)$ denote the recursive OLS estimator of α_1 in (16). Under assumptions (B1)-(B3), $T(a_1(T\lambda) - \alpha_1)$ has the following limiting distribution:*

$$\begin{aligned} T\left(a_1(T\lambda) - \alpha_1\right) &\Rightarrow \alpha_1(T\lambda) \\ &= \frac{\sigma_\varepsilon}{\sigma_\eta}\rho_{\varepsilon\eta}\left\{\frac{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right] dW_\eta(v)}{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right]^2 dv}\right\} \\ &\quad + \frac{\sigma_\varepsilon}{\sigma_\eta}\left(1-\rho_{\varepsilon\eta}^2\right)^{1/2}\left\{\frac{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right] dW_u(v)}{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right]^2 dv}\right\}, \end{aligned} \tag{18}$$

where $\rho_{\varepsilon\eta} = \sigma_{\varepsilon\eta}/(\sigma_\varepsilon\sigma_\eta)$, $W_u(\bullet)$ denotes a Brownian Motion process in the unit interval independent of $W_\eta(\bullet)$ (and $J_c(\bullet)$), and $0 < \lambda \leq 1$. When $\lambda = 1$, the limiting distribution of the full sample estimator in Elliott and Stock (1994) obtains.

Proof. See Appendix B. ■

Corollary 5 *Proposition 5 and the Continuous Mapping Theorem (CMT) imply the following limiting distribution for the maximal absolute fluctuation of the OLS recursive estimator of the slope in (16) relative to its full-sample estimate:*

$$\max_{\lambda_0 \leq \lambda < 1} T\left|a_1(T\lambda) - a_1(T)\right| \Rightarrow \sup_{\lambda_0 \leq \lambda < 1} \left|\alpha_1(\lambda) - \alpha_1(1)\right|, \tag{19}$$

where $0 < \lambda_0 < 1$.

Proof. The proof follows the same lines as the proof of Corollary 1. ■

Proposition 6 *Let $a_1^r(T\lambda)$ denote the reverse-recursive OLS estimator of α_1 in (16). Under assumptions (B1)-(B3), $T(a_1^r(T\lambda)-\alpha_1)$ has the following limiting distribution:*

$$\begin{aligned} T\left(a_1^r\left(T\lambda\right)-\alpha_1\right) &\Rightarrow \alpha_1^r\left(T\lambda\right) \qquad (20) \\ &= \frac{\sigma_\varepsilon}{\sigma_\eta}\rho_{\varepsilon\eta}\left\{\frac{\int_\lambda^1\left[J_c\left(v\right)-\frac{1}{1-\lambda}\int_\lambda^1 J_c\left(v\right)dv\right]dW_\eta\left(v\right)}{\int_\lambda^1\left[J_c\left(v\right)-\frac{1}{1-\lambda}\int_\lambda^1 J_c\left(v\right)dv\right]^2 dv}\right\} \\ &\quad +\frac{\sigma_\varepsilon}{\sigma_\eta}\left(1-\rho_{\varepsilon\eta}^2\right)^{1/2}\left\{\frac{\int_\lambda^1\left[J_c\left(v\right)-\frac{1}{1-\lambda}\int_\lambda^1 J_c\left(v\right)dv\right]dW_u\left(v\right)}{\int_\lambda^1\left[J_c\left(v\right)-\frac{1}{1-\lambda}\int_\lambda^1 J_c\left(v\right)dv\right]^2 dv}\right\}, \end{aligned}$$

where $\rho_{\varepsilon\eta}=\sigma_{\varepsilon\eta}/(\sigma_\varepsilon\sigma_\eta)$, $W_u(\bullet)$ denotes a Brownian Motion process in the unit interval independent of $W_\eta(\bullet)$ (and $J_c(\bullet)$), and $0\leq\lambda<1$.

Proof. See Appendix B. ■

Corollary 6 *Proposition 6 and the Continuous Mapping Theorem (CMT) imply the following limiting distribution for the maximal absolute fluctuation of the OLS reverse-recursive estimator of the slope in (16) relative to its full-sample estimate:*

$$\max_{0<\lambda\leq\lambda_0} T\left|a_1^r\left(T\lambda\right)-a_1\left(T\right)\right| \Rightarrow \sup_{0<\lambda\leq\lambda_0}\left|\alpha_1^r\left(\lambda\right)-\alpha_1\left(1\right)\right|, \qquad (21)$$

where $0<\lambda_0<1$.

Proof. The proof follows the same lines as the proof of Corollary 1. ■

The distributional results in Propositions 5 and 6 are also useful to obtain the limiting distribution of the sequence of Chow test statistic for structural change at dates $\{[T\lambda]:0<\lambda_0\leq\lambda\leq 1-\lambda_0<1\}$. From this distribution we can obtain, by simple application of the CMT, the distribution of the Quandt Likelihood Ratio (QLR) test (Quandt, 1960), which is the max of these statistics. This distribution is given in the corollary to the following proposition.

Proposition 7 *Consider the Chow-Wald statistic to test for a structural break in the slope of (16) at date* $[T\lambda]$,

$$F(T\lambda) = \frac{1}{\widehat{\sigma}_{\varepsilon}^{2}} \frac{\left[a_1(T\lambda) - a_1^r(T\lambda)\right]^2}{\left[\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)^2\right]^{-1} + \left[\sum_{t=[T\lambda]+1}^{T}\left(x_{t-1} - \overline{x}_{[T(1-\lambda)]}\right)^2\right]^{-1}}, \tag{22}$$

where $a_1(T\lambda)$ *is the recursive OLS estimator and* $a_1^r(T\lambda)$ *is the reverse-recursive estimator of* α_1 *in (16),* $\widehat{\sigma}_{\varepsilon}^{2} = \sum_{t=2}^{T}\{(r_t - \overline{r}_{[T\cdot 1]}) - a_1(T)(x_{t-1} - \overline{x}_{[T\cdot 1]})\}^2/(T-1)$, $\overline{z}_{[T\lambda]} = \sum_{t=2}^{[T\lambda]} z_t/([T\lambda]-1)$, $z_t = (r_t, x_{t-1})$ *and* $\overline{x}_{[T(1-\lambda)]} = \sum_{t=[T\lambda]+1}^{T} x_{t-1}/[T(1-\lambda)]$.

Under assumptions (B1)-(B3) and Propositions 5 and 6, $F(T\lambda)$ *converges in distribution to:*

$$F(T\lambda) \Rightarrow F(\lambda) = \frac{1}{\digamma_3(J_c)}\left[\rho_{\varepsilon\eta}\digamma_1(J_c, W_\eta) + \left(1-\rho_{\varepsilon\eta}^2\right)^{1/2}\digamma_2(J_c, W_u)\right]^2,$$

where $\rho_{\varepsilon\eta} = \sigma_{\varepsilon\eta}/(\sigma_\varepsilon\sigma_\eta)$, *and* $\digamma_1$, $\digamma_2$ *and* $\digamma_3$ *are continuous functionals of* J_c, W_η *and* W_u *given in Appendix B,* J_c *is the O-U process defined in Lemma 3 and* (W_η, W_u) *are Brownian Motion processes in the unit interval, with* W_u *independent of* J_c *and* W_η.

Proof. See Appendix B. ■

Corollary 7 *Proposition 7 and the CMT imply the following limiting distribution of the maximal Chow-Wald test statistic (22)over a range of break dates—the QLR statistic:*

$$QLR = \max_{\lambda_0 \le \lambda \le 1-\lambda_0} F(T\lambda) \Rightarrow \sup_{\lambda_0 \le \lambda \le 1-\lambda_0} F(\lambda), \tag{23}$$

where $0 < \lambda_0 < 1$.

Proof. The proof follows the same lines as the proof of Corollary 1. ■

Propositions 5 and 6 show that both the OLS recursive and reverse-recursive estimators of the slope in (16) and the full-sample estimator—which obtains setting $\lambda = 1$ in (18)—are superconsistent: They converge to the true slope α_1 at rate T , which is faster than the usual $\sqrt{T}$-rate. Unlike the fluctuation statistics presented in the previous section, the distribution of the fluctuation statistics (19) and (20) and the QLR statistic (23) depend on a set of nuisance parameters, $\sigma_\varepsilon/\sigma_\eta$, $\rho_{\varepsilon\eta}$ and c. The second moments $\sigma_\varepsilon/\sigma_\eta$ and $\rho_{\varepsilon\eta}$ are consistently estimable, but c is not. However, we can still construct conservative tests for the null of stability. We discuss how to do it in section 3.

3 Monte Carlo Evidence

The asymptotic distributions of the stability test given in section 2 are non-standard distributions for which there are no tables to look for critical values. However, they are easy to evaluate through Monte Carlo experiments. These Monte Carlo experiments proceed by generating repeated samples of discretized versions of the continuous time functionals that describe the distributions for a large enough sample size T.

Tables 1 through 4 report the results of several Monte Carlo experiments that reproduce the asymptotic distributions of the univariate fluctuation statistics (5), (7), (12) and (15) for three values of δ using discrete $10,000$ approximations of size T to these random functionals, for several values of T and δ—we only report results for three values of δ, though we have performed the experiments for all values of δ we use in our empirical analysis. We have considered several values of T in order to evaluate how fast these discrete approximations converge to their asymptotic limits. We choose values of T which are multiples or dividers of 840, the actual number of monthly observations in our sample of one-period log returns (log returns in the NYSE-AMEX-NASDAQ stock markets, from 1926.01 through 1995.12). The values of δ are set so that $[T\delta]$ equals multiples of 12 when $T = 840$. The trimming parameter λ_0 is set so 25% of the effective sample used in the first recursion or reverse-recursion[4].

These tables show that our discrete approximations converge reasonably fast to the asymptotic distributions in all of the univariate fluctuation statistics. The differences in percentile values when $T = 312$ and $T = 2520$ are minimal over all percentiles at all horizons, with the exception perhaps of the extreme right percentiles of the fluctuation statistics based on the Fama-French univariate regression statistic. According to these results, for T equal to the sample size we use in our empirical exercise ($T = 840$) we achieve a good enough to the asymptotic distribution of our univariate fluctuation statistics under the null hypothesis of stability. Hence, the result in Richardson and Stock (1989) on the fast convergence of full sample estimators to their asymptotic distributions under their $k/T \to \delta$ approach is also valid for the recursive and reverse-recursive estimators.

Figure 1 plots histograms of the Monte Carlo distributions—when $T = 840$—of the

[4]For the fluctuation statistics based on the recursive and reverse-recursive variance ratio, we define the effective sample as $[T(1-\delta)]$, i.e., the total number of observations minus those needed to compute the first $[T\delta]$-horizon return observation, that we need to compute the first recursive (or reverse-recursive) variance ratio. For the fluctuation statistics based on the univariate Fama-French regression statistic, the effective sample is $[T(1-2\delta)]$, i.e., the total number of observations minus those needed to compute the first two $[T\delta]$-horizon return observations, that we need to compute the first recursive (or reverse-recursive) regression statistic.

fluctuation statistics (5) and (12) for the same range of values of δ we consider in Tables 1 through 4. Each histogram also reports the mean, standard deviation and coefficient of variation of the distribution. Both tables and figures show the distributions of the fluctuation statistics are skewed to the right and they shift to the right as δ increases, though the percentile values do not increase linearly with δ. Both the mean and standard deviation of the distribution, though they also increase less than proportionally with δ. The coefficient of variation slightly decreases with δ in the fluctuation statistic based on the variance ratio, while it increases with δ in the fluctuation statistic based on the Fama-French univariate regression statistic. Therefore, as we consider larger horizons, the maximum absolute deviations of the recursive (reverse-recursive) statistics from their full sample estimates that are consistent with the null distribution tend to increase. Moreover, in the case of the Fama-French univariate regression, the range of admissible values of these deviations—in the sense of being consistent with the null—also increases with the horizon. However, they do not increase linearly with the horizon.

Overall, our Monte Carlo experiments for the univariate statistics show that large absolute deviations of the recursive and reverse-recursive statistics are consistent with the distributions under the null. For example, the distribution reported in Table 1 implies that the recursive variance ratio may diverge from its full sample estimate, under the null of stability, as much as 47%, 78% and 85%, for horizons of 1, 5 and 8 years respectively. For the reverse-recursive variance ratio the admissible deviations are similar. The distributions for the fluctuation tests based on the Fama-French univariate regression statistic also allow for large deviations of the recursive and reverse-recursive statistics from the full sample estimates to be consistent with the same return generating process.

Table 5 reports the Monte Carlo distributions of the multivariate fluctuation statistics (19), (20) and (23). We have shown in section 2.4 that the asymptotic distributions of these statistics depend on the unknown parameters $\sigma_\varepsilon/\sigma_\eta$, $\rho_{\varepsilon\eta}$ and c. We can substitute them for consistent estimators and the asymptotic results in section 2.4 still hold. However, while $\sigma_\varepsilon/\sigma_\eta$ and $\rho_{\varepsilon\eta}$ are consistently estimable from the data[5], c is not. Nevertheless, we can still put our results to work by using the approach suggested in Wright (1996) to construct critical values for structural stability tests when the regressors are nearly integrated processes. The first step in this approach is to construct a 95% confidence interval for it based on the sample Dickey-Fuller t-statistic for ϕ in (17), following the methodology proposed in Stock (1991) to construct confidence intervals for c. The basic insight of this methodology is to invert the acceptance region of the test statistic whose distribution depends on c[6].

[5] We have shown in Proposition 7 that the full-sample OLS estimator of the residual variance in (16) is consistent for σ_ε^2. Using similar arguments we can also prove that $\sigma_\eta^2 = \sum_{t=2}^T \widehat{\eta}_t^2/(T-1)$ and $\sigma_{\varepsilon\eta} = \sum_{t=2}^T \widehat{\varepsilon}_t \widehat{\eta}_t/(T-1)$, where $\widehat{\eta}_t = x_t - x_{t-1}$, are consistent for σ_η^2 and $\sigma_{\eta\varepsilon}$ respectively.

[6] To do that we construct, through Monte Carlo experiments, a set of distributions of the test statistic

Given this confidence interval, we compute, through Monte Carlo experiments, the asymptotic distributions in Propositions 5, 6 and 7 and Corollaries 5 and 6 for a grid of values of c in the confidence interval and consistent estimators of $\sigma_\varepsilon/\sigma_\eta$ and $\rho_{\varepsilon\eta}$. Hence we end up with a set of distributions for each statistic—one per each value of c. By choosing as critical value the largest 95% percentile of this set of distributions we can construct a conservative test of the null of parameter stability for that particular statistic, since by the Bonferroni inequality, the probability that the statistic takes values above this critical value is always less or equal than 10%.

Panel A in Table 5 reports 10% critical values for (18)—for $\lambda = 1$ and $\alpha_1 = 0$—, (19), (21) and (23) constructed using the method described above. The 95% confidence interval for c shown in the table is based on the sample value of the Dickey-Fuller t-statistic for ϕ in (17) generated by our data set[7]—one-period log returns and log dividend-price ratios in the NYSE-AMEX-NASDAQ stock markets, from 1926.12 through 1995.12. The values of $\sigma_\varepsilon/\sigma_\eta$ and $\rho_{\varepsilon\eta}$ are all based on consistent estimates obtained from our data set. Monte Carlo distributions are based on 10,000 simulations of the continuous time process for the random functionals (18), (19), (21) and (23). To simulate the processes we use discrete approximations of size $T = 829$, and a grid of 11 equally-spaced values of c in its 95% confidence interval. The trimming parameter λ_0 is set to 25%. Panel B reports the average median across the set of distributions generated by the grid of values of c.

The critical values and the medians of the full sample estimator and the fluctuation statistics are very sensitive to changes in the values of the nuisance parameters $\sigma_\varepsilon/\sigma_\eta$ and $\rho_{\varepsilon\eta}$. By contrast, the critical values for the QLR statistic seem to be quite robust to changes in the value of the nuisance parameters. They are also very similar to those reported by Wright (1996) for the case in which the regressor is strictly exogenous. Moreover, Wright (1996) has shown that the QLR statistic is robust to changes in the nuisance parameter c. Hence, the QLR statistic seems to be robust to regressor endogeneity and to changes in the value of c.

τ for an ample grid of values of c. For each percentile p, this generates a function in the (τ, c) space by joining the points $(\tau(p, c_i), c_i)$, where $\tau(p, c_i)$ is the value of τ corresponding to the pth-percentile in the empirical distribution of τ when $c = c_i$. Given an observed value $\widehat{\tau}$ of the test statistic, we look for that interval of values of c which are consistent with the observed value of τ by looking at the ordinates of $\widehat{\tau}$ in the (τ, c) space. For example, Table 5 shows this confidence interval for c based on the sample value of the Dickey-Fuller t-statistic for ϕ in (17) generated by our data set.

[7] We are grateful to Jonathan H. Wright for allowing us to use a set of tables he has constructed that report 90% and 95% confidence intervals for c for a wide range of values of the Dickey-Fuller t-statistic for ϕ.

4 Empirical Results

4.1 The data

Our empirical exercise is based on monthly data from 1925.12 through 1995.12 on returns and dividends on the value-weighted and equally-weighted portfolios inclusive of all securities traded in the NYSE, AMEX and NASDAQ markets during this period, as reported in the Indices files of the Center for Research in Security Prices (CRSP, 1996). Log real returns on each portfolio are log nominal returns on the portfolio minus the log rate of change in the Consumer Price Index for All Urban Consumers, not seasonally adjusted, as reported in the SBBI files of CRSP (1996). Log excess returns on each portfolio are log nominal returns on the portfolio in excess of log returns on US Treasury Bills, as reported in the SBBI files of CRSP (1996). We consider both the value-weighted portfolio and the equally-weighted portfolio because they convey different information: The weighting scheme in the equally-weighted portfolio implies a relatively larger representation of firms with small market capitalization in the computation of returns than in the value-weighted portfolio, whose returns are essentially the returns on firms with large market capitalization.

Following the standard convention in the literature, the dividend-price ratio at month t is computed as in Fama and French (1989). That is we compute a moving average of monthly dividends during the previous year and dividing it by the ex-dividend value of the portfolio at the end of month t: $(D/P)_t = \sum_{t=0}^{11} D_{t-i}/P_t$. This way we remove the seasonality in dividend payments that shows up in the aggregate series.

4.2 Testing for the stability and significance of univariate predictability in stock returns.

Tables 6 through 9 report full sample estimates and recursive and reverse-recursive estimates of the variance ratio (1) and the Fama-French univariate regression statistic (2) for monthly log real returns and log excess returns on the CRSP equally-weighted and value-weighted portfolios. All these statistics have been computed—and their asymptotic distributions evaluated through the Monte Carlo experiments described in section 3—for horizons $[T\delta]/12 =$ 1, 2, 3, 4, 5, 8 and 10 years. The trimming parameter for the recursive and reverse recursive estimators is set to 25% of the effective sample[8]. Since these statistics are not independent, we also compute average statistics across horizons[9]. Figures 2 and 3 plot recursive and

[8] See footnote 5.

[9] For example, we compute the average recursive variance ratio statistic as $vr(\overline{T\delta}, T\lambda) = \sum_{i=1}^{7} vr(T\delta_i, T\lambda)/7$, from which the full-sample variance ratio statistic obtains by setting $\lambda = 1$, and we

reverse-recursive estimates of the recursive and reverse-recursive statistics for horizons 1, 3, 5 and 8 years generated by log real returns on the equally-weighted portfolio and the value-weighted portfolio. The upper four plots correspond to the variance-ratio statistic, while the bottom four correspond to the Fama-French univariate regression statistic. In each plot there are three horizontal solid lines. The middle one crosses the y-axis at the value of the full-sample estimate of the statistic, while the other two lines are drawn at the boundaries of a most-conservative, symmetric 95% confidence interval for the recursive and reverse-recursive statistic[10].

The second and third blocks of Tables 6 through 9 report the stability tests based on the variance ratio. The evidence against the null hypothesis of stability in the autocorrelation structure of returns is almost non-existent for both log real returns and log excess returns on the equally-weighted portfolio, across all horizons (see Tables 6 and 8). Moreover, in some cases the absolute percentage deviations of the recursive and reverse-recursive estimates of the variance ratio from the full-sample estimates are so small that they are on the left tail of the Monte Carlo distribution of the fluctuation statistics. The null of stability is rejected at conventional significance levels only for the reverse-recursive fluctuation statistic at the longest horizons. This result is not surprising, since we have shown in section 3 that large percentage deviations are consistent with the null distributions of the fluctuation statistics. For example, the two upper left plots in Figure 2 show that any recursive or reverse-recursive estimate of the variance ratio between .65 and 1.8 is consistent, in the sense of lying in a 95% confidence interval, with a full-sample variance ratio of 1.23 at a 1-year horizon; at a 5-year horizon, the confidence interval lies between .15 and 1.22 for a full sample estimator of .73.

By contrast, Tables 7 and 9 show evidence against the null in the value-weighted portfolio. Even though the recursive fluctuation test does not reject the null at conventional significance levels at most horizons, the reverse-recursive fluctuation test does reject the

compute the average fluctuation statistic based on the recursive variance ratio as

$$\max_{\delta \le \lambda \le \lambda_0} \left| \frac{vr(\overline{T\delta}, T\lambda) - vr(\overline{T\delta}, T)}{vr(\overline{T\delta}, T)} \right|.$$

The asymptotic representation of the average statistics obtains inmediately from Proposition 1 and the CMT following the same argument we use to obtain Corollary 1. For example, $vr(\overline{T\delta}, T\lambda) \Rightarrow VR(\overline{\delta}, \lambda) = \sum_{i=1}^{7} VR(\delta_i, \lambda)/7$. We compute the other average statistics in the "Avrg." column of Tables 6 through 9 and their asymptotic distributions in a similar way. For each fluctuation test, we report its value, its p-value, the date of its occurrence and the value of the statistic on which the fluctuation test is based on that date.

[10]For the recursive and reverse-recursive variance ratio, this confidence interval is $[vr(T\delta, T) \times (1 - \underline{c}_{.95}), vr(T\delta, T) \times (1 + \overline{c}_{.95})]$, and for the recursive and reverse-recursive Fama-French regression statistic it is $[b(T\delta, T) - \underline{c}_{.95}, b(T\delta, T) + \overline{c}_{.95}]$, where $\underline{c}_{.95} = \min\{c_{.95}, c^r_{.95}\}$, $\overline{c}_{.95} = \max\{c_{.95}, c^r_{.95}\}$ and $c_{.95}$ and $c^r_{.95}$ are the 95% percentiles of the Monte Carlo distributions of the corresponding recursive and reverse-recursive fluctuation statistics.

null. This rejection is specially strong for log excess returns, where the null is also rejected for the average statistic at a 5% significance level. For both log real and log excess value-weighted returns, the rejection of the null of stability at short horizons is signalled by reverse-recursive variance ratios that are well below the full-sample estimates, while at the right end of the horizon spectrum the rejection occurs because the reverse-recursive variance rations are well above the full-sample estimates. Also, at short horizons the occurrence dates of the maximum absolute percentage deviations are around the mid 70's, while for the longest horizons they are all in the 50's. The upper panel in Figure 3 illustrates fairly well this characteristic behavior of the reverse-recursive variance ratio: It increases in the 40's and reach its peak in the 50's, as we remove the years of the Great Depression from the sample, and it falls dramatically in the mid 70's, as we remove from the sample the previous years.

The bottom blocks of Tables 6 through 9 report our empirical results for the fluctuation statistics based on the Fama-French univariate regression statistics. The evidence that emerges from them lends little statistical support against the null of stability in the autocorrelation structure of stock returns on almost every dimension of the data (across horizons, portfolios and measures of return) with only one exception. This exception is the reverse-recursive fluctuation test at a 1-year horizon, that rejects the null of stability across portfolios and measures of return. This rejection occurs because of a large fall of the statistic in the mid 70's, similar to that observed in the reverse-recursive variance ratio. This conclusion is confirmed visually by the four plots in the bottom half of Figures 2 and 3.

Figures 2 and 3 also show another interesting common feature of both the variance ratio statistic and the regression statistic: The reverse-recursive estimates exhibit a larger variability than the recursive estimates. The smoothness of the recursive estimates results from the low variability in the post-World War II returns relative to the pre-World War II returns. Hence, the largest proportion of the total variability in the sample gets incorporated in the very first recursive estimate, which is computed using the first 25% part of the sample, and this in turn makes the recursive estimates to move little relative to the first estimate. Because the reverse-recursive estimator discards the early part of the sample as it moves forward, it allows the relative variability within the postwar period to manifest itself.

The picture that emerges then suggests a structural change towards more univariate predictability in the period after the early 50's, since the reverse-recursive estimates of the variance ratio decrease systematically during the whole period to levels well below one at all horizons. This trend is reinforced by a sudden fall in the mid 70's. The plots for the Fama-French regression statistic also show a pattern of decreasing reverse-recursive estimates. This result defies the commonly held view that most of the univariate predictability in stock returns is induced by the Great Depression period. This view comes mainly from comparing estimates that include the prewar period with estimates that exclude this period. We can easily do this exercise in our plots by comparing the recursive estimate with the reverse-recursive estimate at any date. Thus, by only looking at only one date in the late 40's or early 50's, we observe a variance ratio for the sample starting at that point—the reverse-recursive variance ratio— well above the variance ratio for the sample ending at that point—the recursive variance ratio. The plots for the Fama-French regression statistic also show large disparities between the recursive and reverse-recursive estimates around these dates. However, if we do not limit ourselves to look at only this particular point in time, but the whole set of recursive and reverse-recursive estimates, we observe the pattern we have just described suggesting a change towards more, rather than less, predictability in the latter part of the sample.

Finally, we also report the full-sample estimators of the variance ratio and Fama-French univariate regression statistics as well as their p-values, which are obtained from a Monte Carlo simulation of $VR(\delta, 1)$ and $\beta(\delta, 1)$ similar to the experiments described in section 3. Conditional on structural stability over the whole sample, there is some evidence of predictability—at conventional significance levels—only at horizons of two years, and no much evidence of predictability beyond that horizon.

4.3 Testing for the stability and significance of multivariate predictability in stock returns.

Table 10 reports our empirical results for testing both the statistical significance and the structural stability of the linear forecasting model (16). The testing strategy described in section 3, requires the estimation of a confidence interval for c[11], a measure of how close is the autoregressive parameter in (17) to unity. Panel A reports our full-sample estimates for the AR(1) process fitted to the log dividend yield. It is interesting to note that while the point estimate of ϕ is close to unity, the 95% confidence interval for ϕ includes a unit root only in the extreme right tail of the interval. This result supports the notion that the log dividend yield is a very persistent, but still mean-reverting, process: a near-integrated process.

Panel B report our estimation results for the forecasting equation (16) when estimated for log real returns and log excess returns on the value-weighted and equally weighted portfolios. It is divided in two blocks. The first block reports full-sample estimates of the slope, two statistics to test for the statistical significance of the slope conditional on

[11] See footnote 8.

structural stability and full sample estimates of the standard deviation of the innovations in returns and the correlation between innovations in returns and innovations in the log dividend-yield. The estimated correlation is negative and very high in absolute value. The first statistic to test for the statistical significance of the slope is the conventional t-statistic, which under conventional asymptotics does reject the null $\alpha_1 = 0$ at a 10% significance level for log real value-weighted and equally-weighted returns and at a 5% level for log excess value-weighted and equally-weighted returns. However, conventional asymptotics hold only if the regressor is a strictly stationary process, and the evidence shown in Panel A strongly suggests that, while the log dividend yield is not an integrated process, it is a near-integrated process. Hence we test the significance of the slope under the assumption that the log dividend yield follows a near-integrated process, using the apparatus developed in section 2.4. The third row of Panel B reports T times the full-sample estimate of the slope, whose distribution is given in (18) when $\alpha_1 = 0$. The value on each cell should be compared to the corresponding cell in Panel A of Table 5 for a 10% significance test (see section 3). The estimated statistics are well below these critical values, even well below the median of their asymptotic distributions as reported in Panel B of Table 5. Therefore, the null hypothesis that one-month log returns are not forecastable from the log dividend yield is strongly not rejected when we take into account the persistence in the forecasting variable.

This evidence depends, however, on the forecasting model being stable along the sample period. The second block in Panel B reports the sample estimates of the fluctuation statistics (19) and (21) and the sequential QLR statistic (23). The trimming parameter for the recursive and reverse recursive estimators is set to 25% of the effective sample[12]. All three statistics do not reject the null of stability: Their values are well below the 10% critical values shown in Panel A of Table 5, and even below the medians of the distributions shown in Panel B of Table 5. Therefore, we cannot reject that the model is stable under the null distributions presented in section 2.4, which take into account the persistence in the log dividend yield. But we would not be able to reject this null even under strongest assumptions. For example, if we suppose for a moment that the regressor is exogenous, stationary and that we perform a conventional Chow test for structural change at a break date that happens to maximize the value of the statistic, so the F-statistic (22) has an F distribution, we would still accept the null hypothesis of stability at a 5% level.

By comparing point estimates for the pre-World War II period and the postwar period, it has been argued that the evidence on the forecastability of returns from the dividend yield is stronger in the postwar period. Figure 4 plots recursive and reverse-recursive estimates of the slope and the F-statistic (22) generated by log real returns and log excess returns on the equally-weighted portfolio and the value-weighted portfolio. The horizontal lines in

[12]See footnote 5.

the plots correspond to the full-sample estimates[13]. The plots for the recursive and reverse-recursive estimates of the slope for log excess returns help us to understand why it is claimed in the literature that the postwar period is characterized by more predictability in stock returns: The reverse-recursive estimates are well above the recursive estimates most of the time. That is, most of the time the estimate for the subsample starting at that point is larger than the estimate for the subsample that ends at that point[14]. Hence, by looking only at point estimates, one would conclude there is more predictability in the postwar period than in the prewar period. However, when we compare these estimates with proper critical values, the statistical evidence suggests these differences are due to sample variability rather than population variability. Moreover, the last part of the sample shows a inversion in the relative pattern of the recursive and reverse-recursive estimates of the slope, which also shows up in the plots of the QLR statistic, where the peak of the F-statistic is around 1970 for all log returns except for log excess returns on the value-weighed portfolio.

5 Conclusion

The literature on the predictability of stock returns has been abundant and fruitful during the last decade. Progress has been made on the issue of stock return predictability by considering how this predictability may change as we vary the horizon over which we measure returns and how other variables, particularly the dividend yield, may help forecast returns. However, there are some problems this paper have addressed. On one hand, the inference about predictability on stock returns has been shown to depend on the sample period we use to estimate the test statistics, hence suggesting some kind of structural break in the predictive relations for stock returns. On the other hand, the log dividend yield exhibits high persistence and its innovations are strongly negatively correlated with innovations in stock returns. Recent developments in the time series literature suggest these characteristics of the log dividend yield have important consequences for inference, to the extent that invalidate inference results based on standard asymptotics.

This paper has addressed the issue of stability in the context of recursive estimation. We have derived the asymptotic distributions of fluctuation statistics based on recursive and reverse-recursive estimators of the most common statistics used in the literature to measure persistence in stock returns: the variance ratio, the Fama-French univariate regression statistic for long horizon returns and the slope in the regression equation of log returns onto the log dividend yield. We derive the asymptotic results for the univariate

[13] We have not plotted lines for a 95% confidence interval because they are so distant from the recursive estimates that they would obscure the sample variation in the recursive estimates.

[14] Though not shown, this is true for recursive and reverse-recursive t-ratios as well.

statistics—the variance ratio and the Fama-French univariate regression statistic—under general conditions of autocorrelation and heteroskedasticity, and the asymptotic results for the multivariate regression statistic under the assumption that the log dividend yield is a nearly integrated process. For this statistic we also derive the asymptotic distribution of the sequential QLR statistic—the maximal Chow-Wald F-test for structural breaks—under the same set of assumptions.

The evidence presented here on the stability of univariate predictability in stock returns lends some statistical support to the common presumption that there is a structural break in the behavior of long horizon returns in the 50's, after the Federal Reserve agreement on interest rates. However, the strongest evidence about a structural break dates from the mid 70's, right after the stock market plunge caused by the oil and monetary crisis of the early 70's. Our recursive and reverse-recursive estimates suggest the post-World War II period is characterized by a change towards more univariate predictability in stock returns. By contrast, our empirical evidence on the stability of the log dividend-price ratio forecasting relation does not support the customary practice of using the post-World War II period to estimate regressions of log returns onto lagged log dividend yields. The null hypothesis of stability in the regression coefficients is not rejected at any meaningful significance level, both using fluctuation test and the QLR test.

Our results on the asymptotic distribution of the recursive estimators of the statistics are also useful to test for the statistical significance of the forecasting relations conditional on structural stability by considering the distribution of the full-sample estimators. For the variance ratio and the Fama-French univariate regression statistic these distributions are those reported in Stock (1989). We corroborate his empirical results for a sample period that adds 10 years to his sample (1926-1985): There is some evidence on univariate mean-reversion in stock returns over horizons up to two years, but not much evidence beyond that. With respect to the multivariate forecasting relation involving log dividend yields we find no statistical evidence that one-month log returns, whether real or in excess of returns on T-bills, are predictable from log dividend yields once we take into account the high persistence in the log dividend yield, even though under standard asymptotics we do reject the null hypothesis that the regression coefficient of the log dividend yield is statistically zero.

6 References

Banerjee, A., R.L. Lumsdaine and J.H. Stock, 1992, "Recursive and Sequential Tests of the Unit-Root and Trend-Break Hypotheses: Theory and International Evidence," *Journal of Business and Economic Statistics* 10(3), 271-287.

Campbell, J. Y., 1991, "A Variance Decomposition of Stock Returns," *Economic Journal* 101, 157-179.

Campbell, J. Y., 1997, "Asset Prices, Consumption and the Business Cycle," forthcoming in the *Handbook of Macroeconomics*, Vol. 1, edited by J. B. Taylor and M. Woodford, North-Holland, Amsterdan.

Campbell, J. Y., A. W. Lo and A. C. MacKinlay, 1996, *The Econometrics of Financial Markets*, 1997, Princeton University Press, Princeton, NJ.

Campbell, J. Y. and R. Shiller, 1988a, "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies* 1, 195-227.

Campbell, J. Y. and R. Shiller, 1988b, "Stock Prices, Earnings, and Expected Dividends," *Journal of Finance* 43, 661–676.

Campbell, J. Y. and L. M. Viceira, 1996, "Consumption and Portfolio Decisions When Expected Returs are Time Varying", NBER working paper 5857, National Bureau of Economic Research, Cambridge, MA.

Chan, N.H. and C.Z. Wei, 1987, "Asymptotic Inference for Nearly Nonstationary AR(1) processes", *Annals of Statistics* 15, 1050-1063.

Center for Research and Security Prices, 1996, *1995 File Guides,* Graduate School of Business, University of Chicago, Chicago, IL.

Elliott, G. and J. H. Stock, 1994, "Inference in Time Series Regression When the Order of Integration of a Regressor Is Unknown," *Econometric Theory* 10, 672-700.

Fama, E. and K. French, 1988a, "Permanent and Temporary Components of Stock Prices," *Journal of Political Economy* 96, 246-273.

Fama, E. and K. French, 1988b, "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics* 22, 3-27.

Fama, E. and K. French, 1989, "Business Conditions and Expected Returns on Stocks and Bonds," *Journal of Financial Economics* 25, 23-49.

Hamilton, J. D., 1994, *Time Series Analysis*, Princeton University Press, Princeton, NJ.

Hodrick, R. J., 1992, "Dividend Yields and Expected Stock Returns: Alternative Procedures for Inference and Measurement," *Review of Financial Studies* 5, 357-386.

Kim, M.J., C.R. Nelson and R.Startz, 1991, "Mean Reversion in Stock Prices? A Reappraisal of the Empirical Evidence," *Review of Economic Studies* 58, 515-528.

Kothari, S.P. and J. Shanken, 1995, "Book-to-Market, Dividend Yield and Expected Market Returns: A Time-series Analysis," manuscript, William E. Simon Graduate School of Business Administration, University of Rochester.

Lo, A. W., and A. C. MacKinlay, 1988, "Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test," *Review of Financial Studies* 1, 41-66.

Lo, A. W., and A. C. MacKinlay, 1989, "The Size and Power of the Variance Ratio Test in Finite Samples: A Monte Carlo Investigation," *Journal of Econometrics* 40, 203-238.

Phillips, P. C. B., 1987, "Towards a Unified Asymptotic Theory for Autoregression," *Biometrika* 74, 535-547.

Ploberger, W., W. Krämer and K. Kontrus, 1989, "A New Test for Structural Stability in the Linear Regression Model," *Journal of Econometrics* 40, 307-318.

Poterba, J. and L. Summers, 1989, "Mean Reversion in Stock Returns: Evidence and Implications," *Journal of Financial Economics* 22, 27-60.

Quandt, R.E., 1960, "Tests of the Hypothesis That a Linear Regression Obeys Two Separate Regimes," *Journal of the American Statistical Association* 55, 324-330.

Richardson, M., J. H. Stock, 1989, "Drawing Inferences from Statistics Based on Multiyear Asset Returns," *Journal of Financial Economics* 25, 323-348.

Stock, J. H., 1991, "Confidence Intervals for the Largest Autoregressive Root in U.S. Economic Time Series," *Journal of Monetary Economics* 10, 91-115.

Stock, J. H., 1994, "Unit Roots, Structural Breaks and Trends," *Handbook of Econometrics*, Vol. 4, edited by R.F. Engle and D.L. McFadden, North-Holland, Amsterdan.

Wright, J. H., 1996, "Structural Stability Tests in the Linear Regression Model When the Regressors Have Roots Local to Unity," *Economic Letters* 52, 257-262.

A Appendix A

In this appendix we state some lemmas and prove the propositions in section 2.3 of the paper.

A.1 Proof of Lemma 2

Using the definition of $[T\delta]$-horizon log return we can decompose $(r_{[Tv]}^{([T\delta])} - [T\delta]\,\mu)$ as follows:

$$\sqrt{\frac{1}{T}}\left(r_{[Tv]}^{([T\delta])} - [T\delta]\,\mu\right) = \sqrt{\frac{1}{T}}\sum_{t=1}^{[Tv]}(r_t-\mu) - \sqrt{\frac{1}{T}}\sum_{t=1}^{[T(v-\delta)]}(r_t-\mu).$$

But, from Lemma 1 (FCLT for general $I\,(0)$ processes), the first term in the sum above converges in distribution to $V^{1/2}W(v)$, and the second term converges to $V^{1/2}W(v-\delta)$. The CMT states that if $y_{[Tv]} \Rightarrow y(v)$, where $y(v)$ is a random function of $v \in [0,1]$, and $f(\cdot)$ is a continuous functional, then $f(y_{[Tv]}) \Rightarrow f(y(v))$ (Hamilton, 1994 and Stock, 1994). Lemma 1 shows that the partial sums of demeaned one-period returns have a limiting representation as a random function in $C[0,1]$. Since the addition of random functions defines a continuous functional, the result stated in the lemma follows applying the CMT. ■

A.2 Proof of Proposition 1

To prove this proposition, first note that Lemma 1 (FCLT) and the definition of $\mu\,(1,T\lambda)$ (the recursive estimator of the mean of r_t) implies the following limiting representation for $\sqrt{T}$ times the deviations of $\mu\,(1,T\lambda)$ from μ (the unconditional mean of r_t):

$$\begin{aligned} \sqrt{T}\,(\mu\,(1,T\lambda)-\mu) &= \frac{1}{\lambda}\sqrt{\frac{1}{T}}\sum_{t=1}^{[T\lambda]}(r_t-\mu) \\ &\Rightarrow \frac{1}{\lambda}V^{1/2}W_r\,(\lambda)\,. \end{aligned} \tag{24}$$

To obtain the limiting distribution of $vr(T\delta, T\lambda)$, note that this distribution equals the ratio of the limiting distributions of numerator and denominator, by the CMT. Multiply both numerator and denominator by T^{-2}. Adding to and substracting from the terms in brackets in the numerator $[T\delta]\mu$, the resulting expression for the numerator is:

$$\begin{aligned} &\left(\frac{1}{T}\right)^2 \sum_{t=[T\delta]}^{[T\lambda]}\left[\left(r_t^{([T\delta])} - [T\delta]\,\mu\right) - [T\delta]\,(\mu\,(1,T\lambda)-\mu)\right]^2 \\ = &\ \frac{1}{T}\sum_{t=[T\delta]}^{[T\lambda]}\left[\sqrt{\frac{1}{T}}\left(r_t^{([T\delta])} - [T\delta]\,\mu\right) - \delta\sqrt{T}\,(\mu\,(1,T\lambda)-\mu)\right]^2. \end{aligned}$$

Lemma 2 gives us the limiting distribution of the first term in brackets, while (24) gives us the limiting distribution for the second term. Finally, the CMT and the definition of integral imply the following distribution for T^{-2} times the numerator of $vr(T\delta, T\lambda)$:

$$\left(\frac{1}{T}\right)^2 \sum_{t=[T\delta]}^{[T\lambda]}\left[r_t^{([T\delta])} - [T\delta]\,\mu\,(1,T\lambda)\right]^2 \Rightarrow V\int_\delta^\lambda \left[W_r\,(v) - W_r\,(v-\delta) - \frac{\delta}{\lambda}W_r\,(\lambda)\right]^2 dv.$$

Also, the resulting expression for the numerator is:

$$\left(\frac{1}{T}\right)^2 [T\delta] \sum_{t=1}^{[T\lambda]} [r_t - \mu(1, T\lambda)]^2 = \delta\lambda \frac{1}{[T\lambda]} \sum_{t=1}^{[T\lambda]} [r_t - \mu(1, T\lambda)]^2 \Rightarrow \delta\lambda \cdot \gamma_0,$$

from (A3). Collecting terms we obtain the desired result. ■

A.3 Proof of Proposition 2

The proof of this proposition is similar to that of Proposition 1. It follows from simple algebraic manipulations of the expression for the reverse-recursive estimator (6), Lemma 2, the FCLT and CMT and the definition of integral.

From Lemma 1 we have:

$$\sqrt{T}\left(\mu^r(1, T(\lambda-\delta)) - \mu\right) = \frac{1}{1-\lambda+\delta}\left(\sqrt{\frac{1}{T}}\sum_{t=1}^{T}(r_t-\mu) - \sqrt{\frac{1}{T}}\sum_{t=1}^{[T(\lambda-\delta)]}(r_t-\mu)\right) \Rightarrow V^{1/2}\frac{1}{1-\lambda+\delta}\left[W_r(1) - W_r(\lambda-\delta)\right]. \quad (25)$$

Multiplying both numerator and denominator of $vr^r(T\delta, T\lambda)$ by T^{-2} we find that, using (25), the numerator converges in distribution to:

$$\left(\frac{1}{T}\right)^2 V_T^{-1} \sum_{t=[T\lambda]}^{T}\left[r_t^{([T\delta])} - [T\delta]\,\mu^r(1, T(\lambda-\delta))\right]^2$$

$$= \frac{1}{T}\sum_{t=[T\lambda]+1}^{T}\left[\sqrt{\frac{1}{T}}V_T^{-1/2}\left(r_t^{([T\delta])} - [T\delta]\,\mu\right) - \delta\sqrt{T}\left(\mu^r(1, T(\lambda-\delta)) - \mu\right)\right]^2$$

$$\Rightarrow \int_\lambda^1 \left\{W_r(v) - W_r(v-\delta) - \frac{\delta}{1-\lambda+\delta}\left[W_r(1) - W_r(\lambda-\delta)\right]\right\}^2 dv.$$

Similarly, T^{-2} times the denominator equals $\delta(1-\lambda+\delta)\gamma_0^r(1, T(\lambda-\delta))^{-1}$ which converges in probability to $\delta(1-\lambda+\delta)\gamma_0$ by (A3'). Collecting terms we obtain the result stated in the Proposition. ■

A.4 Proof of Proposition 3

Write the recursive OLS estimator of $\beta(\delta)$ in (8) as

$$b(T\delta, T\lambda) = \frac{T^{-1}\sum_{t=2[T\delta]}^{[T\lambda]}\left[T^{-1/2}r_{t-[T\delta]}^{([T\delta])} - T^{-1/2}\mu_{-[T\delta]}(T\delta, T\lambda)\right]\left[T^{-1/2}r_t^{([T\delta])} - T^{-1/2}\mu(T\delta, T\lambda)\right]}{T^{-1}\sum_{t=2[T\delta]}^{[T\lambda]}\left[T^{-1/2}r_{t-[T\delta]}^{([T\delta])} - T^{-1/2}\mu_{-[T\delta]}(T\delta, T\lambda)\right]^2}.$$

From Lemma 2, the CMT and the definition of integral,

$$\begin{aligned}\sqrt{\frac{1}{T}}\mu\left(T\delta,T\lambda\right) &= V^{1/2}\frac{1}{[T\lambda]-2\left[T\delta\right]+1}\sum_{t=2[T\delta]}^{[T\lambda]}\sqrt{\frac{1}{T}}V^{-1/2}r_{t}^{([T\delta])} \\ &\Rightarrow V^{1/2}\frac{1}{\lambda-2\delta}\int_{2\delta}^{\lambda}\left[W_{r}\left(v\right)-W_{r}\left(v-\delta\right)\right]dv \\ &\Rightarrow V^{1/2}\mu\left(\delta,\lambda\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\sqrt{\frac{1}{T}}\mu_{-[T\delta]}\left(T\delta,T\lambda\right) &= V^{1/2}\frac{1}{[T\lambda]-2\left[T\delta\right]+1}\sum_{t=2[T\delta]}^{[T\lambda]}\sqrt{\frac{1}{T}}V^{-1/2}r_{t-[T\delta]}^{([T\delta])} \\ &\Rightarrow V^{1/2}\frac{1}{\lambda-2\delta}\int_{2\delta}^{\lambda}\left[W_{r}\left(v-\delta\right)-W_{r}\left(v-2\delta\right)\right]dv \\ &= V^{1/2}\mu_{-\delta}\left(\delta,\lambda\right).\end{aligned}$$

Applying again Lemma 2, the CMT and the definition of integral to the numerator and denominator of $b(T\delta,T\lambda)$ we obtain (9) in Proposition 3. Note that V cancels out in both numerator and denominator.

To find the limiting distribution of $t(T\delta,T\lambda)$, write the statistic as

$$\frac{1}{T}t\left(T\delta,T\lambda\right)=\frac{b\left(T\delta,T\lambda\right)-b\left(T\delta,T\right)}{\sqrt{TQ(T\delta,T\lambda)\,T^{-1}\Omega\left(T\delta,T\right)\;TQ(T\delta,T\lambda)}}.$$

The asymptotic distribution of the numerator obtains from (9). For the denominator we have that

$$\left[TQ(T\delta,T\lambda)\right]^{-1}=\frac{1}{[T\lambda]-2[T\delta]-1}\sum_{t=2[T\delta]+1}^{[T\lambda]}\left[T^{-1/2}r_{t-[T\delta]}^{[T\delta]}-T^{-1/2}\mu_{-[T\delta]}\left(T\delta,T\lambda\right)\right]^{2}$$

equals the denominator of $b(T\delta,T\lambda)$ up to a scaling factor. Hence, its asymptotic distribution is given by:

$$\begin{aligned}\left[TQ(T\delta,T\lambda)\right]^{-1} &\Rightarrow V\frac{1}{\lambda-2\delta}\int_{2\delta}^{\lambda}\left[W_{r}\left(v-\delta\right)-W_{r}\left(v-2\delta\right)-\mu_{-\delta}\left(\delta,\lambda\right)\right]^{2} \\ &= Q\left(\delta,\lambda\right)^{-1}.\end{aligned}$$

We need to find the asymptotic representation of $T^{-1}\Omega_{T}$, where

$$\Omega\left(T\delta,T\right)=\gamma_{z,T}\left(0\right)+2\sum_{j=1}^{[T\delta]}w\left(j,T\delta\right)\gamma_{z,T}\left(j\right),$$

$w(j,T\delta)$ is some weighting scheme that depends on $j/[T\delta]$—for example, $w(j,T\delta)=1-j/([T\delta]+1)$ if we use the Newey-West estimator—, $\gamma_{z,T}(j)$ is a full-sample estimator of $\gamma_{z}(j)$, the $j-th$ order autocovariance of z_{t}, and

$$z_{t}=r_{t-[T\delta]}^{[T\delta]}\widehat{u}_{t}^{[T\delta]}=r_{t-[T\delta]}^{[T\delta]}\left[\left(r_{t}^{[T\delta]}-\mu\left(T\delta,T\right)\right)-b\left(T\delta,1\right)\left(r_{t-[T\delta]}^{[T\delta]}-\mu_{-[T\delta]}\left(T\delta,T\right)\right)\right].\qquad(26)$$

Assume that $\lim_{T\to\infty}(j/T) = J > 0$. Then $\lim_{T\to\infty} w(j, T\delta) = w(J, \delta)$—for example, $w(J, \delta) = 1 - J/(\delta + j)$ in the Newey-west weighting scheme. Also, from Lemma 2, the CMT, the definition of integral and (26), we have that

$$\begin{aligned}\frac{1}{T}\gamma_{z,T}(j) &= \frac{1}{T-2[T\delta]-1}\sum_{t=2[T\delta]+j}^{T}\left(T^{-1/2}z_t\right)\left(T^{-1/2}z_{t-j}\right) \\ &\Rightarrow V^2\int_{2\delta+J}^{1}\xi(v,\delta,0)\,\xi(v,\delta,J)\,dv = \gamma_{z,J},\end{aligned}$$

where

$$\begin{aligned}\xi(v,\delta,J) &= [W_r(v-\delta-J) - W_r(v-2\delta-J)][W_r(v-J) - W_r(v-\delta-J) - \mu_{-\delta}(\delta,1)] \\ &\quad -\beta(\delta,1)[W_r(v-\delta-J) - W_r(v-2\delta-J) - \mu_{-\delta}(\delta,1)]^2,\end{aligned}$$

and $\xi(v, \delta, 0)$ follows from above for $J = 0$. Hence, the limiting representation of $T^{-1}\Omega(T\delta, T)$ is going to be

$$\frac{1}{T}\Omega(T\delta,T) \Rightarrow V^2\Omega(\delta,1) = V^2\left[\gamma_z(0) + 2\sum_{J=1}^{\delta} w(J,\delta)\,\gamma_z(J)\right]. \tag{27}$$

Collecting terms for the numerator and denominator of $T^{-1}t(T\delta, T\lambda)$, we obtain (10). ■

A.5 Proof of Proposition 4

The proof for this proposition follows the same arguments as the proof for Proposition 3. Write the reverse-recursive OLS estimator of $\beta(\delta)$ in (8) as

$$b^r(T\delta,T\lambda) = \frac{T^{-1}\sum_{t=[T\lambda]}^{T}\left[T^{-1/2}r_{t-[T\delta]}^{([T\delta])} - T^{-1/2}\mu_{-[T\delta]}^{r}(T\delta,T\lambda)\right]\left[T^{-1/2}r_t^{([T\delta])} - T^{-1/2}\mu^r(T\delta,T\lambda)\right]}{T^{-1}\sum_{t=[T\lambda]}^{T}\left[T^{-1/2}r_{t-[T\delta]}^{([T\delta])} - T^{-1/2}\mu_{-[T\delta]}^{r}(T\delta,T\lambda)\right]^2},$$

where

$$\begin{aligned}\sqrt{\frac{1}{T}}\mu^r(T\delta,T\lambda) &= V^{1/2}\frac{1}{[T(1-\lambda)]+1}\sum_{t=[T\lambda]}^{T}\sqrt{\frac{1}{T}}V^{-1/2}r_t^{([T\delta])} \\ &\Rightarrow V^{1/2}\frac{1}{1-\lambda}\int_{\lambda}^{1}[W_r(v) - W_r(v-\delta)]\,dv \\ &= V^{1/2}\mu^r(\delta,\lambda),\end{aligned}$$

and,

$$\begin{aligned}\sqrt{\frac{1}{T}}\mu_{-[T\delta]}^{r}(T\delta,T\lambda) &= V^{1/2}\frac{1}{[T(1-\lambda)]+1}\sum_{t=[T\lambda]}^{T}\sqrt{\frac{1}{T}}V^{-1/2}r_{t-[T\delta]}^{([T\delta])} \\ &\Rightarrow V^{1/2}\frac{1}{1-\lambda}\int_{\lambda}^{1}[W_r(v-\delta) - W_r(v-2\delta)]\,dv \\ &= V^{1/2}\mu_{-\delta}^{r}(\delta,\lambda).\end{aligned}$$

Applying again Lemma 2, the CMT and the definition of integral to the numerator and denominator of $b^r(T\delta, T\lambda)$ we obtain (13) in Proposition 4. Note that V cancels out in both numerator and denominator.

To find the limiting distribution of $t^r(T\delta, T\lambda)$, write the statistic as

$$\frac{1}{T}t^r\left(T\delta, T\lambda\right) = \frac{b^r\left(T\delta, T\lambda\right) - b\left(T\delta, T\right)}{\sqrt{TQ^r\left(T\delta, T\lambda\right)\, T^{-1}\Omega\left(T\delta, T\right)\, TQ^r\left(T\delta, T\lambda\right)}}.$$

The asymptotic distribution of the numerator obtains from (13) and the asymptotic distribution of $\Omega\left(T\delta, T\right)$ is given in (27). We also have that

$$\begin{aligned}
\left[TQ^r\left(T\delta, T\lambda\right)\right]^{-1} &= \frac{1}{\left[T\left(1-\lambda\right)\right]}\sum_{t=[T\lambda]+1}^{T}\left[T^{-1/2}r_{t-[T\delta]}^{[T\delta]} - T^{-1/2}\mu_{-[T\delta]}^{r}\left(T\delta, T\left(1-\lambda\right)\right)\right]^2 \\
&\Rightarrow V\frac{1}{1-\lambda}\int_{\lambda}^{1}\left[W_r\left(v-\delta\right) - W_r\left(v-2\delta\right) - \mu_{-\delta}^{r}\left(\delta, 1-\lambda\right)\right]^2 \\
&= Q^r\left(\delta, \lambda\right)^{-1}.
\end{aligned}$$

Collecting terms for the numerator and denominator of $T^{-1}t(T\delta, T\lambda)$, we obtain (14). ■

B Appendix B

In this appendix we state some lemmas and prove the propositions in section 2.4 of the paper.

B.1 Proof of Proposition 5

Writing the OLS recursive estimator of α_1 in deviations with respect to the sample means we have that:

$$\begin{aligned}
a_1\left(T\lambda\right) &= \frac{\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)\left(r_{t-1} - \overline{r}_{[T\lambda]}\right)}{\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)^2} \\
&= \alpha_1 + \frac{\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)\left(\varepsilon_t - \overline{\varepsilon}_{[T\lambda]}\right)}{\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)^2} \\
&= \alpha_1 + \frac{\sum_{t=2}^{[T\lambda]}x_{t-1}\varepsilon_t - \left([T\lambda]-1\right)\overline{x}_{[T\lambda]}\overline{\varepsilon}_{[T\lambda]}}{\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)^2},
\end{aligned}$$

where $\overline{z}_{[T\lambda]} = \sum_{t=2}^{[T\lambda]} z_t/([T\lambda]-1)$, for $z_t = (x_{t-1}, r_t, \varepsilon_t)$.

Consider now

$$T\left(a_1\left(T\lambda\right) - \alpha_1\right) = \frac{\frac{1}{T}\sum_{t=2}^{[T\lambda]}x_{t-1}\varepsilon_t - \lambda\overline{x}_{[T\lambda]}\overline{\varepsilon}_{[T\lambda]} + \frac{1}{T}\overline{x}_{[T\lambda]}\overline{\varepsilon}_{[T\lambda]}}{\frac{1}{T^2}\sum_{t=2}^{[T\lambda]}\left(x_{t-1} - \overline{x}_{[T\lambda]}\right)^2}. \tag{28}$$

Using the CMT and the FCLT, the limiting representation of the ratio in (28) will be ratio of the limiting representations of numerator and denominator.

For the denominator we have:

$$\begin{aligned} \frac{1}{T^2}\sum_{t=2}^{[T\lambda]}\left(x_{t-1}-\overline{x}_{[T\lambda]}\right)^2 &= \frac{1}{T}\sum_{t=2}^{[T\lambda]}\left(\frac{x_{t-1}}{\sqrt{T}}-\frac{T}{[T\lambda]-1}\frac{1}{T}\sum_{t=1}^{[T\lambda]}\frac{x_{t-1}}{\sqrt{T}}\right)^2 \\ &\Rightarrow \sigma_\eta^2\int_0^\lambda\left[J_c(v)-\frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right]^2 dv, \end{aligned} \tag{29}$$

where the second line follows immediately from Lemma 3, the CMT and the definition of integral.

To find the limiting representation of the numerator it is convenient to use the linear projection of ε_t onto η_t, so we can decompose ε_t in terms of uncorrelated processes. This decomposition is given by

$$\varepsilon_t=\frac{\sigma_{\varepsilon\eta}}{\sigma_\eta^2}\eta_t+u_t, \tag{30}$$

where u_t is a m.d.s. with $\mathrm{Var}\,(u_t)=\sigma_\varepsilon^2-(\sigma_{\varepsilon\eta}^2/\sigma_\eta^2)$ and $\mathrm{Cov}\,(\eta_t,u_t)=0$.

Given (30) we have

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{[T\lambda]}x_{t-1}\varepsilon_t &= \frac{\sigma_{\varepsilon\eta}}{\sigma_\eta^2}\frac{1}{T}\sum_{t=1}^{[T\lambda]}x_{t-1}\eta_t+\frac{1}{T}\sum_{t=1}^{[T\lambda]}x_{t-1}u_t \\ &\Rightarrow \sigma_{\varepsilon\eta}\int_0^\lambda J_c(v)\,dW_\eta(v)+\left(\sigma_\varepsilon^2\sigma_\eta^2-\sigma_{\varepsilon\eta}^2\right)^{1/2}\int_0^\lambda J_c(v)\,dW_u(v), \end{aligned}$$

where the second line follows from Theorem 2.4 in Chan and Wei (1987)—see also Lemma 1 in Phillips (1987).

Also, from (30), the FCLT and the CMT,

$$\begin{aligned} \overline{x}_{[T\lambda]}\overline{\varepsilon}_{[T\lambda]} &= \left(\frac{T}{[T\lambda]-1}\right)^2\left(\frac{1}{T}\sum_{t=1}^{[T\lambda]}\frac{x_{t-1}}{\sqrt{T}}\right)\left(\sum_{t=1}^{[T\lambda]}\frac{\varepsilon_{t-1}}{\sqrt{T}}\right) \\ &\Rightarrow \frac{1}{\lambda^2}\left(\sigma_\eta\int_0^\lambda J_c(v)\,dv\right)\left(\frac{\sigma_{\varepsilon\eta}}{\sigma_\eta}W_\eta(\lambda)+\left(\sigma_\varepsilon^2-\frac{\sigma_{\varepsilon\eta}^2}{\sigma_\eta^2}\right)^{1/2}W_u(\lambda)\right) \end{aligned}$$

which in turn implies that $T^{-1}\overline{x}_{[T\lambda]}\overline{\varepsilon}_{[T\lambda]}$ is $o_p(1)$.

Finally, collecting terms and noting that $W(\lambda)=\int_0^\lambda dW(v)$ we obtain (18). ■

B.2 Proof of Proposition 6

The proof for this proposition is very similar to that of Proposition 5, so we will skip some intermediate steps. First, note that we can write the OLS reverse-recursive estimator of α_1 as:

$$T\left(a_1^r(T\lambda)-\alpha_1\right)=\frac{\frac{1}{T}\sum_{t=[T\lambda]+1}^T x_{t-1}\varepsilon_t-(1-\lambda)\,\overline{x}_{[T(1-\lambda)]}\overline{\varepsilon}_{[T(1-\lambda)]}}{\frac{1}{T^2}\sum_{t=[T\lambda]+1}^T\left(x_{t-1}-\overline{x}_{[T(1-\lambda)]}\right)^2}, \tag{31}$$

where $\overline{z}_{[T(1-\lambda)]} = \sum_{t=[T\lambda]+1}^{T} z_t/[T(1-\lambda)]$, for $z_t = (x_{t-1}, r_t, \varepsilon_t)$.

By the CMT, the limiting representation of (31) is the ratio of the limiting representations of numerator and denominator.

For the denominator of (31) we have:

$$\begin{aligned}\frac{1}{T^2}\sum_{t=[T\lambda]+1}^{T}\left(x_{t-1}-\overline{x}_{[T\lambda]}\right)^2 &= \frac{1}{T}\sum_{t=[T\lambda]+1}^{T}\left(\frac{x_{t-1}}{\sqrt{T}}-\frac{1}{1-\lambda}\frac{1}{T}\sum_{t=[T\lambda]+1}^{T}\frac{x_{t-1}}{\sqrt{T}}\right)^2 \\ &\Rightarrow \sigma_\eta^2\int_\lambda^1\left[J_c(v)-\frac{1}{1-\lambda}\int_\lambda^1 J_c(v)\,dv\right]^2 dv,\end{aligned} \tag{32}$$

where the second line follows immediately from Lemma 3, the CMT and the definition of integral.

Using (30) we can obtain the limiting distributions of the objects in the numerator of (31):

$$\begin{aligned}\frac{1}{T}\sum_{t=[T\lambda]+1}^{T}x_{t-1}\varepsilon_t &= \frac{\sigma_{\varepsilon\eta}}{\sigma_\eta^2}\frac{1}{T}\sum_{t=[T\lambda]+1}^{T}x_{t-1}\eta_t+\frac{1}{T}\sum_{t=[T\lambda]+1}^{T}x_{t-1}u_t \\ &\Rightarrow \sigma_{\varepsilon\eta}\int_\lambda^1 J_c(v)\,dW_\eta(v)+\left(\sigma_\varepsilon^2\sigma_\eta^2-\sigma_{\varepsilon\eta}^2\right)^{1/2}\int_\lambda^1 J_c(v)\,dW_u(v),\end{aligned}$$

and

$$\begin{aligned}\overline{x}_{[T(1-\lambda)]}\overline{\varepsilon}_{[T(1-\lambda)]} &= \left(\frac{1}{1-\lambda}\right)^2\left(\frac{1}{T}\sum_{t=[T\lambda]+1}^{T}\frac{x_{t-1}}{\sqrt{T}}\right)\left(\sum_{t=[T\lambda]+1}^{T}\frac{\varepsilon_{t-1}}{\sqrt{T}}-\sum_{t=1}^{[T\lambda]}\frac{\varepsilon_{t-1}}{\sqrt{T}}\right) \\ &\Rightarrow \left(\frac{1}{1-\lambda}\right)^2\left(\sigma_\eta\int_\lambda^1 J_c(v)\,dv\right)\left(\frac{\sigma_{\varepsilon\eta}}{\sigma_\eta}\left[W_\eta(1)-W_\eta(\lambda)\right]+\left(\sigma_\varepsilon^2-\frac{\sigma_{\varepsilon\eta}^2}{\sigma_\eta^2}\right)^{1/2}\left[W_u(1)-W_u(\lambda)\right]\right).\end{aligned}$$

Finally, collecting terms and noting that $W(1)-W(\lambda)=\int_\lambda^1 dW(v)$ we obtain (20). ■

B.3 Proof of Proposition 7

We first show that $\widehat{\sigma}_\varepsilon^2$ is consistent for σ_ε^2. We have that

$$r_t-\overline{r}_{[T\cdot 1]}-a_1(T)\left(x_{t-1}-\overline{x}_{[T\cdot 1]}\right)=\varepsilon_t-\overline{\varepsilon}_{[T\cdot 1]}-\left(a_1(T)-\alpha_1\right)\left(x_{t-1}-\overline{x}_{[T\cdot 1]}\right).$$

Hence,

$$\begin{aligned}\widehat{\sigma}_\varepsilon^2 &= \frac{1}{T}\sum_{t=2}^{T}\left(\varepsilon_t-\overline{\varepsilon}_{[T\cdot 1]}\right)^2+T^2\left(a_1(T)-\alpha_1\right)^2\frac{1}{T^3}\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_{[T\cdot 1]}\right)^2 \\ &\quad -2T\left(a_1(T)-\alpha_1\right)\frac{1}{T^2}\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_{[T\cdot 1]}\right)\left(\varepsilon_t-\overline{\varepsilon}_{[T\cdot 1]}\right) \\ &\to \sigma_\varepsilon^2+O_p(1)\times\frac{1}{T}O_p(1)-2\times O_p(1)\times\frac{1}{T}O_p(1) \\ &= \sigma_\varepsilon^2+o_p(1),\end{aligned}$$

so $\widehat{\sigma}_{\varepsilon}^{2}$ is consistent for σ_{ε}^{2}.

The limiting distribution of $F(T\lambda)$ then follows immediately from Propositions 5 and 6, (29) and (32) after multiplying both numerator and denominator by T^2. From Propositions 5 and 6 we get, after simple algebraic manipulations, $\digamma_1(J_c, W_\eta)$ and $\digamma_2(J_c, W_u)$,

$$\digamma_1(J_c, W_\eta) = \frac{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right] dW_\eta(v)}{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right]^2 dv} - \frac{\int_\lambda^1 \left[J_c(v) - \frac{1}{1-\lambda}\int_\lambda^1 J_c(v)\,dv\right] dW_\eta(v)}{\int_\lambda^1 \left[J_c(v) - \frac{1}{1-\lambda}\int_\lambda^1 J_c(v)\,dv\right]^2 dv},$$

$$\digamma_2(J_c, W_u) = \frac{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right] dW_u(v)}{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right]^2 dv} - \frac{\int_\lambda^1 \left[J_c(v) - \frac{1}{1-\lambda}\int_\lambda^1 J_c(v)\,dv\right] dW_u(v)}{\int_\lambda^1 \left[J_c(v) - \frac{1}{1-\lambda}\int_\lambda^1 J_c(v)\,dv\right]^2 dv},$$

while (29) and (32) give $\digamma_3(J_c)$,

$$\digamma_3(J_c) = \left\{\int_0^\lambda \left[J_c(v) - \frac{1}{\lambda}\int_0^\lambda J_c(v)\,dv\right]^2 dv\right\}^{-1} + \left\{\int_\lambda^1 \left[J_c(v) - \frac{1}{1-\lambda}\int_\lambda^1 J_c(v)\,dv\right]^2 dv\right\}^{-1}.$$

■

TABLE 1

Monte Carlo Distributions of the Fluctuation Statistic Based on the Recursive Variance Ratio

$$\sup_{\lambda_0 \leq \lambda \leq 1} \left| \frac{VR(\delta,\lambda) - VR(\delta,1)}{VR(\delta,1)} \right|$$

T	Percentiles								
	0.025	0.05	0.10	.25	0.50	0.75	0.90	0.95	0.975
					Horizon: $\delta = 12/840$				
312	0.100	0.112	0.130	0.169	0.232	0.316	0.398	0.454	0.505
840	0.103	0.117	0.135	0.177	0.241	0.328	0.413	0.467	0.509
1680	0.103	0.115	0.135	0.177	0.241	0.325	0.415	0.468	0.511
2520	0.103	0.115	0.135	0.175	0.242	0.330	0.418	0.468	0.517
					Horizon: $\delta = 60/840$				
312	0.175	0.202	0.240	0.326	0.455	0.596	0.715	0.779	0.829
840	0.178	0.202	0.240	0.325	0.457	0.599	0.716	0.779	0.820
1680	0.176	0.201	0.241	0.328	0.458	0.598	0.714	0.776	0.832
2520	0.180	0.204	0.241	0.329	0.464	0.604	0.720	0.777	0.821
					Horizon: $\delta = 96/840$				
312	0.202	0.234	0.280	0.384	0.525	0.674	0.785	0.847	0.896
840	0.207	0.241	0.288	0.393	0.530	0.681	0.793	0.851	0.899
1680	0.209	0.241	0.287	0.388	0.530	0.675	0.791	0.848	0.903
2520	0.208	0.244	0.291	0.397	0.540	0.685	0.794	0.851	0.902

Note: See Notes following Table 10.

TABLE 2

Monte Carlo Distributions of the Fluctuation Statistic Based on the Reverse-Recursive Variance Ratio

$$\sup_{\delta \leq \lambda \leq \lambda_0} \left| \frac{VR^r(\delta,\lambda) - VR(\delta,1)}{VR(\delta,1)} \right|$$

T	Percentiles								
	0.025	0.05	0.10	.25	0.50	0.75	0.90	0.95	0.975
					Horizon: $\delta = 12/840$				
312	0.103	0.115	0.132	0.170	0.233	0.315	0.400	0.455	0.504
840	0.103	0.116	0.134	0.177	0.242	0.328	0.418	0.471	0.520
1680	0.102	0.116	0.135	0.176	0.242	0.327	0.417	0.474	0.525
2520	0.102	0.115	0.133	0.174	0.239	0.325	0.412	0.469	0.519
					Horizon: $\delta = 60/840$				
312	0.175	0.199	0.241	0.341	0.482	0.632	0.743	0.800	0.842
840	0.176	0.204	0.245	0.342	0.485	0.634	0.753	0.804	0.843
1680	0.176	0.203	0.244	0.340	0.487	0.637	0.750	0.802	0.841
2520	0.177	0.204	0.244	0.340	0.484	0.634	0.747	0.804	0.843
					Horizon: $\delta = 96/840$				
312	0.207	0.242	0.294	0.416	0.583	0.734	0.829	0.871	0.903
840	0.212	0.248	0.302	0.427	0.590	0.738	0.831	0.874	0.901
1680	0.207	0.245	0.300	0.427	0.596	0.742	0.836	0.876	0.904
2520	0.209	0.246	0.304	0.428	0.595	0.738	0.833	0.874	0.903

Note: See Notes following Table 10.

TABLE 3

Monte Carlo Distributions of the Fluctuation Statistic Based on the Recursive Fama-French Regression Statistic
$\sup_{\lambda_0 \leq \lambda \leq 1} |\beta(\delta, \lambda) - \beta(\delta, 1)|$

T	Percentiles								
	0.025	0.05	0.10	.25	0.50	0.75	0.90	0.95	0.975
				Horizon: $\delta = 12/840$					
312	0.070	0.080	0.094	0.124	0.172	0.239	0.310	0.359	0.402
840	0.071	0.081	0.096	0.127	0.177	0.248	0.324	0.374	0.418
1680	0.072	0.082	0.096	0.127	0.178	0.244	0.323	0.374	0.425
2520	0.072	0.081	0.096	0.129	0.179	0.248	0.329	0.383	0.430
				Horizon: $\delta = 60/840$					
312	0.122	0.150	0.188	0.275	0.414	0.604	0.790	0.910	1.036
840	0.118	0.145	0.187	0.279	0.424	0.606	0.808	0.939	1.058
1680	0.123	0.150	0.189	0.278	0.422	0.606	0.798	0.936	1.047
2520	0.119	0.149	0.188	0.282	0.426	0.617	0.827	0.966	1.090
				Horizon: $\delta = 96/840$					
312	0.140	0.179	0.238	0.364	0.562	0.825	1.112	1.297	1.472
840	0.141	0.180	0.233	0.360	0.559	0.833	1.139	1.321	1.511
1680	0.139	0.182	0.243	0.368	0.570	0.838	1.131	1.316	1.503
2520	0.140	0.176	0.233	0.357	0.564	0.845	1.143	1.331	1.513

Note: See Notes following Table 10.

TABLE 4

Monte Carlo Distributions of the Fluctuation Statistic Based on the Reverse-Recursive Fama-French Regression Statistic
$\sup_{\lambda_0 \leq \lambda \leq 1} |\beta^r(\delta, \lambda) - \beta(\delta, 1)|$

T	Percentiles								
	0.025	0.05	0.10	.25	0.50	0.75	0.90	0.95	0.975
				Horizon: $\delta = 12/840$					
312	0.070	0.079	0.091	0.121	0.167	0.232	0.303	0.351	0.397
840	0.069	0.079	0.094	0.124	0.173	0.240	0.314	0.362	0.407
1680	0.071	0.081	0.094	0.124	0.173	0.242	0.315	0.365	0.414
2520	0.070	0.080	0.094	0.124	0.173	0.240	0.315	0.366	0.407
				Horizon: $\delta = 60/840$					
312	0.123	0.146	0.181	0.258	0.379	0.546	0.727	0.842	0.963
840	0.121	0.145	0.182	0.258	0.380	0.546	0.726	0.846	0.944
1680	0.123	0.150	0.185	0.266	0.389	0.560	0.735	0.861	0.966
2520	0.122	0.148	0.184	0.261	0.385	0.545	0.727	0.849	0.955
				Horizon: $\delta = 96/840$					
312	0.160	0.191	0.240	0.343	0.505	0.740	0.998	1.180	1.365
840	0.157	0.193	0.239	0.346	0.507	0.737	1.007	1.192	1.374
1680	0.156	0.189	0.236	0.340	0.509	0.750	1.025	1.196	1.374
2520	0.161	0.197	0.241	0.342	0.507	0.742	1.014	1.191	1.351

Note: See Notes following Table 10.

TABLE 5

10% Critical Values for Multivariate Mean-Reversion Tests
and Average Median of Their Asymptotic Distributions
When 95% Confidence Interval for c is $[-21.41, 2.11]$

Statistic	$\sigma_\epsilon/\sigma_\eta = 0.963$, $\rho_{\epsilon,\eta} = -0.948$	$\sigma_\epsilon/\sigma_\eta = 1.270$, $\rho_{\epsilon,\eta} = -0.884$	$\sigma_\epsilon/\sigma_\eta = 0.960$, $\rho_{\epsilon,\eta} = -0.947$	$\sigma_\epsilon/\sigma_\eta = 1.270$, $\rho_{\epsilon,\eta} = -0.881$
(A) 10% Critical Values				
$T\,a_1(T)$	498.217	642.854	496.871	641.667
$\max_{\lambda_0 \le \lambda < 1} T\,\lvert a_1(T\lambda) - a_1(T)\rvert$	1567.919	2018.844	1561.833	2016.238
$\max_{0 < \lambda \le \lambda_0} T\,\left\lvert a_1^r(T\lambda) - a_1(T)\right\rvert$	1598.715	2033.578	1593.597	2030.080
$QLR = \max_{\lambda_0 \le \lambda \le 1-\lambda_0} F(T\lambda)$	11.330	10.883	11.322	10.873
(B) Average Median				
$T\,a_1(T)$	89.356	109.766	89.029	109.342
$\max_{\lambda_0 \le \lambda < 1} T\,\lvert a_1(T\lambda) - a_1(T)\rvert$	446.309	574.910	444.723	574.405
$\max_{0 < \lambda \le \lambda_0} T\,\left\lvert a_1^r(T\lambda) - a_1(T)\right\rvert$	457.476	582.819	456.019	581.889
$QLR = \max_{\lambda_0 \le \lambda \le 1-\lambda_0} F(T\lambda)$	3.640	3.520	3.638	3.515

Note: See notes following Table 10.

TABLE 6

Recursive and Reverse-Recursive Estimates of Fluctuation Statistics for Log Real Returns on the CRSP Equally-Weighted Portfolio (1926.01-1995.12)

	Horizon in Months ($T\delta$)							
	12	24	36	48	60	96	120	Avrg.
	Full Sample Variance Ratio							
$vr(T\delta, T)$	1.226	1.140	0.970	0.861	0.729	0.349	0.267	0.792
p-value	0.040	0.149	0.372	0.506	0.657	0.964	0.979	0.607
	Recursive Variance Ratio							
$\text{Max}\left\lvert\frac{vr(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.105	0.090	0.111	0.121	0.218	0.671	0.734	0.143
p-value	0.972	0.998	0.997	0.996	0.931	0.265	0.232	0.937
$vr(T\delta, T\lambda)$	1.097	1.038	0.862	0.757	0.570	0.115	0.071	0.679
Date (Tλ)	1944.06	1944.12	1945.09	1971.05	1972.02	1950.01	1951.01	1967.06
	Reverse-Recursive Variance Ratio							
$\text{Max}\left\lvert\frac{vr^r(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.273	0.542	0.524	0.468	0.443	1.134	1.214	0.347
p-value	0.395	0.107	0.246	0.449	0.567	0.001	0.001	0.600
$vr^r(T\delta, T\lambda)$	0.891	0.522	0.462	0.458	0.406	0.745	0.591	0.517
Date (Tλ)	1975.04	1975.10	1976.06	1977.06	1978.06	1956.11	1958.11	1980.07
	Full Sample Fama-French Regression Statistic							
$b(T\delta, T)$	-0.068	-0.233	-0.367	-0.484	-0.550	-0.379	-0.193	-0.325
p-value	0.040	0.149	0.372	0.506	0.657	0.964	0.979	0.607
	Recursive Fama-French Regression Statistic							
$\text{Max}\lvert b(T\delta,T\lambda) - b(T\delta,T)\rvert$	0.059	0.073	0.184	0.207	0.205	0.188	0.340	0.042
p-value	0.993	0.992	0.822	0.822	0.873	0.943	0.801	0.977
$b(T\delta, T\lambda)$	-0.009	-0.306	-0.551	-0.691	-0.755	-0.191	0.147	-0.283
Date (Tλ)	1975.04	1971.06	1950.05	1950.01	1975.02	1975.06	1977.02	1982.08
	Reverse-Recursive Fama-French Regression Statistic							
$\text{Max}\lvert b^r(T\delta,T\lambda) - b(T\delta,T)\rvert$	0.387	0.233	0.405	0.311	0.254	0.342	0.507	0.098
p-value	0.035	0.516	0.261	0.545	0.760	0.756	0.563	0.921
$b^r(T\delta, T\lambda)$	-0.455	-0.001	0.038	-0.172	-0.296	-0.721	-0.700	-0.227
Date (Tλ)	1978.12	1978.04	1973.04	1975.06	1977.06	1973.04	1973.12	1951.08

Note: See notes following Table 10.

TABLE 7

Recursive and Reverse-Recursive Estimates of Fluctuation Statistics for Log Real Returns on the CRSP Value-Weighted Portfolio (1926.01-1995.12)

	Horizon in Months ($T\delta$)							
	12	24	36	48	60	96	120	Avrg.
	Full Sample Variance Ratio							
$vr(T\delta, T)$	1.166	1.079	0.936	0.799	0.707	0.547	0.556	0.827
p-value	0.087	0.228	0.428	0.608	0.692	0.767	0.676	0.541
	Recursive Variance Ratio							
$\max\left\lvert\frac{vr(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.072	0.124	0.121	0.141	0.350	0.817	0.825	0.203
p-value	0.999	0.982	0.994	0.989	0.705	0.074	0.107	0.795
$vr(T\delta, T\lambda)$	1.250	1.213	1.049	0.686	0.459	0.100	0.097	0.659
Date ($T\lambda$)	1975.04	1975.12	1957.07	1948.11	1949.05	1949.06	1950.12	1950.12
	Reverse-Recursive Variance Ratio							
$\max\left\lvert\frac{vr^r(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.435	0.679	0.716	0.698	0.657	1.042	1.110	0.531
p-value	0.082	0.024	0.041	0.108	0.218	0.003	0.002	0.238
$vr^r(T\delta, T\lambda)$	0.659	0.347	0.266	0.241	0.242	1.117	1.172	0.388
Date ($T\lambda$)	1975.05	1976.04	1977.06	1978.06	1979.06	1955.04	1958.11	1980.09
	Full Sample Fama-French Regression Statistic							
$b(T\delta, T)$	-0.067	-0.228	-0.290	-0.173	-0.076	-0.022	-0.191	-0.150
p-value	0.087	0.228	0.428	0.608	0.692	0.767	0.676	0.541
	Recursive Fama-French Regression Statistic							
$\max\lvert b(T\delta, T\lambda) - b(T\delta, T)\rvert$	0.074	0.079	0.362	0.473	0.510	0.317	0.553	0.070
p-value	0.967	0.987	0.393	0.315	0.367	0.800	0.562	0.913
$b(T\delta, T\lambda)$	0.008	-0.306	-0.652	-0.646	-0.587	-0.339	0.361	-0.219
Date ($T\lambda$)	1956.10	1951.08	1952.10	1949.09	1950.12	1956.04	1964.10	1958.06
	Reverse-Recursive Fama-French Regression Statistic							
$\max\lvert b^r(T\delta, T\lambda) - b(T\delta, T)\rvert$	0.439	0.160	0.555	0.577	0.374	0.394	0.714	0.198
p-value	0.015	0.780	0.097	0.146	0.515	0.673	0.332	0.649
$b^r(T\delta, T\lambda)$	-0.506	-0.067	0.265	0.405	0.297	-0.417	-0.906	0.048
Date ($T\lambda$)	1978.12	1975.02	1953.03	1954.10	1956.09	1973.04	1974.09	1954.10

Note: See notes following Table 10.

TABLE 8

Recursive and Reverse-Recursive Estimates of Fluctuation Statistics for Log Excess Returns on the CRSP Equally-Weighted Portfolio (1926.01-1995.12)

	Horizon in Months ($T\delta$)							
	12	24	36	48	60	96	120	Avrg.
	Full Sample Variance Ratio							
$vr(T\delta, T)$	1.269	1.238	1.111	1.036	0.910	0.458	0.399	0.917
p-value	0.022	0.072	0.184	0.259	0.388	0.874	0.882	0.381
	Recursive Variance Ratio							
$\text{Max}\left\lvert\frac{vr(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.066	0.071	0.059	0.074	0.149	0.525	0.603	0.105
p-value	0.999	1.000	1.000	1.000	0.990	0.512	0.447	0.984
$vr(T\delta, T\lambda)$	1.185	1.325	1.177	0.959	0.774	0.218	0.158	0.821
Date (Tλ)	1944.06	1974.12	1978.09	1971.05	1972.02	1950.01	1951.01	1968.05
	Reverse-Recursive Variance Ratio							
$\text{Max}\left\lvert\frac{vr^r(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.245	0.502	0.475	0.420	0.383	0.821	0.798	0.371
p-value	0.490	0.159	0.331	0.539	0.678	0.113	0.228	0.548
$vr^r(T\delta, T\lambda)$	0.958	0.617	0.583	0.601	0.561	0.834	0.717	0.577
Date (Tλ)	1975.04	1975.10	1976.06	1977.06	1978.06	1948.05	1950.05	1980.12
	Full Sample Fama-French Regression Statistic							
$b(T\delta, T)$	-0.023	-0.156	-0.291	-0.412	-0.435	-0.119	0.052	-0.198
p-value	0.022	0.072	0.184	0.259	0.388	0.874	0.882	0.381
	Recursive Fama-French Regression Statistic							
$\text{Max}\lvert b(T\delta, T\lambda) - b(T\delta, T)\rvert$	0.072	0.053	0.144	0.146	0.126	0.210	0.191	0.068
p-value	0.973	0.999	0.908	0.930	0.967	0.921	0.943	0.918
$b(T\delta, T\lambda)$	0.049	-0.209	-0.435	-0.558	-0.561	0.092	-0.139	-0.266
Date (Tλ)	1946.08	1971.06	1972.12	1973.11	1974.12	1981.01	1970.08	1971.07
	Reverse-Recursive Fama-French Regression Statistic							
$\text{Max}\lvert b^r(T\delta, T\lambda) - b(T\delta, T)\rvert$	0.379	0.264	0.359	0.258	0.208	0.681	0.739	0.191
p-value	0.039	0.424	0.341	0.676	0.856	0.300	0.309	0.670
$b^r(T\delta, T\lambda)$	-0.402	0.108	0.068	-0.153	-0.226	-0.799	-0.687	-0.389
Date (Tλ)	1978.12	1978.02	1979.11	1947.01	1948.01	1973.04	1974.03	1975.12

Note: See Notes following Table 10.

TABLE 9

Recursive and Reverse-Recursive Estimates of Fluctuation Statistics for Log Excess Returns on the CRSP Value-Weighted Portfolio (1926.01-1995.12)

	Horizon in Months ($T\delta$)							
	12	24	36	48	60	96	120	Avrg.
	Full Sample Variance Ratio							
$vr(T\delta, T)$	1.209	1.184	1.084	0.977	0.883	0.615	0.646	0.943
p-value	0.051	0.110	0.214	0.330	0.423	0.674	0.557	0.341
	Recursive Variance Ratio							
$\text{Max}\left\lvert\frac{vr(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.084	0.170	0.221	0.215	0.139	0.598	0.661	0.116
p-value	0.995	0.908	0.862	0.911	0.994	0.383	0.349	0.974
$vr(T\delta, T\lambda)$	1.310	1.386	1.324	1.187	1.006	0.247	0.219	1.052
Date ($T\lambda$)	1974.12	1956.08	1957.07	1957.07	1959.08	1949.06	1950.12	1978.03
	Reverse-Recursive Variance Ratio							
$\text{Max}\left\lvert\frac{vr^r(T\delta,T\lambda)-vr(T\delta,T)}{vr(T\delta,T)}\right\rvert$	0.462	0.741	0.818	0.837	0.822	0.739	0.822	0.687
p-value	0.058	0.011	0.012	0.018	0.037	0.248	0.185	0.049
$vr^r(T\delta, T\lambda)$	0.651	0.307	0.198	0.159	0.157	0.161	1.177	0.295
Date ($T\lambda$)	1975.05	1976.04	1977.04	1978.05	1979.05	1980.06	1952.04	1979.07
	Full Sample Fama-French Regression Statistic							
$b(T\delta, T)$	-0.016	-0.155	0.245	-0.172	-0.051	0.219	0.076	-0.049
p-value	0.051	0.110	0.214	0.330	0.423	0.674	0.557	0.341
	Recursive Fama-French Regression Statistic							
$\text{Max}\lvert b(T\delta, T\lambda) - b(T\delta, T)\rvert$	0.107	0.048	0.256	0.315	0.304	0.162	0.321	0.082
p-value	0.847	0.999	0.627	0.599	0.704	0.961	0.821	0.876
$b(T\delta, T\lambda)$	0.092	-0.107	-0.501	-0.487	-0.354	0.382	0.397	-0.131
Date ($T\lambda$)	1956.08	1957.08	1947.12	1949.06	1950.12	1957.11	1960.03	1974.02
	Reverse-Recursive Fama-French Regression Statistic							
$\text{Max}\lvert b^r(T\delta, T\lambda) - b(T\delta, T)\rvert$	0.540	0.266	0.500	0.588	0.437	0.623	0.838	0.304
p-value	0.003	0.417	0.140	0.138	0.402	0.356	0.232	0.354
$b^r(T\delta, T\lambda)$	-0.556	-0.421	0.254	0.416	0.386	-0.403	-0.762	-0.353
Date ($T\lambda$)	1978.12	1978.10	1952.10	1950.10	1951.08	1973.04	1974.08	1983.04

Note: See notes following Table 10.

TABLE 10

Multivariate Mean-Reversion Tests on the CRSP Portfolio (1926.12-1995.12)

(A) Process for the Log Dividend-Price Ratio: $x_t = \mu + \phi x_{t-1} + \eta_t$, $\phi = 1 + c/T$

$\hat{\phi}$	0.983
Dickey-Fuller t-stat for $\phi = 1$	-2.520
95% Confidence Interval for c	[-21.413,2.107]
95% Confidence Interval for ϕ	[0.974,1.002]
$\hat{\sigma}_\eta$	0.057

(B) Expected Return Equation: $r_t = \alpha_0 + \alpha_1 x_{t-1} + \varepsilon_t$

	Real VW Ret.	Real EW Ret.	Excess VW Ret.	Excess EW Ret.
$a_1(T)$	0.011	0.018	0.010	0.018
t-statistic	1.591	2.108	1.575	2.093
$T\,a_1(T)$	8.716	15.231	8.601	15.116
$\hat{\sigma}_\epsilon$	0.056	0.073	0.055	0.073
$\hat{\rho}_{\epsilon,\eta}$	-0.948	-0.884	-0.947	-0.881
$\max_{\lambda_0 \leq \lambda < 1} T\,\lvert a_1(T\lambda) - a_1(T)\rvert$	5.194	10.488	8.496	6.481
$a_1(T\lambda)$	0.004	0.031	0.000	0.010
Date (Tλ)	1946.05	1951.01	1946.05	1968.11
$\max_{0 < \lambda \leq \lambda_0} T\,\left\lvert a_1^r(T\lambda) - a_1(T)\right\rvert$	11.757	13.971	12.343	13.450
$a_1^r(T\lambda)$	0.025	0.035	0.025	0.034
Date (Tλ)	1953.10	1969.02	1932.07	1969.01
$QLR = \max_{\lambda_0 \leq \lambda \leq 1-\lambda_0} F(T\lambda)$	0.905	1.223	1.485	1.241
Date (Tλ)	1968.11	1968.12	1949.06	1968.12

Note: See Notes following Table 10.

NOTES:

Table 1: Monte Carlo distributions are based on 10,000 simulations of the continuous time process for the random functional in the heading of the table. To simulate the process we use discrete approximations of size T. We consider values of T which are multiples of 840,the actual number of monthly observations in our sample of one-period log returns (log returns in the NYSE-AMEX-NASDAQ stock markets, from 1926.01 through 1995.12). The values of δ are set so that $[T\delta]$ equals multiples of 12 when T=840. The trimming parameter λ_0 is set so 25% of the effective sample is used in the first recursion. We define the effective sample for the recursive variance-ratio statistic as $[T(1-\delta)]$, i.e. the total number of observations minus those needed to compute the first $[T\delta]$-horizon return observation.

Table 2: Monte Carlo distributions are based on 10,000 simulations of the continuous time process for the random functional in the heading of the table. To simulate the process we use discrete approximations of size T. We consider values of T which are multiples of 840,the actual number of monthly observations in our sample of one-period log returns (log returns in the NYSE-AMEX-NASDAQ stock markets, from 1926.01 through 1995.12). The values of δ are set so that $[T\delta]$ equals multiples of 12 when T=840. The trimming parameter λ_0 is set so 25% of the effective sample is used in the first recursion. We define the effective sample for the reverse-recursive variance-ratio statistic as $[T(1-\delta)]$, i.e. the total number of observations minus those needed to compute the first $[T\delta]$-horizon return observation.

Table 3: Monte Carlo distributions are based on 10,000 simulations of the continuous time process for the random functional $\beta(\delta, \lambda)$. To simulate the process we use discrete approximations of size T. We consider values of T which are multiples of 840,the actual number of monthly observations in our sample of one-period log returns (log returns in the NYSE-AMEX-NASDAQ stock markets, from 1926.01 through 1995.12). The values of δ are set so that $[T\delta]$ equals multiples of 12 when T=840. The trimming parameter λ_0 is set so 25% of the effective sample is used in the first recursion. We define the effective sample for the recursive Fama-French univariate mean-reversion regression statistic as $[T(1-2\delta)]$, i.e. the total number of observations minus those needed to compute the first two non-overlapping $[T\delta]$-horizon return observations.

Table 4: Monte Carlo distributions are based on 10,000 simulations of the continuous time process for the random functional $\beta^r(\delta, \lambda)$. To simulate the process we use discrete approximations of size T. We consider values of T which are multiples of 840,the actual number of monthly observations in our sample of one-period log returns (log returns in the NYSE-AMEX-NASDAQ stock markets, from 1926.01 through 1995.12). The values of δ are set so that $[T\delta]$ equals multiples of 12 when T=840. The trimming parameter λ_0 is set so 25% of the effective sample is used in the first recursion. We define the effective sample for the reverse-recursive Fama-French univariate mean-reversion regression statistic as $[T(1-2\delta)]$, i.e. the total number of observations minus those needed to compute the first two non-overlapping $[T\delta]$-horizon return observations.

Table 5: Monte Carlo distributions are based on 10,000 simulations of the continuous time process for the random functionals to which the statistics in the first column of the table converge. These limiting representations are given in Propositions 5, 6 and 7 and Corollaries 5, 6 and 7 in text, and they depend on the nuisance parameters c, $\sigma_\epsilon/\sigma_\eta$ and $\rho_{\epsilon,\eta}$. To simulate the processes we use discrete approximations of size T=829, consistent estimates of σ_ϵ, σ_η and $\rho_{\epsilon,\eta}$, and a grid of values of c in a 95% confidence interval, all based on the data we use in our empirical application—one-period log returns and log dividend-price ratios in the NYSE-AMEX-NASDAQ stock markets, from 1926.12 through 1995.12. The trimming parameter λ_0 is set to $\lambda_0 = 25\%$. The 95% confidence interval for c was computed following the methodology proposed in Stock (1991) and described in text. Also, the critical values for each statistic reported in Panel A are the maximum 95% percentile of the set of distributions for each statistic generated from the grid of values of c, following the approach suggested in Wright (1996). This grid contains 11 equally-spaced values in the 95% confidence interval for c and it includes the extreme values in the interval. Panel B reports the average median across the set of distributions generated by the grid of values of c.

Tables 6 through 9: Monthly log excess returns are log nominal returns on the CRSP (value-weighted or equally weighted) portfolio inclusive of all securities traded in the NYSE, AMEX and NASDAQ markets in excess of log returns on U.S. Treasury Bills, as reported in the Indices and SBBI files of the Center for Research in Security Prices (1996). Monthly log real returns are log nominal returns on this portfolio minus the log rate of change in the Consumer Price Index for All Urban Consumers, not seasonally adjusted, as reported in the Indices and SBBI files of the Center of Research in Security Prices (1996). For each recursive and reverse-recursive fluctuation statistic we report the value of the maximum fluctuation statistic, its p-value according to its Montecarlo distribution, the value of the variance-ratio or Fama-Frech univariate regression statistic associated with the maximum fluctuation statistic and the date of the maximum fluctuation statistic. p-values are based on Monte Carlo distributions involving 10,000 simulations of each test statistic. Each simulation is based on a discrete time approximation 840 observations long (the number of months in our sample) to a standard Brownian Motion process in the unit interval (Row T=840 in Tables 1 through 4 report the deciles for some of these distributions). The Monte Carlo distributions replicate the asymptotic distributions of the full sample statistics given in Propositions 1 through 4 (for $\lambda = 1$) and the maximum fluctuation statistics given in Corollaries 1 through 4. The trimming parameter λ_0 in the recursive and reverse-recursive statistics is set so 25% of the effective sample (the minimum number of observations needed to compute a recursive estimate of each estatistic) is used in the first recursion (and reverse-recursion). The "Avrg." column refers to the mean across horizons of the statisc in the corresponding row. For example, the average full sample variance ratio statistic in the first row equals $\sum_{\forall\delta} vr(T\delta, T)$. The asymptotic distribution of the average statistics obtains immediately from Propositions 1 through 4 and the CMT.

Table 10: Monthly log excess returns are log nominal returns on the CRSP (value-weighted or equally weighted) portfolio inclusive of all securities traded in the NYSE, AMEX and NASDAQ markets in excess of log returns on U.S. Treasury Bills, as reported in the Indices and SBBI files of the Center for Research in Security Prices (1996). Monthly log real returns are log nominal returns on this portfolio minus the log rate of change in the Consumer Price Index for All Urban Consumers, not seasonally adjusted, as reported in the Indices and SBBI files of the Center of Research in Security Prices (1996). Following the standard convention in the literature, the dividend price ratio at month t is computed as in Fama and French (1989), i.e., by computing the moving average $(D/P)_t = \sum_{t=0}^{11} D_{t-i}/P_t$, where D_t and P_t are month t dividend and ex-dividend value of the portfolio. For each recursive and reverse-recursive fluctuation statistic we report the value of the maximum fluctuation statistic, the estimate of the slope associated with the maximum fluctuation statistic and the date of the maximum fluctuation statistic. For the QLR statistic we also report the date it occurrs. The sample values of the full sample statistic, fluctuation statistics and QLR statistic in each column should be compared with the corresponding column in Table 5 for a 10% significance test of the null $\alpha_1 = 0$–full sample statistics—and no structural breaks—fluctuation statistics and QLR statistic.

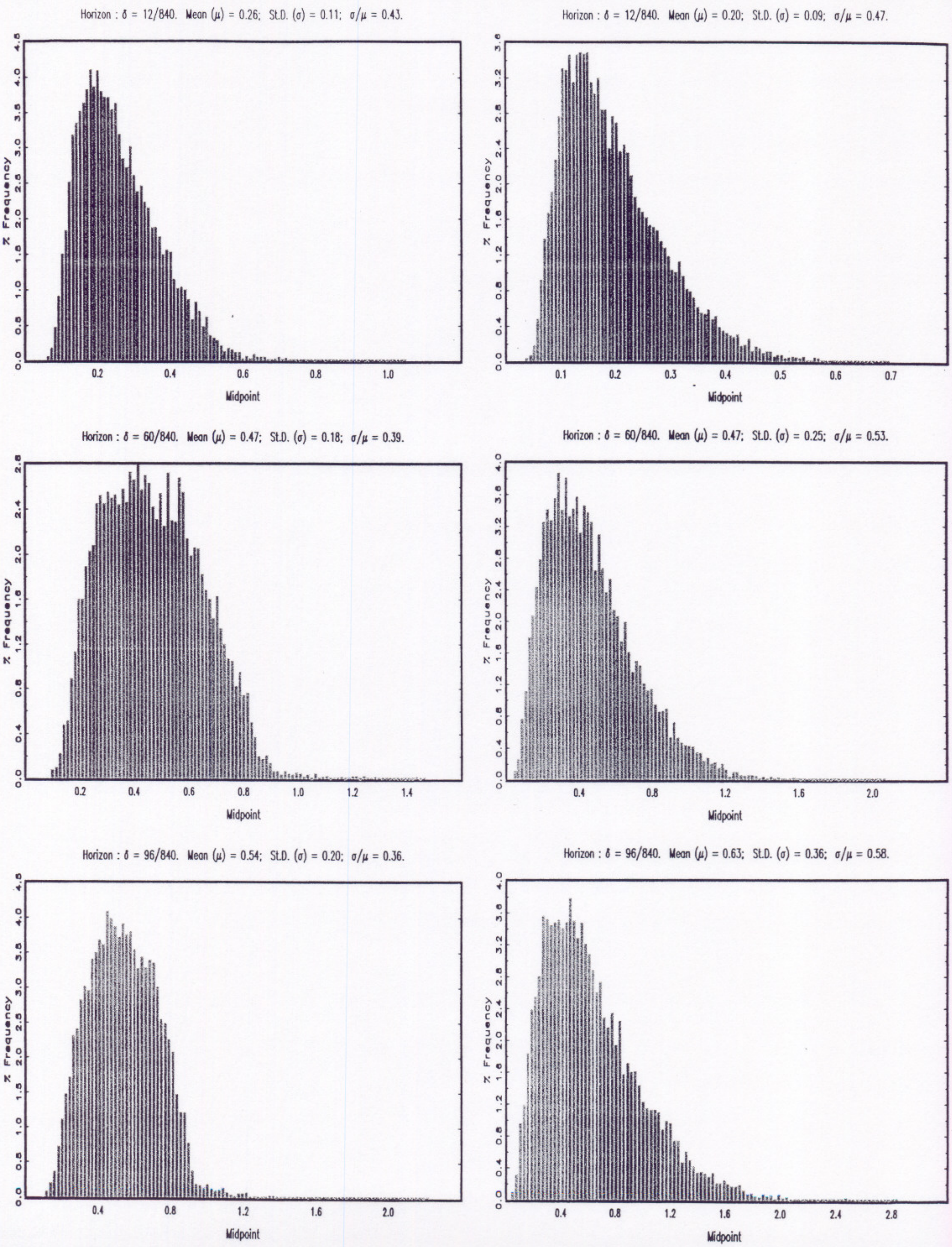

Figure 1. Histograms of Monte Carlo distributions of the recursive fluctuation statistics based on the variance ratio (left column) and the Fama-French univariate regression statistic (right column).

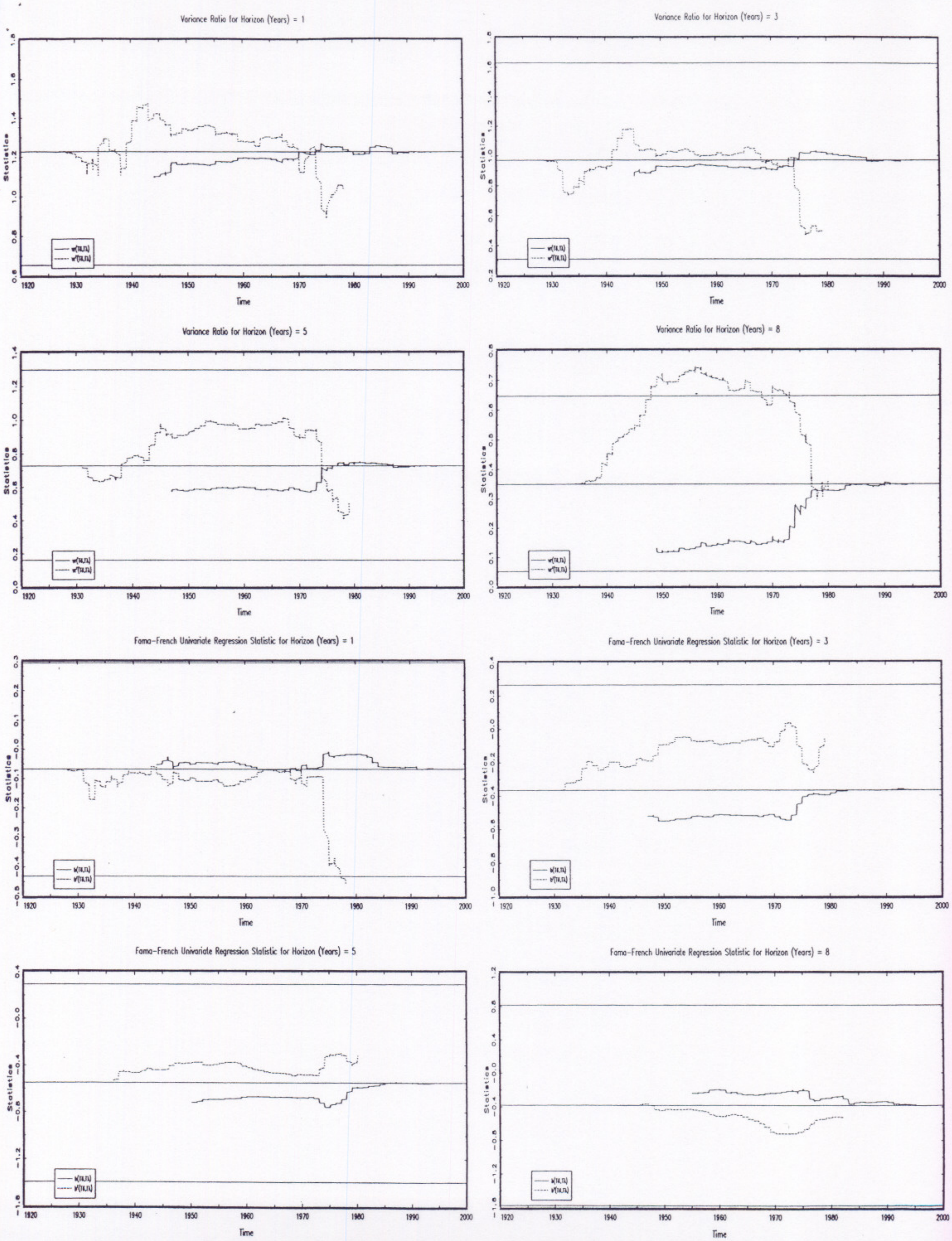

Figure 2. Recursive and reverse-recursive estimates of univariate mean-reversion statistics for log real returns on the CRSP equally-weighted portfolio.

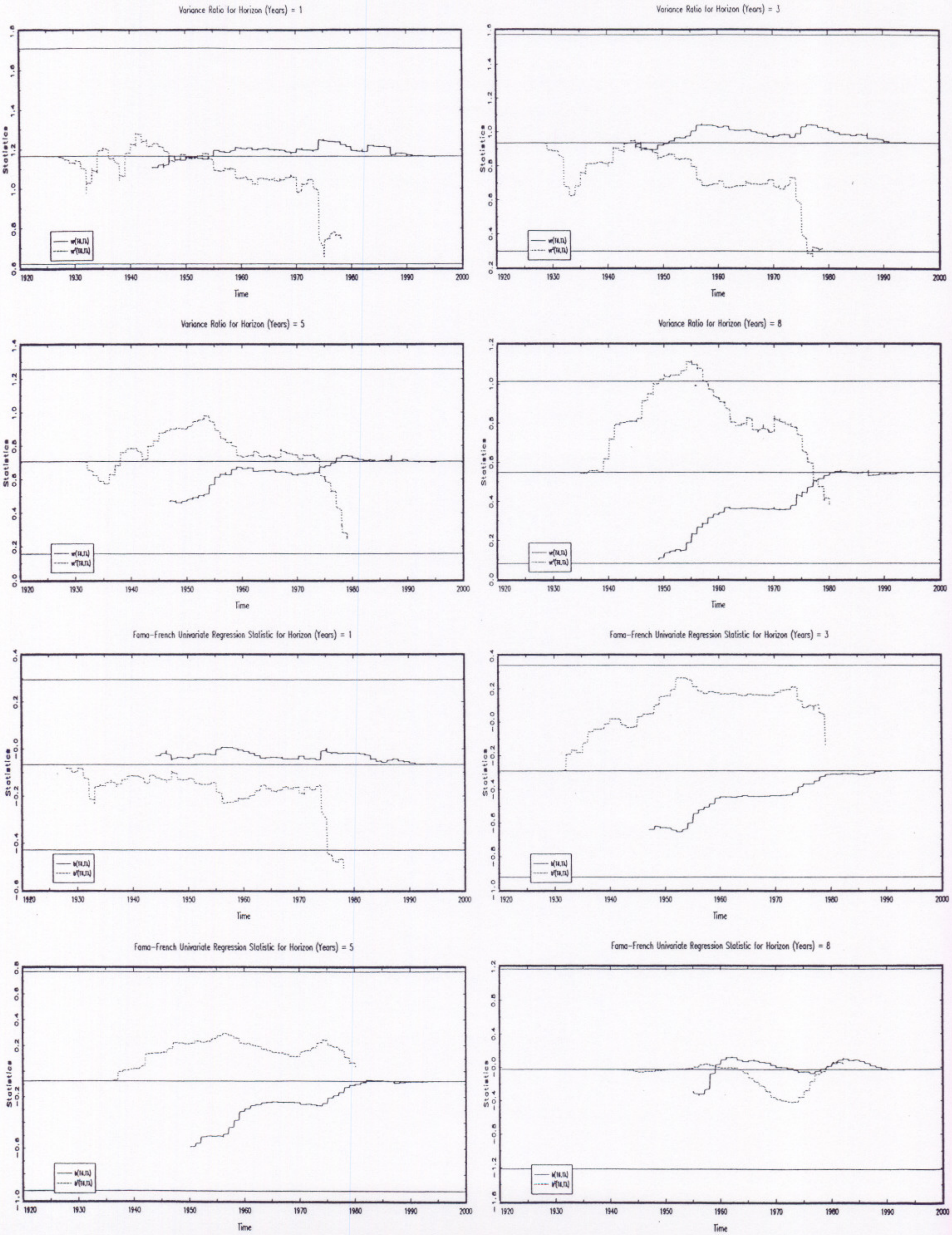

Figure 3. Recursive and reverse-recursive estimates of univariate mean-reversion statistics for log real returns on the CRSP value-weighted portfolio.

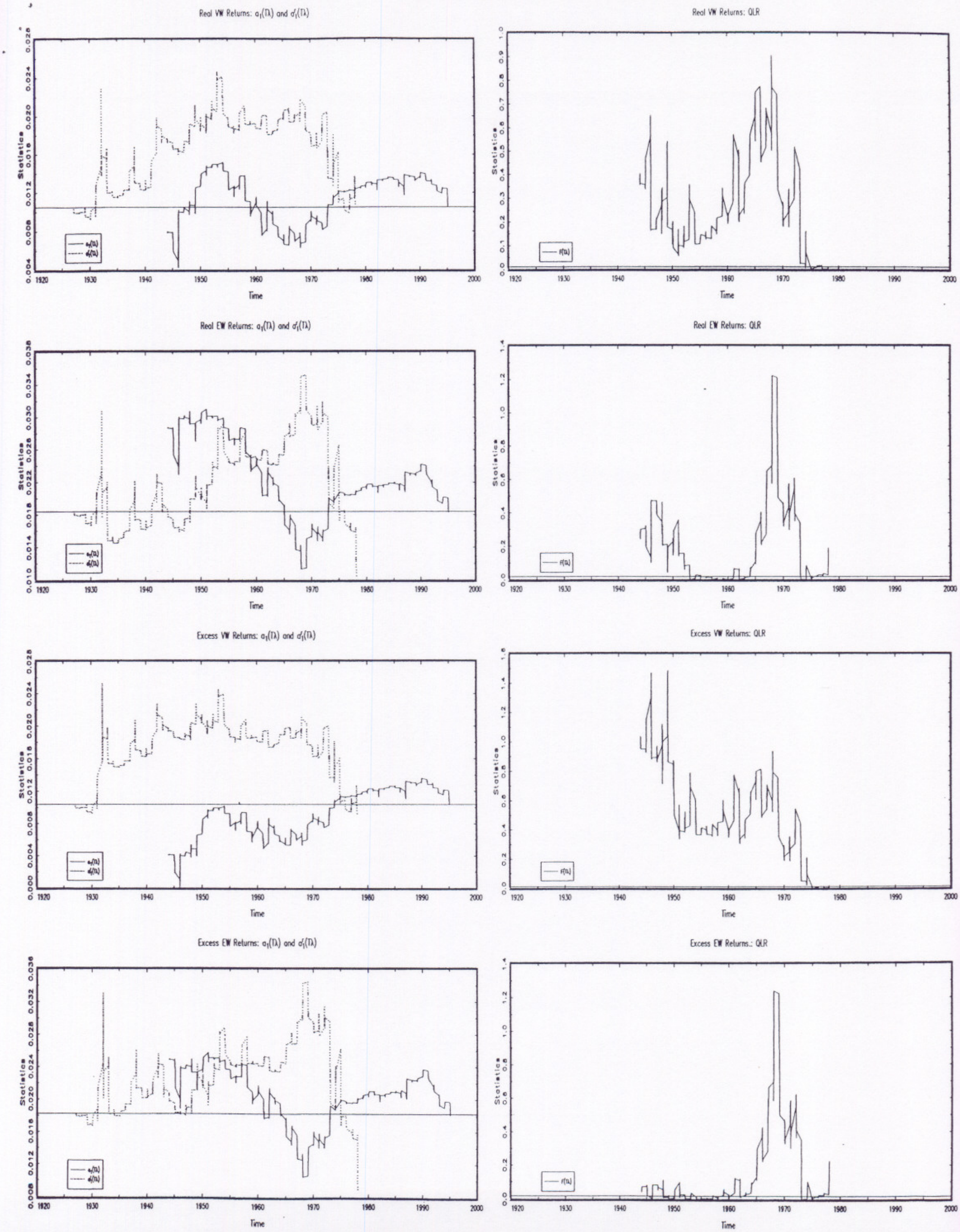

Figure 4. Recursive and reverse-recursive estimates of the slope in the regression of log returns on the log dividend-yield and QLR statistic.