Pathwise Stationary Solutions and Random Periodic Solutions of Random Dynamical Systems (SPDEs, Random Maps)

Huaizhong Zhao (Loughborough University)

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1. The problem

There are two fundamental concepts in deterministic dynamical systems: fixed point (equilibrium point) and periodic solutions. For stochastic dynamical systems, they correspond to fundamental ideas: pathwise stationary solutions and random periodic solutions. The question is to give their proper definitions, conditions for their existence and stability (stable and unstable manifolds).

Stationary solution: concept: well-known, results: a lot to do.

Random periodic solution: did not exist as far as we know.

2. The concept of stationary solutions

Consider a *random dynamical system* on a measurable space $(S, \mathcal{B})$ over a metric DS $(\Omega, \mathcal{F}, P, \{\theta(t)\}_{t\in \mathrm{T}})$ with time T

$$\phi : \mathrm{T} \times \Omega \times S \to S, \quad (t, \omega, h) \to \phi(t, \omega, h).$$

A *stationary solution* is a $\mathcal{F}$-measurable random variable $Y^* : \Omega \to S$ such that

$$\phi(t, \omega, Y^*(\omega)) = Y^*(\theta_t \omega), \quad t \in T \ a.s.$$

This is the corresponding notion of a steady or equilibrium state in deterministic dynamical systems $\phi : T \times S \to S$.

Example 1 Simplest ever nontrivial example:

As a random perturbation to the deterministic equation

$$\frac{dy}{dt} = -y, \quad y(0) = h,$$

we consider the Ornstein-Uhlenbeck process

$$dy = -ydt + dB_t, \quad y(0) = h.$$

Variation of constant formula gives the following solution:

$$\phi_t^{\omega} h = h\mathrm{e}^{-t} + \int_0^t \mathrm{e}^{-(t-s)} dB_s(\omega).$$

It is easy to check that

$$Y^*(\omega) = \int_{-\infty}^0 \mathrm{e}^s dB_s(\omega)$$

is the stationary solution of the equation and for any $h \in R^1$, as $t \to \infty$

$$|\phi_t^{\omega} h - Y^*(\theta_t \omega)|$$

$$
\begin{aligned}
&= e^{-t}|h - \int_{-\infty}^{0} e^{s} dB_s(\omega)| \\
&\to 0.
\end{aligned}
$$

It is well known that

●

$$
\begin{aligned}
Y^*(\theta(t)\omega) &= \phi(t,\omega)Y^*(\omega) \\
&\Updownarrow \\
\mu(dx, d\omega) &= \delta_{Y^*(\omega)}(dx)P(d\omega) \\
\text{is} \quad & \text{an} \quad \text{invariant measure}
\end{aligned}
$$

● Every ergodic invariant measure μ of a RDS on R^1 is a random Dirac measure.

In general, this is not true. However, the following is also well known:

● Every invariant measure is Dirac by considering the extended probability space:

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, (\bar{\theta}(t))_{t\in\mathbb{T}}) = (\Omega \times S, \mathcal{F} \otimes \mathscr{B}(S), \mu, (\Theta(t))_{t\in\mathbb{T}})$$

and

$$\bar{\phi}(t, \bar{\omega}) = \phi(t, \omega).$$

But, by considering the extended probability space, one regards the dynamical system as noise as well, so the dynamics is different.

Remarks

(i) The stationary solution is not a fixed point in the deterministic sense, but a random moving fixed point or equilibrium of the stochastic system in the state space. It describes the invariance over time along the measurable and P-preserving transformation θ_t: $\Omega \to \Omega$.

(ii) For SPDEs, a stationary solution consists of infinitely many random moving surfaces on the configuration space due to the random external force pumped to the systems constantly.

3. Work on stationary solution of SPDEs

Previous work: The existence of stationary solutions of SPDEs is one of the basic problems: no general methods.

• Sinai (1991, 1996), Stochastic Burgers equations with additive C^3 noise, Feynman-Kac formula and Hopf-Cole transformation, so good regularity is needed.

• E, Khanin, Mazel and Sinai, Annals of Mathematics (2000), Stochastic inviscid Burgers equations with additive C^3 noise (minimizing method)

• Mattingly, 2D Stochastic Navier-Stokes equation with additive noise, CMP (1999)

• Caraballo, Kloeden and Schmalfuss (2004), stochastic evolution equations with small Lipschitz constant and linear noise.

• Mohammed, Zhang and Zhao, Memoirs of AMS (2008). Stochastic evolution equations and SPDEs with discrete spectrum, integral equation with linear or additive noise, stable/unstable manifolds.

A basic assumption in invariant manifold theory: there exists a stationary solution.

• Arnold, Mohammed, Scheutzow (SDE cases)

• Duan, Lu and Schmalfuss, Annals of probability (2003).

• Mohammed, Zhang and Zhao (2008).

Our more recent work:

The method of infinite horizon integral equation:

Consider the following semilinear see on H:

$$\begin{aligned} du(t) &= -Au(t)\,dt + F(u(t))\,dt + B_0 dW(t), \quad t \geq 0, \\ u(0) &= x \in H. \end{aligned} \tag{1}$$

In the above equation, let $F : H \to H$ be a globally Lipschitz map with Lipschitz constant L:

$$|F(v_1) - F(v_2)| \leq L|v_1 - v_2|, \quad v_1, v_2 \in H.$$

Let A be a self-adjoint operator on H admit a discrete non-vanishing spectrum $\{\mu_n, n \geq 1\}$ which is bounded below. Let $\{e_n,\ n \geq 1\}$ denote a basis for H consisting of eigenvectors of A, viz. $Ae_n = \mu_n e_n$, $n \geq 1$. Suppose $B_0 \in L_2(K, H)$.

Let $W(t), t \in R$, be a Brownian motion on the canonical complete filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and with covariance Hilbert space K. Let $T_t = e^{-At}$ stand for the strongly continuous semigroup generated by $-A$. Then (**??**) has a unique mild solution given by

$$u(t,x) = T_t x + \int_0^t T_{t-s} F(u(s,x))ds + \int_0^t T_{t-s} B_0 dW(s), \quad t \geq 0$$

Denote by μ_m the largest negative eigenvalue of A and by μ_{m+1} its smallest positive eigenvalue. Thus there is an orthogonal $\{T_t\}_{t\geq 0}$-invariant splitting of H using the negative eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_m\}$ and the positive eigenvalues $\{\mu_n : n \geq m+1\}$ of A:

$$H = H^+ \oplus H^-$$

where H^+ is a closed linear subspace of H and H^- is a finite-dimensional subspace. Denote by $p^+ : H \to H^+$ and $p^- : H \to H^-$ the corresponding projections onto H^+ and H^- respectively.

Suppose that $F : H \to H$ is globally bounded, and its Lipschitz constant L satisfies

$$L[\mu_{m+1}^{-1} - \mu_m^{-1}] < 1. \tag{2}$$

Theorem 1 *(Mohammed, Zhang and Zhao (2008)) Assume the above conditions on A, B_0, F together with (**??**). Then there is a unique $\mathcal{F}$-measurable map $Y : \Omega \to H$ satisfying*

$$\begin{aligned} Y(\omega) = &\int_{-\infty}^{0} T_{-s}p^{+}F(Y(\theta(s,\omega)))ds - \int_{0}^{\infty} T_{-s}p^{-}F(Y(\theta(s,\omega)))ds \\ &+[\int_{-\infty}^{0} T_{-s}p^{+}B_0\, dW(s)](\omega) - [\int_{0}^{\infty} T_{-s}p^{-}B_0\, dW(s)](\omega) \end{aligned} \tag{3}$$

for all $\omega \in \Omega$. Moreover, $u(t, \omega, Y(\omega)) = Y(\theta_\omega)$ for all t a.s.

Theorem 2 *(Zhao and Zhou (2009)) Under the same conditions about A and B_0 and F is globally bounded and locally Lipschitz. Then there exists at least* *one $\mathcal{F}$-measurable map $Y : \Omega \to H$ satisfying (**??**). Moreover, $u(t, \omega, Y(\omega)) = Y(\theta_\omega)$ for all t a.s.*

The method of backwards doubly stochastic differential equations:

The problem is to find the stationary solution of the following SPDEs

$$\begin{aligned} dv(t,x) &= [\mathcal{L}v(t,x) + f(x, v(t,x), \sigma^*(x)Dv(t,x))]dt \\ &\quad +g(x, v(t,x), \sigma^*(x)Dv(t,x))dB_t, \\ v(0,x) &= h(x). \end{aligned} \tag{4}$$

Here

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with $(a_{ij}(x)) = \sigma\sigma^*(x)$.

Fix notation:

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\hat{B}_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ be two mutually independent standard Brownian motion processes with values on U and R^d. Let $\mathcal{N}$ denote the class of P-null sets. For each $t \geq 0$, we define

$$\begin{aligned}\mathcal{F}_{t,s,T} &= \mathcal{F}^{\hat{B}}_{s,T} \otimes \mathcal{F}^{W}_{t,s} \bigvee \mathcal{N}, t \leq s \leq T; \\ \mathcal{F}_{t,s} &= \mathcal{F}_{t,s,\infty}, s \geq t \geq 0.\end{aligned}$$

Here for any process $\{\eta_t\}$, $\mathcal{F}^{\eta}_{s,t} = \sigma\{\eta_r - \eta_s; 0 \leq s \leq r \leq t\}$, $\mathcal{F}^{\eta}_{t,s,\infty} = \bigvee_{T\geq 0} \mathcal{F}^{\eta}_{t,s,T}$.

Let $\rho(x) = (1 + |x|)^q$, $q > 3$. Denote $I = [t, T]$ or $[0, \infty)$, $S^{2,-K}(I; L^2_\rho(R^d, R^1))$ the set of jointly measurable and adapted continuous random processes $\{\psi_t, t \geq 0\}$ with values on $L^2_\rho(R^d, R^m)$ satisfying

$$E[\sup_{t\in I} \mathrm{e}^{-Kt} \|\psi_t\|^2_{L^2_\rho(R^d,R^m)}] < \infty;$$

and $M^{2,-K}(I;L^2_\rho(R^d,R^m))$ the set of jointly measurable and adapted random processes $\{\psi_t, t\geq 0\}$ with values on $L^2_\rho(R^d,R^m)$ satisfying

$$E[\int_I \mathrm{e}^{-Ks}\|\psi_s\|^2_{L^2_\rho(R^d,R^m)}ds] < \infty.$$

Take $\hat{B}_s = B_{T-s} - B_T$. Consider for $s\geq 0$, $K>0$

$$\begin{aligned} &\mathrm{e}^{-Ks}Y_s \\ = \ & \int_s^\infty \mathrm{e}^{-Kr} f(X_r^{t,\cdot}, Y_r, Z_r)dr + \int_s^\infty K\mathrm{e}^{-Kr}Y_r dr \\ & - \int_s^\infty \mathrm{e}^{-Kr} g(X_r^{t,\cdot}, Y_r, Z_r)d^\dagger \hat{B}_r - \int_s^\infty \mathrm{e}^{-Kr}\langle Z_r, dW_r\rangle, \end{aligned} \tag{5}$$

Theorem 3 *(Zhang and Zhao, JFA (2007)) $v(t,\omega) = Y^*(\theta_t\omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ is a stationary solution of the SPDE (**??**).*

Theorem 4 *Under a monotonicity condition, Eq.(**??**) has a unique solution*

$$
\begin{aligned}
(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \ &\in \ S^{2,-K}([0,\infty); L_\rho^2(\mathrm{R}^d, \mathrm{R}^1)) \\
&\quad \times M^{2,-K}([0,\infty); L_\rho^2(\mathrm{R}^d, \mathrm{R}^d)),
\end{aligned}
$$

and $u(t,\cdot) = Y_t^{t,\cdot}$ is a weak solution of (2) and $u(t,\cdot)$ is continuous almost surely with respect to t in $L_\rho^2(\mathrm{R}^d, \mathrm{R}^1)$.

4. Random periodic solutions

For a deterministic dynamical system $\Phi_t : X \to X$ over time $t \in I$, where X is the state space, I is the set of all real numbers, or discrete real numbers, a periodic solution is a periodic function $\phi : I \to X$ with period $T \neq 0$ such that

$$\phi(t+T) = \phi(t) \text{ and } \Phi_t(\phi(t_0)) = \phi(t+t_0) \text{ for all } t, t_0 \in I. \tag{6}$$

To see the motivation for such a definition, let's first note two obvious but fundamental truth in the definition of periodic solution (**??**) of the deterministic systems when I is the set of real numbers:

(i) The function ϕ (given in the parametric form here) is a closed curve in the phase space;

(ii) If the dynamical system starts at a point on the closed curve, the orbit will remain on the same closed curve.

But note, in the case of stochastic dynamical systems, although the function ϕ may still be a periodic function, one would expect that ϕ depend on ω. In other words, we would expect infinitely many periodic functions ϕ^{ω}, $\omega \in \Omega$. Moreover, even the random dynamical system starts at a point on the curve ϕ^{ω}, it will not stay in the same periodic curve when time is running. In fact, the periodic curve actually is not the orbit of the random dynamical system, but the random dynamical system will move from one periodic curve ϕ^{ω} to another periodic curve $\phi^{\theta_t \omega}$ at time $t \in I$.

Here in order to illustrate the concept, as a simple example, we consider the random dynamical system generated by a perturbation to the following deterministic ordinary differential equation in R^2:

$$\begin{cases} \frac{dx(t)}{dt} = x(t) - y(t) - x(t)(x^2(t) + y^2(t)), \\ \frac{dy(t)}{dt} = x(t) + y(t) - y(t)(x^2(t) + y^2(t)). \end{cases} \tag{7}$$

It is well-known that above equation has a limit cycle

$$x^2(t) + y^2(t) = 1.$$

Consider a random perturbation

$$\begin{cases} dx = (x - y - x(x^2 + y^2))dt + x \circ dW(t), \\ dy = (x + y - y(x^2 + y^2))dt + y \circ dW(t). \end{cases} \tag{8}$$

Here $W(t)$ is a one-dimensional motion on the canonical probability space $(\Omega, \mathcal{F}, P)$ with the P-preserving map θ being taken to the shift operator $(\theta_t\omega)(s) = W(t+s) - W(t)$. Using polar coordinates

$$x = \rho\cos 2\pi\alpha, \quad y = \rho\sin 2\pi\alpha,$$

then we can write

$$\begin{cases} d\rho(t) = (\rho(t) - \rho^3(t))dt + \rho(t) \circ dW(t), \\ d\alpha = \frac{1}{2\pi}dt. \end{cases} \tag{9}$$

This equation has a unique close form solution as follows:

$$\rho(t, \alpha_0, \rho_0, \omega) = \frac{\rho_0 e^{t+W_t(\omega)}}{(1 + 2\rho_0^2 \int_0^t e^{2(s+W_s(\omega))}ds)^{\frac{1}{2}}}, \quad \alpha(t, \alpha_0, \rho_0, \omega) = \alpha_0 + \frac{t}{2\pi}.$$

It is easy to check that

$$\rho^*(\omega) = (2\int_{-\infty}^0 e^{2s+2W_s(\omega)}ds)^{-\frac{1}{2}}$$

is the stationary solution of the first equation of (**??**) i.e.

$$\rho(t, \alpha_0, \rho^*(\omega), \omega) = \rho^*(\theta_t \omega)$$

and

$$\Phi(t, \omega)(\alpha_0, \rho_0) = (\alpha_0 + \frac{t}{2\pi}\ mod\ 1, \rho(t, \alpha_0, \rho_0, \omega))$$

defines a random dynamical system $\Phi(t, \omega) = (\Phi_1(t, \omega), \Phi_2(t, \omega)) : [0, 1] \times R^1 \longrightarrow [0, 1] \times R^1$.

Define

$$L^\omega = \{(\alpha, \rho^*(\omega)) : \ 0 \le \alpha \le 1\},$$

then

$$L^{\theta_t \omega} = \{(\alpha, \rho^*(\theta_t \omega)) : \ 0 \le \alpha \le 1\}.$$

It is noticed that

$$\begin{aligned} \Phi(t, \omega) L^\omega &= \{(\alpha + \frac{t}{2\pi}\ mod\ 1, \rho^*(\theta_t \omega)) : \ 0 \le \alpha \le 1\} \\ &= \{(\alpha, \rho^*(\theta_t \omega)) : \ 0 \le \alpha \le 1\}. \end{aligned}$$

Therefore

$$\Phi(t,\omega)L^{\omega} = L^{\theta_t\omega},$$

i.e. $L^{\cdot}$ is invariant under Φ. Moreover

$$\Phi(2\pi,\omega)(\alpha,\rho^*(\omega)) = (\alpha,\rho^*(\theta_{2\pi}\omega)).$$

Define for $(x,y) \in R^2,\ x = \rho\cos 2\pi\alpha,\ y = \rho\sin 2\pi\alpha$

$$\begin{aligned}
& \tilde{\Phi}(t,\omega)(x,y) \\
= \ & (\Phi_2(t,\omega)(\alpha,\rho)\cos(2\pi\Phi_1(t,\omega)(\alpha,\rho)),\ \Phi_2(t,\omega)(\alpha,\rho)\sin(2\pi\Phi_1(t,\omega)(\alpha,\rho))),
\end{aligned}$$

and

$$\phi^{\omega}(t) = (\rho^*(\omega)\cos(2\pi\alpha + t), \rho^*(\omega)\sin(2\pi\alpha + t)).$$

It is obvious that

$$\phi^{\omega}(2\pi + t) = \phi^{\omega}(t),$$

and

$$
\begin{aligned}
\tilde{\Phi}(t,\omega)\phi^{\omega}(0) &= \tilde{\Phi}(t,\omega)(\rho^*(\omega)\cos(2\pi\alpha), \rho^*(\omega)\sin(2\pi\alpha)) \\
&= (\rho^*(\theta_t\omega)\cos(2\pi\alpha+t), \rho^*(\theta_t\omega)\sin(2\pi\alpha+t)) \\
&= \phi^{\theta_t\omega}(t).
\end{aligned}
$$

From this we can tell that the random dynamical system generated by the stochastic differential equation (**??**) has a periodic invariant solution. Moreover if $x^2(0)+y^2(0)\neq 0$, then

$$
x^2(t,\theta(-t,\omega)) + y^2(t,\theta(-t,\omega)) \to \rho^*(\omega)^2
$$

as $t\to\infty$.

Andrei Yevik's movie.

Definition 1 *An invariant random periodic solution is an $\mathcal{F}$-measurable periodic function $\phi:\Omega\times I\to X$ of period T such that*

$$
\phi^{\omega}(t+T) = \phi^{\omega}(t) \text{ and } \Phi_t^{\omega}(\phi^{\omega}(t_0)) = \phi^{\theta_t\omega}(t+t_0) \text{ for all } t, t_0 \in I. \tag{10}
$$

5. Random maps–stationary solutions and random periodic solutions

Consider the random map

$$f_\theta(x) = \theta x(1-x),$$

where

$$P(\theta = \lambda) = p, \quad P(\theta = \mu) = 1-p.$$

$(0 < p < 1)$. Define

$$\Omega = \{(\cdots, \varepsilon_{-2}, \varepsilon_{-1}; \varepsilon_0, \varepsilon_1, \varepsilon_2 \cdots) : \varepsilon_k = \lambda, \mu, \ for\ k = 0, \pm 1, \pm 2, \cdots\}$$

as the sample space of the random dynamical system. For

$$\omega = (\cdots, \varepsilon_{-2}, \varepsilon_{-1}; \varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots),$$

Define

$$\begin{aligned} \theta\omega &= (\cdots, \varepsilon_{-1}, \varepsilon_0; \varepsilon_1, \varepsilon_2, \cdots), \\ \theta^{-1}\omega &= (\cdots, \varepsilon_{-3}, \varepsilon_{-2}; \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots), \end{aligned}$$

and for $x \in [0, 1]$,

$$\begin{aligned} \Phi_0(\omega, x) &= x, \\ \Phi(\omega, x) &= f_{\varepsilon_0}(x), \\ \Phi_k(\omega, x) &= \Phi(\theta^{k-1}\omega) \cdots \Phi(\theta\omega)\Phi(\omega, x). \end{aligned}$$

Theorem 5 *(Lian and Zhao (2009)): When $1 < \mu < \lambda \leq 3$, there exists a unique $Y(\omega) \in (0, 1)$ such that*

$$\Phi_k(\omega, Y(\omega)) = Y(\theta^k \omega)$$

$$\Phi_n(\theta_{-n}\omega, x) \to Y(\omega)$$

as $n \to \infty$.

The idea is to prove there exist constants $M > 0$, $L < 1$ and a function $\xi(n)$ ($\xi(n) \to \infty$ as $n \to \infty$) such that for any $x, y \in [1 - \frac{1}{\mu}, 1 - \frac{1}{\lambda}]$

$$|f_{\varepsilon_{-1}} f_{\varepsilon_{-2}} \cdots f_{\varepsilon_{-n}}(x) - f_{\varepsilon_{-1}} f_{\varepsilon_{-2}} \cdots f_{\varepsilon_{-n}}(x)| \leq ML^{\xi(n)}|x - y|. \qquad (11)$$

Random periodic solution of period 2 is a pair of the random variable $\{\alpha(\omega), \beta(\omega)\}$ such that when n is even

$$\begin{aligned}\Phi_n(\omega, \alpha(\omega)) &= \alpha(\theta^n(\omega)),\\ \Phi_n(\omega, \beta(\omega)) &= \beta(\theta^n(\omega)),\end{aligned}$$

and when n is odd,

$$\begin{aligned}\Phi_n(\omega, \alpha(\omega)) &= \beta(\theta^n(\omega)),\\ \Phi_n(\omega, \beta(\omega)) &= \alpha(\theta^n(\omega)).\end{aligned}$$

Recall deterministic case, for $3 < \theta < 1 + \sqrt{5}$,

$$\begin{aligned}\alpha_\theta &= \frac{\theta + 1 - \sqrt{(\theta+1)(\theta-3)}}{2\theta},\\ \beta_\theta &= \frac{\theta + 1 + \sqrt{(\theta+1)(\theta-3)}}{2\theta}.\end{aligned}$$

For the random case, define

$$F_\theta = f_\theta f_\theta, \quad \theta = \lambda, \mu,$$

Consider

$$\omega = (\cdots, \varepsilon_{-2}, \varepsilon_{-1}; \varepsilon_0, \varepsilon_1, \cdots)$$

with

$$\varepsilon_{2n} = \varepsilon_{2n+1}, n = 0, \pm 1, \pm 2, \cdots. \quad (12)$$

Proposition 1 *Assume* $3.0176 \leq \mu < \lambda \leq 1 + \sqrt{5}$, *there exists* $\alpha(\omega)$, $\beta(\omega)$ *such that for any* $x \in [\alpha_\lambda, \alpha_\mu]$

$$F_{\varepsilon_{-1}} F_{\varepsilon_{-2}} \cdots F_{\varepsilon_{-n}}(x) \to \alpha(\omega),$$

and for any $x \in [\beta_\mu, \beta_\lambda]$

$$F_{\varepsilon_{-1}} F_{\varepsilon_{-2}} \cdots F_{\varepsilon_{-n}}(x) \to \beta(\omega).$$

Define

$$\alpha(\theta\omega) = f_{\varepsilon_0}(\beta(\omega)), \beta(\theta\omega) = f_{\varepsilon_0}(\alpha(\omega)).$$

Then

Theorem 6 *(Lian and Zhao (2009)) For* ω *satisfying (**??**),* $\{(\alpha(\omega),\beta(\omega)\}$ *is a random periodic solution of period 2.*