Random Attractors for Stochastic Wave Equations with Critical Exponents on $\mathbb{R}^n$

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June 8-12, 2009 Tianjin

Outline

- **Stochastic wave equations on $\mathbb{R}^n$.**
- **Random attractors theory.**
- **Random absorbing sets for wave equations.**
- **Pullback asymptotic compactness.**
- **Existence of random attractors.**

The stochastic wave equation on $\mathbb{R}^n$:

$$u_{tt} + \alpha u_t - \Delta u + \lambda u + f(x,u) = g(x) + h(x)\frac{dw}{dt}.$$

- λ **and** α **are positive constants.**
- g **and** h **are given functions on** $\mathbb{R}^n$**.**
- w **is a two-sided Wiener process.**
- f **is a smooth nonlinear function satisfying certain growth conditions.**

Question: Long term behavior of solutions?

- Stochastic PDEs on **bounded domains:**
  - Crauel and Flandoli (1994);
  - Flandoli and Schmalfuss (1996);
  - Arnold (1998);
  - Caraballo, Langa and Robinson (2001);
  - Chueshov and Scheutzow (2004);
  - Chueshov and Schmalfuss (2007);
  - Li and Guo (2008), etc.

- Stochastic systems on **unbounded domains:**
  - Bates, Lisei and Lu (2006);
  - Brzezniak and Li (2006);
  - Bates, Lu and Wang (2008).

Difficulty: Sobolev embeddings are not compact on unbounded domains.

Some methods to overcome the difficulty:

- **Weighted spaces:**
  Abergel (1989); Babin and Vishik (1990), etc.

- **Energy equation approach:**
  Ball (1997, 2004); Rosa (1998); Moise, Rosa and X.Wang (1998); Ju (2000, 2001); Goubet and Rosa (2002); Brzezniak and Li (2006), etc.

- **Tail estimates approach:**
  B.Wang (1999); Antoci and Prizzi (2001, 2002); Morillas and Valero (2005); Prizzi (2005); Yang, Sun and Zhong (2007); Bates, Lu and Wang (2008), etc.

Goal: Prove existence of a random attractor for the stochastic wave equation.

The concept of random attractors for stochastic PDEs is an extension of global attractors of deterministic PDEs. This extension was developed by Crauel-Flandoli (1994) and Flandoli-Schmalfuss (1996).

(X, d): metric space with Borel σ-algebra $\mathcal{B}(X)$.
$(\Omega, \mathcal{F}, P)$: probability space.
$\mathcal{D}$: collection of some random subsets of X.

Definition (Shift Operators). **Let $\theta : \mathbb{R} \times \Omega \to \Omega$ be $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable. Then $(\theta_t)_{t \in \mathbb{R}}$ is called a family of shift operators on Ω if**

- **θ_0 is the identity on Ω;**
- **$\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$.**

Definition (Metric Dynamical System). **Let $(\theta_t)_{t \in \mathbb{R}}$ be a family of shift operators. Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system if $(\theta_t)_{t \in \mathbb{R}}$ is measure preserving, i.e., $\theta_t P = P$ for all $t \in \mathbb{R}$.**

Definition (Random Dynamical System). Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system, and Φ a mapping:

$$\Phi : \mathbb{R}^+ \times \Omega \times X \to X \quad (t, \omega, x) \mapsto \Phi(t, \omega, x).$$

Then Φ is called a continuous random dynamical system on X if Φ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$,

(i) $\Phi(0, \omega, \cdot)$ is the identity on X;

(ii) $\Phi(t+s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)\ \forall t, s \geq 0$;

(iii) $\Phi(t, \omega, \cdot) : X \to X$ is continuous $\forall t \in \mathbb{R}^+$.

Definition (Tempered Sets). **Let $\{B(\omega)\}_{\omega\in\Omega}$ be a random bounded subset of X. Then $\{B(\omega)\}_{\omega\in\Omega}$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if there exists $x_0 \in X$ such that for P-a.e. $\omega \in \Omega$,**

$$\lim_{t\to-\infty} e^{\beta t} d(x_0, B(\theta_t\omega)) = 0 \quad \textbf{for all } \beta > 0.$$

All bounded deterministic sets are tempered.

Definition (Random Absorbing Sets). Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set of Φ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $T(\omega, B) > 0$ such that

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq T(\omega, B).$$

Definition (Pullback Asymptotic Compactness). Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\Phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition (Random Attractor). **A random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is called a $\mathcal{D}$-random attractor for Φ if for P-a.e. $\omega \in \Omega$ and all $t \geq 0$,**

- **$\mathcal{A}(\omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;**

- **$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant: $\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega)$;**

- **$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$: for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,**

$$\lim_{t\to\infty} d(\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d is the Hausdorff semi-distance.

Definition (Inclusion-Closed Collection). A collection $\mathcal{D}$ of random subsets of X is called inclusion closed if $\{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ imply $\tilde{D} \in \mathcal{D}$.

Proposition (see, e.g., Bates-Lisei-Lu, 2006). Let $\mathcal{D}$ be inclusion closed and Φ be continuous on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. If Φ has a closed absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$ and is also $\mathcal{D}$-pullback asymptotically compact in X, then Φ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$:

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

The stochastic wave equation on $\mathbb{R}^3$:

$$u_{tt} + \alpha u_t - \Delta u + \lambda u + f(x,u) = g(x) + h(x)\frac{dw}{dt}.$$

The nonlinearity f and its antiderivative F satisfy, for some $\gamma \in [1,3]$,

$$|f(x,u)| \le c_1 |u|^\gamma + \phi_1(x),$$

$$|f'_u(x,u)| \le c_2 |u|^{\gamma-1} + \phi_2(x),$$
$$f(x,u)u - c_3 F(x,u) \ge \phi_3(x),$$
$$F(x,u) \ge c_4 |u|^{\gamma+1} - \phi_4(x),$$

Example. $f(u) = |u|^{\gamma-1}u$**,** $\quad \gamma = 3$**: critical.**

The probability space $(\Omega, \mathcal{F}, P)$ is given by

- $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : w(0) = 0\}$.
- $\mathcal{F}$ **is the Borel σ-algebra induced by the compact-open topology of Ω.**
- P **is the Wiener measure on $(\Omega, \mathcal{F})$.**

The shift operator is given by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

$(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ **is a metric dynamical system.**

Change of variables:

$$z = u_t + \delta u, \quad \delta > 0.$$

The system for (u, z) is given by

$$\frac{\partial u}{\partial t} + \delta u = z,$$

$$\frac{\partial z}{\partial t} + (\alpha - \delta) z + (\lambda + \delta^2 - \alpha\delta) u - \Delta u + f(x, u) = g + h\frac{dw}{dt},$$

with the initial conditions:

$$u(x, \tau) = u_\tau(x), \quad z(x, \tau) = z_\tau(x).$$

Change of variables:

$$v(t, \tau, \omega, v_\tau) = z(t, \tau, \omega, z_\tau) - h\omega(t).$$

Stochastic Wave Equation

The system for (u, v) is given by

$$\frac{\partial u}{\partial t} + \delta u - v = h\omega(t),$$

$$\frac{\partial v}{\partial t} + (\alpha - \delta)v + (\lambda + \delta^2 - \alpha\delta)u - \Delta u + f(x, u) = g + (\delta - \alpha)h\omega(t),$$

with the initial conditions:

$$u(x, \tau) = u_\tau(x), \; v(x, \tau) = v_\tau(x) = z_\tau(x) - h(x)\omega(\tau).$$

This problem is well-posed in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

z **is given by** $z(t, \tau, \omega, z_\tau) = v(t, \tau, \omega, v_\tau) + h\omega(t)$.

Let Φ be a mapping given by

$$\Phi : \mathbb{R}^+ \times \Omega \times (H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)) \to H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

$$\Phi(t, \omega, (u_0, v_0)) = (u(t, 0, \omega, u_0), z(t, 0, \omega, z_0)),$$

where $z(t, 0, \omega, z_0) = v(t, 0, \omega, v_0) + h\omega(t)$.

Φ **is a continuous random dynamical system on** $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ **over** $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

$\mathcal{D} = \{\{B(\omega)\}_{\omega\in\Omega} : B$ **is temped in** $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\}$.

Lemma. **For every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $T = T(\omega, B) < 0$ such that for all $\tau \le T$ and $t \in [\tau, 0]$,**

$$\|u(t,\tau,\omega,u_\tau)\|^2_{H^1(\mathbb{R}^3)} + \|v(t,\tau,\omega,v_\tau)\|^2 \le e^{-\sigma t} r(\omega),$$

where $\sigma > 0$ is a constant, and $r(\omega)$ is tempered.

Particularly, for $t = 0$, we have:

$$\|u(0,\tau,\omega,u_\tau)\|^2_{H^1(\mathbb{R}^3)} + \|v(0,\tau,\omega,v_\tau)\|^2 \le r(\omega).$$

A random absorbing set is given by:

$$K(\omega) = \{(u,v) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : \|u\|^2_{H^1} + \|v\|^2 \le r(\omega)\}$$

Lemma. **For every** $\epsilon > 0$, $B = \{B(\omega)\}_{\omega \in \Omega}$ **and** P**-a.e.** $\omega \in \Omega$, **there exist** $T = T(\epsilon, \omega, B) < 0$ **and** $K(\epsilon, \omega) > 0$ **such that for all** $\tau \leq T$ **and** $t \in [\tau, 0]$,

$$\int\limits_{|x| \geq K} \left(|u(t, \tau, \omega, u_\tau)|^2 + |\nabla u(t, \tau, \omega, u_\tau)|^2 \right) dx \leq \epsilon e^{-\sigma t},$$

$$\int\limits_{|x| \geq K} |v(t, \tau, \omega, v_\tau)|^2 dx \leq \epsilon e^{-\sigma t},$$

where σ **is a positive deterministic constant.**

The estimates for $t = 0$ **are of particular interest.**

Asymptotic Compactness: Subcritical Case

Asymptotic compactness: For P-a.e. $\omega \in \Omega$, $\{\Phi(t_n, \theta_{-t_n}\omega, (u_{0,n}, v_{0,n}))\}$ has a convergent subsequence in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ provided $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $(u_{0,n}, v_{0,n}) \in B(\theta_{-t_n}\omega)$.

Idea of proof:

- By tail estimates, $\forall \epsilon > 0$, $\exists K(\epsilon), N(\epsilon) > 0$:

  $$\|\Phi(t_n, \theta_{-t_n}\omega, (u_{0,n}, v_{0,n}))\|_{H^1(\mathbb{R}^3\setminus Q_K)\times L^2(\mathbb{R}^3\setminus Q_K)} \le \epsilon,$$

  for $n \ge N$ and $Q_K = \{x \in \mathbb{R}^3 : |x| \le K(\epsilon)\}$.

- Prove Φ is asymptotically compact in $H^1(Q_K) \times L^2(Q_K)$ by a decomposition trick.

Asymptotic Compactness: Subcritical Case

Let ψ be a smooth function such that

$$\psi(s) = 1 \quad \textbf{if } |s| \leq 1; \qquad \psi(s) = 0 \quad \textbf{if } |s| \geq 2.$$

Given $k \geq 1$, set $\tilde{u} = \psi(\frac{|x|}{k})u$ and $\tilde{v} = \psi(\frac{|x|}{k})v$.

The system for $(\tilde{u}, \tilde{v})$ is defined on Q_{2k}:

$$\tilde{u}_t + \delta\tilde{u} - \tilde{v} = \psi h\omega(t),$$

$$\tilde{v}_t + (\alpha - \delta)\tilde{v} + (\lambda + \delta^2 - \alpha\delta)\tilde{u} - \Delta\tilde{u} + \psi f(x, u)$$
$$= \psi g + (\delta - \alpha)\psi h\omega(t) - u\Delta\psi - 2\nabla\psi\nabla u,$$

with zero boundary conditions.

Asymptotic Compactness: Subcritical Case

Consider the eigenvalue problem:

$$-\Delta \tilde{u} = \lambda \tilde{u} \quad \textbf{in} \quad Q_{2k}, \quad \textbf{with} \quad \tilde{u}|_{\partial Q_{2k}} = 0.$$

Eigenvalues: $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \to \infty.$

Eigenfunctions: $\{e_n\}_{n=1}^{\infty}$**, a basis of** $L^2(Q_{2k})$**.**

$X_n =$ **span**$\{e_1, \cdots, e_n\}$ **and** $P_n : L^2(Q_{2k}) \to X_n$**.**

Lemma. For every $\epsilon > 0$**,** $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ **and** P**-a.e.** $\omega \in \Omega$**, there exist** $K = K(\omega, \epsilon) > 0$**,** $N = N(\omega, \epsilon) > 0$ **and** $T = T(B, \omega, \epsilon) < 0$ **such that for all** $k \ge K$**,** $n \ge N$ **and** $\tau \le T$**,**

$$\|(I - P_n)\tilde{u}(0, \tau, \omega)\|_{H_0^1(Q_{2k})} + \|(I - P_n)\tilde{v}(0, \tau, \omega)\|_{L^2(Q_{2k})} \le \epsilon.$$

Asymptotic Compactness: Subcritical Case

Proof. Set $\tilde{u}_n = (I - P_n)\tilde{u}$ **and** $\tilde{v}_n = (I - P_n)\tilde{v}$**.**

$$\frac{d}{dt}\left(\|\tilde{v}_n\|^2 + \alpha_1\|\tilde{u}_n\|^2 + \|\nabla\tilde{u}_n\|^2 + 2(\psi f(x,u), \tilde{u}_n)\right)$$

$$+\sigma\left(\|\tilde{v}_n\|^2 + \alpha_1\|\tilde{u}_n\|^2 + \|\nabla\tilde{u}_n\|^2 + 2(\psi f(x,u), \tilde{u}_n)\right)$$
$$\leq c\lambda_{n+1}^{\frac{\gamma-3}{2}}\left(\|v\|^6 + \|u\|_{H^1}^{3\gamma-3}\right) + \frac{c}{k^2}\|u\|_{H^1}^2 + \dots.$$

For $\gamma < 3$ **and sufficiently large** n **and** k**:**

$$\frac{d}{dt}\left(\|\tilde{v}_n\|^2 + \alpha_1\|\tilde{u}_n\|^2 + \|\nabla\tilde{u}_n\|^2 + 2(\psi f(x,u), \tilde{u}_n)\right)$$

$$+\sigma\left(\|\tilde{v}_n\|^2 + \alpha_1\|\tilde{u}_n\|^2 + \|\nabla\tilde{u}_n\|^2 + 2(\psi f(x,u), \tilde{u}_n)\right)$$
$$\leq \epsilon\left(\|v\|^6 + \|u\|_{H^1}^{3\gamma-3}\right) + \epsilon\|u\|_{H^1}^2 + \dots.$$

Asymptotic Compactness: Subcritical Case

Asymptotic compactness: $\{\Phi(t_n, \theta_{-t_n}\omega, (u_{0,n}, v_{0,n}))\}$ **has a convergent subsequence in** $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ **if** $t_n \to \infty$ **and** $(u_{0,n}, v_{0,n}) \in B(\theta_{-t_n}\omega)$**.**

Proof:

- **Tail estimates:** $\forall \epsilon$, $\exists K$, $\exists N$ **s.t. for** $n \geq N$,

$$\|\Phi(t_n, \theta_{-t_n}\omega, (u_{0,n}, v_{0,n}))\|_{H^1(\mathbb{R}^3 \setminus Q_K) \times L^2(\mathbb{R}^3 \setminus Q_K)} \leq \epsilon,$$

- **There is** $m > 0$ **such that for** $n \geq N$,

$$\|(I - P_m)\Phi(t_n, \theta_{-t_n}\omega, (u_{0,n}, v_{0,n}))\|_{H^1(Q_K) \times L^2(Q_K)} \leq \epsilon.$$

- $P_m(\Phi(t_n, \theta_{-t_n}\omega, (u_{0,n}, v_{0,n})))$ **is bounded in the finite dimensional space** $P_m(H^1(Q_K) \times L^2(Q_K))$**.**

Asymptotic Compactness: Critical Case

Method: Tail estimates and energy equations.

Idea of the energy equation approach:

$$u_n \to u \text{ in } L^2 \Longleftrightarrow u_n \rightharpoonup u \text{ in } L^2 \text{ and } \|u_n\|_{L^2} \to \|u\|_{L^2}$$

The energy equation approach was

- **introduced by J. M. Ball (1997, 2004);**
- **used by Rosa (1998); Moise, Rosa and X.Wang (1998); Ju (2000, 2001); Goubet and Rosa (2002); Brzezniak and Li (2006), and many others.**

Asymptotic Compactness: Critical Case

$H^1 \times L^2$ **energy of wave equation:**

$$\frac{d}{dt}E(u,v) + 4\sigma E(u,v) = \Psi(u,v),$$

$$E(u,v) = \|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^3} F(x,u)dx,$$

$$\Psi(u,v) = -2(\alpha-\delta-2\sigma)\|v\|^2 - 2(\delta-2\sigma)(\lambda+\delta^2-\alpha\delta)\|u\|^2$$

$$-2(\delta-2\sigma)\|\nabla u\|^2 + 8\sigma\int_{\mathbb{R}^3} F(x,u)dx - 2\delta\int_{\mathbb{R}^3} f(x,u)udx$$

$$+2(\lambda+\delta^2-\alpha\delta)(u,h)\omega(t) + 2(\nabla u, \nabla h)\omega(t)$$

$$+2\omega(t)\int_{\mathbb{R}^3} f(x,u)h(x)dx + 2(g,v) + 2(\delta-\alpha)(v,h)\omega(t).$$

Asymptotic Compactness: Critical Case

$H^1 \times L^2$ **energy equation:**

$$E(u(t,\tau,\omega,u_\tau),v(t,\tau,\omega,v_\tau)) = e^{-4\sigma(t-\tau)}E(u_\tau,v_\tau)$$

$$+\int_\tau^t e^{4\sigma(\xi-t)}\Psi(u(\xi,\tau,\omega,u_\tau),v(\xi,\tau,\omega,v_\tau))d\xi.$$

Lemma (asymptotic compactness)
$\{\Phi(t_n,\theta_{-t_n}\omega,(u_{0,n},v_{0,n}))\}$ **is precompact in** $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ **if** $t_n \to \infty$ **and** $(u_{0,n},v_{0,n}) \in B(\theta_{-t_n}\omega)$**; that is,**

$\{(u(0,-t_n,\omega,u_{0,n}),v(0,-t_n,\omega,v_{0,n}))\}$ **has a convergent subsequence in** $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ **if** $t_n\to\infty$ **and** $(u_{0,n},v_{0,n})\in B(\theta_{-t_n}\omega)$**.**

Asymptotic Compactness: Critical Case

Proof.

- **There is** N **such that for all** $n \geq N$**,**

  $$\|u(0,-t_n,\omega,u_{0,n})\|_{H^1}^2 + \|v(0,-t_n,\omega,v_{0,n})\|^2 \leq R(\omega).$$

- **There is** $(\tilde{u},\tilde{v}) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ **such that, up to a subsequence,**

  $$(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n})) \to (\tilde{u},\tilde{v}) \text{ **weakly**},$$

  This implies that

  $$\liminf_{n\to\infty} \|(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n}))\| \geq \|(\tilde{u},\tilde{v})\|$$

  We only need to prove

  $$\limsup_{n\to\infty} \|(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n}))\| \leq \|(\tilde{u},\tilde{v})\|$$

Asymptotic Compactness: Critical Case

- **Notice that, for any fixed $m \geq 1$,**

$$u(0, -t_n, \omega, u_{0,n}) = u(0, -m, \omega, u(-m, -t_n, \omega, u_{0,n})),$$

and

$$v(0, -t_n, \omega, v_{0,n}) = v(0, -m, \omega, v(-m, -t_n, \omega, v_{0,n})).$$

- **Energy equation in $H^1 \times L^2$:**

$$E(u(t, \tau, \omega, u_\tau), v(t, \tau, \omega, v_\tau)) = e^{-4\sigma(t-\tau)} E(u_\tau, v_\tau)$$

$$+ \int_\tau^t e^{4\sigma(\xi - t)} \Psi(u(\xi, \tau, \omega, u_\tau), v(\xi, \tau, \omega, v_\tau)) d\xi.$$

Asymptotic Compactness: Critical Case

$$E(u(0,-t_n,\omega,u_{0,n}),v(0,-t_n,\omega,v_{0,n}))$$

$$= e^{-4\sigma m}E(u(-m,-t_n,\omega,u_{0,n}),v(-m,-t_n,\omega,v_{0,n}))$$

$$-2(\alpha-\delta-2\sigma)\int_{-m}^{0}e^{4\sigma\xi}\|v(\xi,-m,\omega,v(-m,-t_n,\omega,v_{0,n}))\|^2 d\xi$$

$$-2(\delta-2\sigma)(\lambda+\delta^2-\alpha\delta)\int_{-m}^{0}e^{4\sigma\xi}\|u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\|^2 d\xi$$

$$-2(\delta-2\sigma)\int_{-m}^{0}e^{4\sigma\xi}\|\nabla u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\|^2 d\xi$$

$$+8\sigma\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}F(x,u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n})))dxd\xi$$

$$-2\delta\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\times f(x,u(\xi,-m,\omega,u))$$

$$+2(\lambda+\delta^2-\alpha\delta)\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}h(x)u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\omega(\xi)$$

$$+2\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}\nabla h(x)\cdot\nabla u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\omega(\xi)$$

$$+2\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}h(x)f(x,u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n})))\omega(\xi)$$

$$+2\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}g(x)v(\xi,-m,\omega,v(-m,-t_n,\omega,v_{0,n}))dxd\xi$$

$$+2(\delta-\alpha)\int_{-m}^{0}e^{4\sigma\xi}\int_{\mathbb{R}^3}h(x)v(\xi,-m,\omega,v(-m,-t_n,\omega,v_{0,n}))\omega(\xi)dxd\xi.$$

Asymptotic Compactness: Critical Case

- **For any fixed m, there is $(\tilde{u}_m, \tilde{v}_m) \in H^1 \times L^2$:**

$$\tilde{u} = u(0, -m, \omega, \tilde{u}_m) \quad \textbf{and} \quad \tilde{v} = v(0, -m, \omega, \tilde{v}_m),$$

- **Energy equation in $H^1 \times L^2$:**

$$E(u(t, \tau, \omega, u_\tau), v(t, \tau, \omega, v_\tau)) = e^{-4\sigma(t-\tau)} E(u_\tau, v_\tau)$$

$$+ \int_\tau^t e^{4\sigma(\xi - t)} \Psi(u(\xi, \tau, \omega, u_\tau), v(\xi, \tau, \omega, v_\tau)) d\xi.$$

Asymptotic Compactness: Critical Case

$$
\begin{aligned}
E(\tilde{u},\tilde{v}) = e^{-4\sigma m} E(\tilde{u}_m,\tilde{v}_m) \\
-2(\alpha-\delta-2\sigma)\int_{-m}^{0} e^{4\sigma\xi}\|v(\xi,-m,\omega,\tilde{v}_m)\|^2 d\xi \\
-2(\delta-2\sigma)(\lambda+\delta^2-\alpha\delta)\int_{-m}^{0} e^{4\sigma\xi}\|u(\xi,-m,\omega,\tilde{u}_m)\|^2 d\xi \\
-2(\delta-2\sigma)\int_{-m}^{0} e^{4\sigma\xi}\|\nabla u(\xi,-m,\omega,\tilde{u}_m)\|^2 d\xi \\
+8\sigma\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} F(x,u(\xi,-m,\omega,\tilde{u}_m))dxd\xi \\
-2\delta\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} u(\xi,-m,\omega,\tilde{u}_m)\times f(x,u(\xi,-m,\omega,\tilde{u}_m)) \\
+2(\lambda+\delta^2-\alpha\delta)\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} h(x)u(\xi,-m,\omega,\tilde{u}_m)\omega(\xi) \\
+2\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} \nabla h(x)\cdot\nabla u(\xi,-m,\omega,\tilde{u}_m)\omega(\xi)dxd\xi \\
+2\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} h(x)f(x,u(\xi,-m,\omega,\tilde{u}_m))\omega(\xi) \\
+2\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} g(x)v(\xi,-m,\omega,\tilde{v}_m)dxd\xi \\
+2(\delta-\alpha)\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} h(x)v(\xi,-m,\omega,\tilde{v}_m)\omega(\xi)dxd\xi.
\end{aligned}
$$

Asymptotic Compactness: Critical Case

$$
\begin{aligned}
&E(u(0,-t_n,\omega,u_{0,n}),v(0,-t_n,\omega,v_{0,n}))\\
&= e^{-4\sigma m}E(u(-m,-t_n,\omega,u_{0,n}),v(-m,-t_n,\omega,v_{0,n}))\\
&-2(\alpha-\delta-2\sigma)\int_{-m}^{0} e^{4\sigma\xi}\|v(\xi,-m,\omega,v(-m,-t_n,\omega,v_{0,n}))\|^2 d\xi\\
&-2(\delta-2\sigma)(\lambda+\delta^2-\alpha\delta)\int_{-m}^{0} e^{4\sigma\xi}\|u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\|^2 d\xi\\
&-2(\delta-2\sigma)\int_{-m}^{0} e^{4\sigma\xi}\|\nabla u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\|^2 d\xi\\
&+8\sigma\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} F(x,u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n})))dxd\xi\\
&-2\delta\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\times f(x,u(\xi,-m,\omega,u))\\
&+2(\lambda+\delta^2-\alpha\delta)\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} h(x)u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\omega(\xi)\\
&+2\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} \nabla h(x)\cdot\nabla u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n}))\omega(\xi)dxd\xi\\
&+2\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} h(x)f(x,u(\xi,-m,\omega,u(-m,-t_n,\omega,u_{0,n})))\omega(\xi)\\
&+2\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} g(x)v(\xi,-m,\omega,v(-m,-t_n,\omega,v_{0,n}))dxd\xi\\
&+2(\delta-\alpha)\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} h(x)v(\xi,-m,\omega,v(-m,-t_n,\omega,v_{0,n}))\omega(\xi)dxd\xi.
\end{aligned}
$$

Asymptotic Compactness: Critical Case

For instance, the following convergence holds:

$$\lim_{n\to\infty}\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} F(x, u(\xi, -m, \omega, u(-m, -t_n, \omega, u_{0,n})))$$

$$= \int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3} F(x, u(\xi, -m, \omega, \tilde{u}_m))dxd\xi,$$

which is implied by

$$|\int_{-m}^{0} e^{4\sigma\xi}\int_{\mathbb{R}^3}\Big(F(x, u(\xi, -m, \omega, u(-m, -t_n, \omega, u_{0,n}))) - F(x, u(\xi, -m, \omega, \tilde{u}_m))\Big)|$$

$$\leq \int_{-m}^{0} e^{4\sigma\xi}\int_{|x|>k} |F(x, u(\xi, -m, \omega, u(-m, -t_n, \omega, u_{0,n})))-F(x, u(\xi, -m, \omega, \tilde{u}_m))|$$

$$+|\int_{-m}^{0} e^{4\sigma\xi}\int_{|x|<k} F(x, u(\xi, -m, \omega, u(-m, -t_n, \omega, u_{0,n})))-F(x, u(\xi, -m, \omega, \tilde{u}_m))|.$$

Asymptotic Compactness: Critical Case

Finally, we have

$$\limsup_{n\to\infty} E(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n})) \le E(\tilde{u},\tilde{v}),$$

which implies that

$$\limsup_{n\to\infty} \|(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n}))\| \le \|(\tilde{u},\tilde{v})\|.$$

We also have

$$\liminf_{n\to\infty} \|(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n}))\| \ge \|(\tilde{u},\tilde{v})\|.$$

Then it follows that

$$\lim_{n\to\infty} \|(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n}))\| = \|(\tilde{u},\tilde{v})\|,$$

which along with the weak convergence yields

$(u(0,-t_n,\omega,u_{0,n}), v(0,-t_n,\omega,v_{0,n})) \to (\tilde{u},\tilde{v})$ **strongly**.

Theorem. **The random dynamical system Φ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, i.e., for P-a.e. $\omega\in\Omega$,**

- **$\mathcal{A}(\omega)$ is compact in $L^2(\mathbb{R}^n)$.**

- **$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant:**

$$\phi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \quad \forall\ t\geq 0.$$

- **$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every tempered random subset $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$:**

$$\lim_{t\to\infty} \mathrm{d}_{H^1\times L^2}(\phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),\mathcal{A}(\omega)) = 0.$$

Remarks:

- **Existence of attractors in**

$$\mathcal{D}_\sigma = \{D = \{D(\omega)\}_{\omega\in\Omega} : \lim_{t\to\infty} e^{-\sigma t}\|D(\theta_{-t}\omega)\| = 0\}.$$

- **Existence of invariant measures.**

Future work:

- **Uniqueness of invariant measures.**
- **Structures of attractors.**